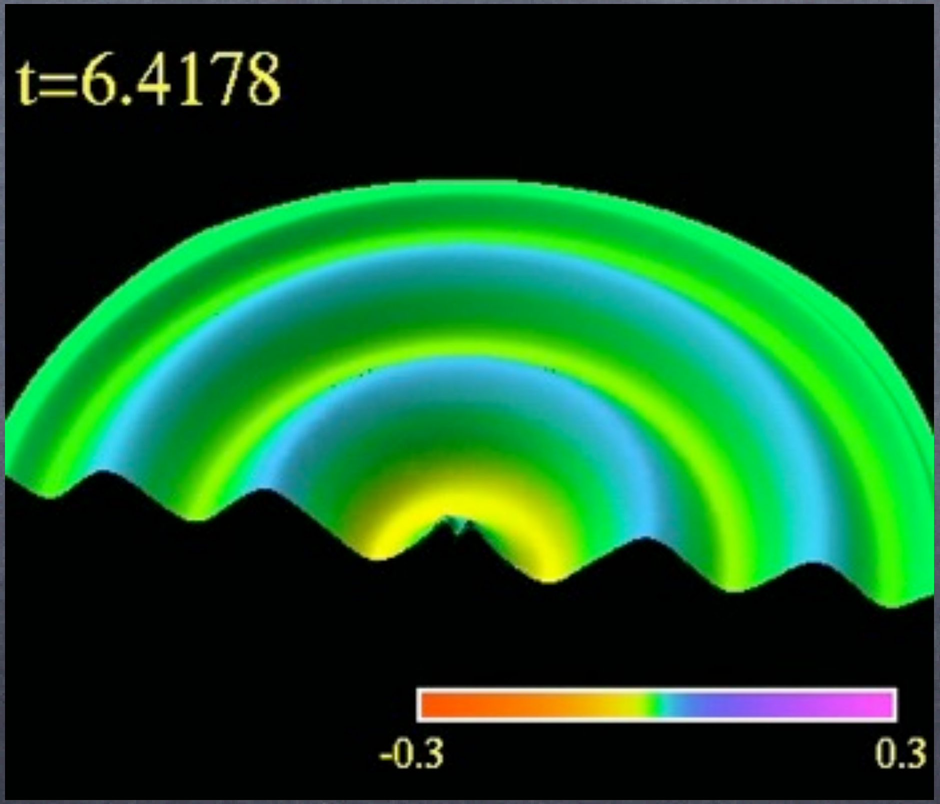
Introduction to Theory and Numerics of Partial Differential Equations VI: Wave equation and Einstein equations in spherical symmetry/1+1 dimensions



Sascha Husa

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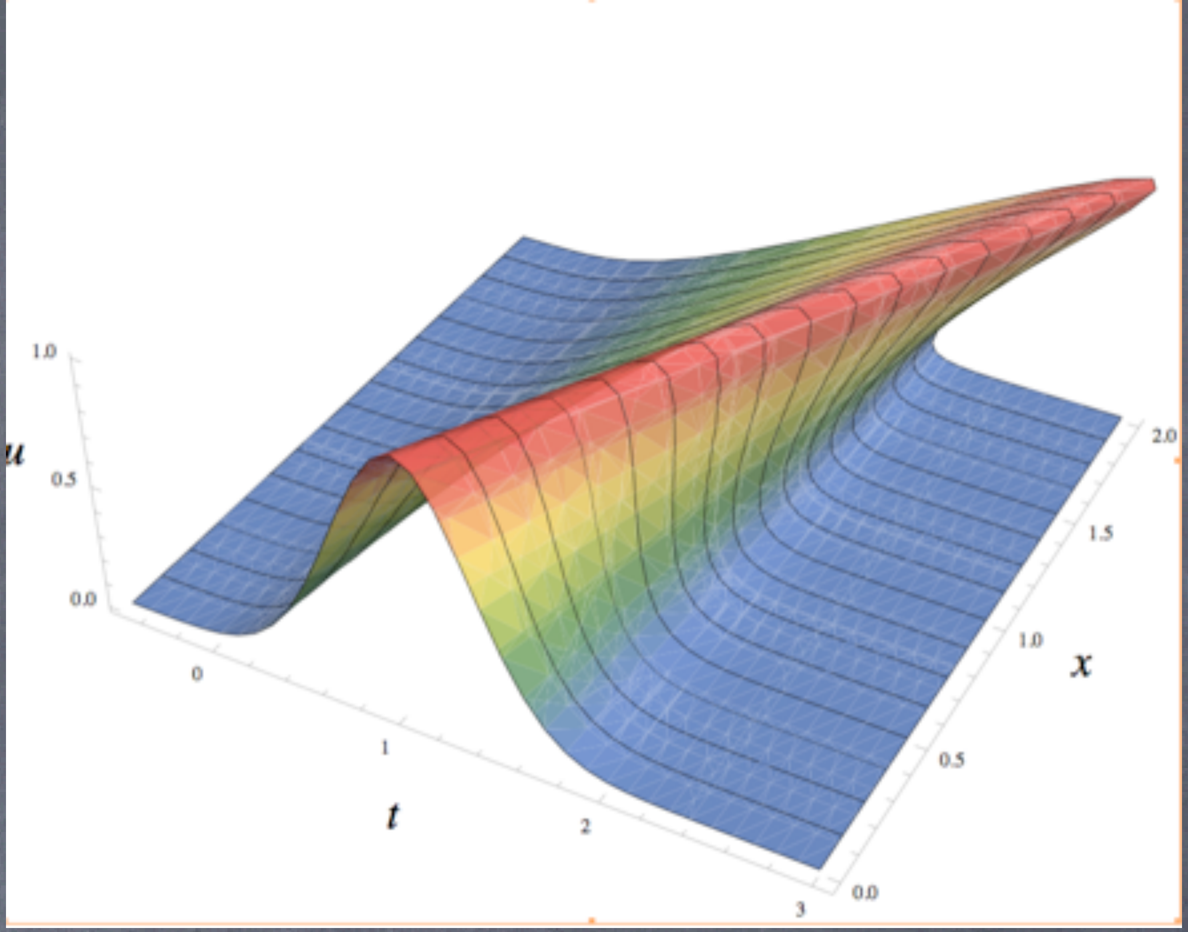
Lab goals for today

- Look at energy, energy conservation and characteristic variables in your wave equation code.
- Upgrade your code to fourth order centered finite differences. What is the effect on accuracy/ efficiency?
- Change boundary conditions to reflecting, incoming signal, and "outgoing".
- Implement the shifted wave equation.
- Start planning your coupled Einstein code (who can form a black hole tomorrow?)

smooth and distributional solutions

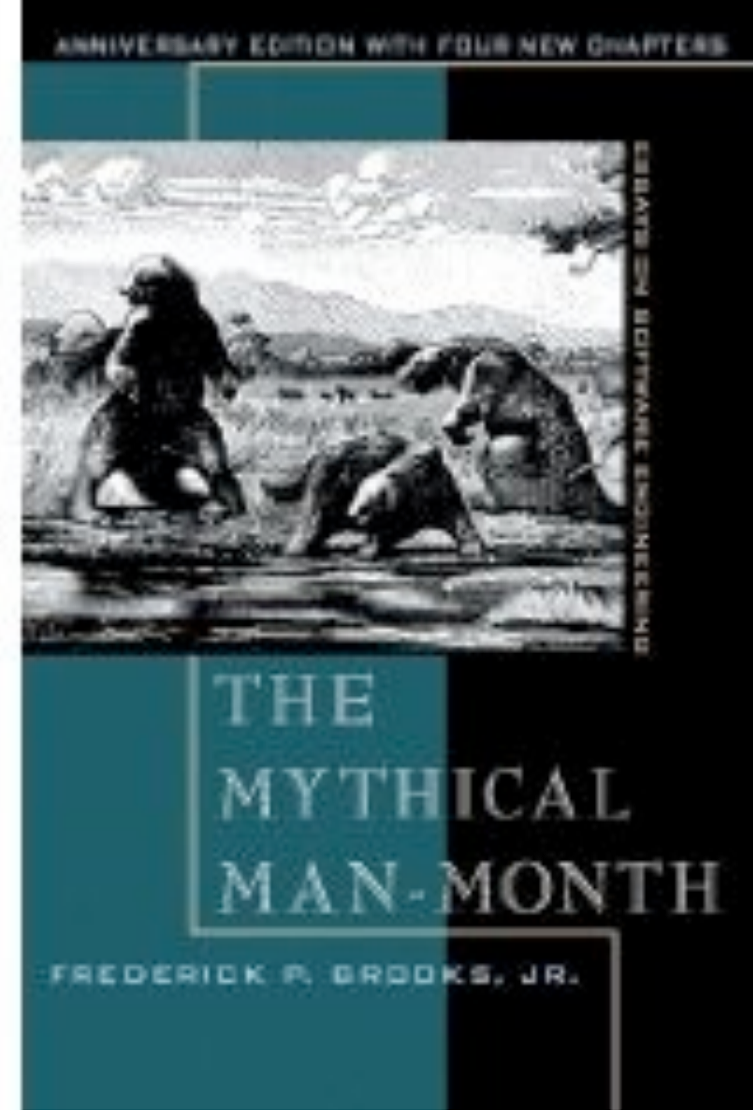
- Burger's equation: $u_t = u\, u_x$.
- Characteristic speeds depend on u, peak velocity overtakes rest of the wave after some time.



- More generally: characteristics can cross, typically signifies physical breakdown of underlying PDE, like in fluid dynamics.
- Unless a PDE is linearly degenerate (speeds independent of solution), shocks can form from smooth data in a finite time.
- Vacuum EE: can be written in linearly degenerate form, do not expect physical shocks, but shocks can form due to bad gauge conditions.
- Numerical methods for fluid dynamics are dominated by methods that deal with shocks - e.g. propagate shocks at correct physical speed.
- Solutions of vacuum GR are smooth except due to bad gauges or physical singularities, high order FD or spectral ideal!

Cost & error

- Want: approximate solution to a PDE with an error estimate at affordable cost.
  - affordable cost in running the code,
  - affordable cost in developing the code,
  - affordable cost modifying the code for new problems.
- We will only need a certain accuracy, but it may not be easy to understand what it should be, in particular if we want to study new phenomena.
- Don't waste too much time trying to interpret poor numerical data, use the time to produce better data. Don't become obsessed with machine accuracy (double precision $\sim 10^{-16}$, 64 bit). Usually this is more than enough, sometimes not!
- If you have a good idea and a working code, computer time will come to you.
- Often most of the human time is spent on debugging the code, and on trying to figure out "the physics" when numerical data are poor (inaccurate, noisy, ...)
- Defensive programming & good enough resolution & enough output!



Avoid problems early ...

Defensive programming is a form of defensive design intended to ensure the continuing function of a piece of software in spite of unforeseeable usage of said software. The idea can be viewed as reducing or eliminating the prospect of Murphy's Law having effect. Defensive programming techniques are used especially when a piece of software could be misused mischievously or inadvertently to catastrophic effect.

Defensive programming is an approach to improve software and source code, in terms of:

- General quality - Reducing the number of software bugs and problems.
- Making the source code comprehensible - the source code should be readable and understandable so it is approved in a code audit.
- Making the software behave in a predictable manner despite unexpected inputs or user actions.

- Keep it Simple!

Finite difference stencils in Fourier space

- Example: second order centered finite difference stencils.

$$\partial_x f \approx \frac{f_3 - f_1}{2h} \qquad \partial_{xx} f \approx \frac{f_3 - 2f_2 + f_1}{h^2}$$

- apply them to a wave of frequency ω:

$$f(x) = e^{i\omega x}$$

- Apply finite difference operator to function:

$$\frac{e^{ihw} - e^{-ihw}}{2h}$$

- Simplify expression

$$\hat{D}_2 = \frac{i \sin(hw)}{h}$$

Numerical stability for first order hyperbolic systems

- P : linear constant coefficient differential operator

$$\partial_t u = P(\partial_x)\, u \qquad \hat{P}(i\omega) : \quad \partial/\partial x_j \to i\omega_j = i\frac{\xi_j}{h} \quad (\text{i.e. } \hat{P} = i\omega_i A^i)$$

- WP is equivalent to $|e^{\hat{P}(i\omega)t}| \leq K e^{\alpha t}$ -> need $\hat{P}$ diagonalizable
- discretize, e.g. 2nd order centered: $\partial_x \Rightarrow \frac{i}{h}\sin\xi$ (exercise!)
- n-th order Runge Kutta: $v^{n+1} = Q\, v^n = p(\Delta t\, P) v^n \quad p(x) = \sum_{l=0}^{l=n} \frac{x^l}{l!}$
- Fourier: $\hat{v}^{n+1}(\xi) = \hat{Q}(\xi)\, v^n(\xi) = p(\Delta t\, \hat{P}(\xi))\hat{v}^n(\xi)$
- now we can solve: $\hat{v}^n(\xi) = \hat{Q}(\xi)^n\, v^0(\xi)$
- amplification matrix $\hat{Q}$ diagonalizable if $\hat{P}$ is!
- stability if eigenvalues satisfy: $|q_\mu| \leq 1$, $q_\mu = p(\Delta t\, p_\mu)$
- PDE does not explicitly depend on direction or dimension d

$$\lambda = \frac{\Delta t}{\Delta x} \leq \frac{\alpha_0}{\sigma(A)\sqrt{d}}, \qquad \alpha_0 = 2(ICN), \sqrt{3}(RK3), \sqrt{8}(RK4)$$

nonlinear systems and dissipation

- Numerical schemes for quasi-linear hyperbolic PDEs: can use the same numerical methods, but need to dissipate high frequency modes to achieve numerical stability.

- Standard procedure: add Kreiss-Oliger dissipation for 2r-2 accurate scheme, dissipation strength $\sigma > 0$:

$$\partial_t u \to \partial_t u + Qu, \quad Q_{2r} = \sigma \frac{(-\Delta x)^{2r-1}}{2^{2r}} (D_+)^r (D_-)^r$$

  - does not degrade convergence order!

- Adding too much dissipation decreases time-step limit (makes equations behave more and more like heat equation).

- Artificial dissipation in fluid dynamics has traditionally been used to smear out shocks, superseded by "High resolution shock capturing" methods.

Second order in space systems: motivation

- Can we discuss well-posedness for second order in space systems like YADM and g-harmonic without first order reduction?
- Reduction to first order in time -> new evolution equations
- Reduction to first order in space -> new evolution & constraint equations.
  - enlarges solution space, new unphysical d.o.f. may give rise to instabilities (remember EM on curved background).
- General theory for WP of 2nd order in space only > 2004
  - How about accuracy of 1st vs. 2nd order in space?
- generalized wave equations: WP

example: mixed order wave equation

- Time domain: $h_{,t} = k, \quad k_{,t} = h_{,xx}$
- Frequency domain, t -> ω: $\hat{h}_{,t} = \hat{k}, \quad \hat{k}_{,t} = -\omega^2 \hat{h}$
- Introduce new variable λ as the square root of $h_{,xx}$:

$$\hat{\lambda} := i\omega\hat{h} \quad \Rightarrow \quad \hat{\lambda}_{,t} = i\omega\hat{k}, \quad \hat{k}_{,t} = i\omega\hat{\lambda} \quad \hat{h}_{,t} = \hat{k}$$

$$\partial_t \begin{pmatrix} h \\ k \\ \lambda \end{pmatrix} = A \begin{pmatrix} h \\ k \\ \lambda \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & i\omega \end{pmatrix}$$

- Characteristic speeds are -1,1,0; problem is symmetric hyperbolic and well posed in the norm (L^2 does not always work!):

$$||u||^2 = \int \left(|h|^2 + |k|^2 + |\partial_x h|^2 \right) \, dx$$

- In the Fourier domain this system could be treated in analogy with first order in space systems, using a “pseudo-differential reduction” - but variables play different roles depending on how often they are differentiated.
- In the discrete case, we will have to choose an appropriate discretization for the derivative in the norm!

second order in space hyperbolic systems

- normal form: P takes second derivatives of u, but not v.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}, \quad P = \begin{pmatrix} A^i\partial_i + B & C \\ D^{ij}\partial_i\partial_j + E^i\partial_i + F & G^i\partial_i + J \end{pmatrix}$$

- Second order principal symbol $\hat{P} = \begin{pmatrix} i\omega A^n & C \\ -\omega^2 D^{nn} & i\omega G^n \end{pmatrix}$
- Analyze WP & numerical stability by pseudo-differential reduction (first order reduction in Fourier space).
- WP reduces to diagonalizability of $\hat{P}_{\text{reduced}} = i\omega \begin{pmatrix} A^n & C \\ D^{nn} & G^n \end{pmatrix}$
- Discrete stability is **not** implied by WP + centered FD + small Δt
  - $\partial_{xx} = \partial_x\partial_x$ does not carry over from continuum, e.g.

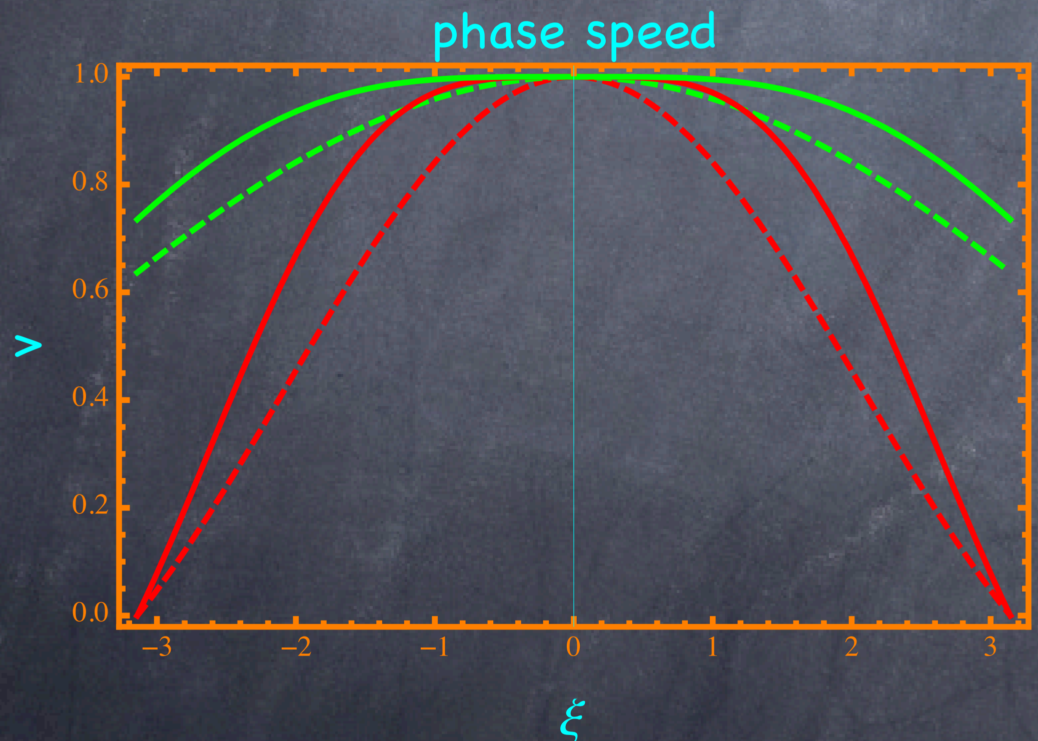
$$\hat{D}^{(2)} = -\frac{4}{\Delta x^2}\sin^2\frac{\xi}{2} \neq \left(\frac{i}{\Delta x}\sin\xi\right)^2$$

- discrete norm: $\|u\|_h^2 + \|v\|_h^2 + \sum_{i=1}^{d} \|D_{+i}u\|_h^2, \; D_+v_j = \frac{v_{j+1} - v_j}{\Delta x}$

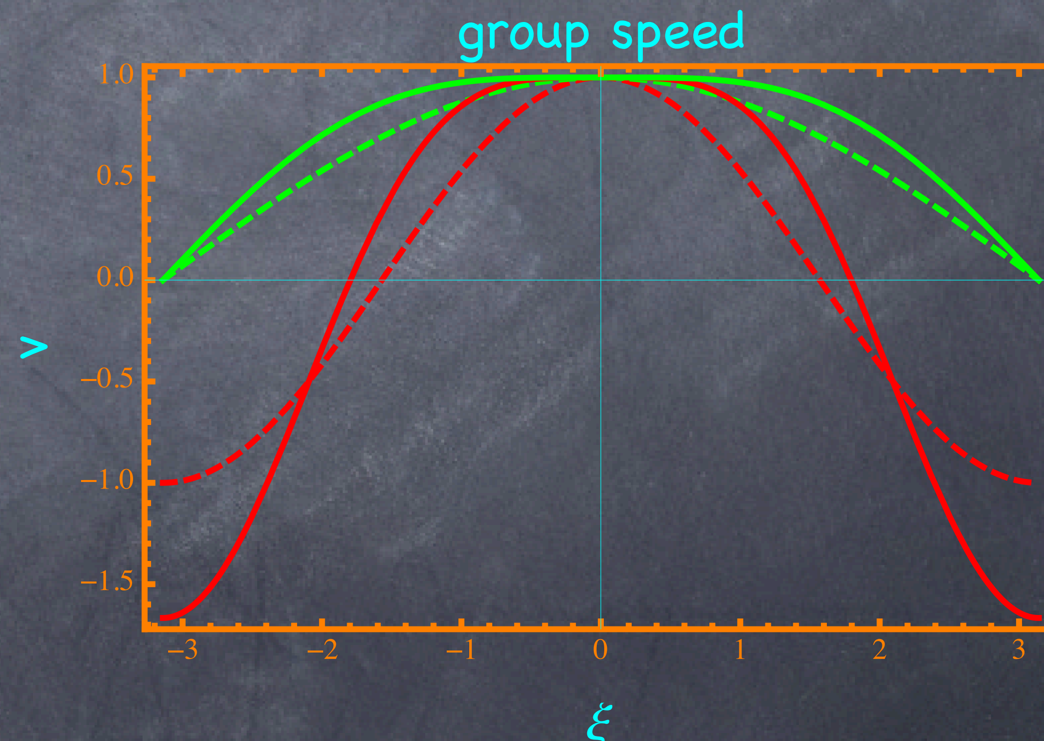
comparison 1st vs 2nd order in space

- λ(ξ) eigenval. of $\hat{P}(\xi)$
- phase velocity $v_p = i\frac{\lambda}{\omega}$
- group vel. $v_g = i\frac{d\lambda}{d\omega}$

	2nd order accurate	
	advective	wave
v_p	$\frac{\sin\xi}{\xi} \approx 1 - \frac{\xi^2}{6} + O(\xi^4)$	$\frac{2}{\xi}\sin\frac{\xi}{2} \approx 1 - \frac{\xi^2}{24} + O(\xi^4)$
v_g	$\cos\xi \approx 1 - \frac{\xi^2}{2} + O(\xi^4)$	$\cos\frac{\xi}{2} \approx 1 - \frac{\xi^2}{8} + O(\xi^4)$
C.I.	α_0	$\alpha_0/2$
u.m.	$0, \pi$	0
f.u.m.	$\pm\frac{\pi}{2} \approx \pm 1.571$	π



advection eq.
wave eq.
- - 2nd order
--- 4th order



- modes with speeds of the wrong sign will come out of BHs!
- second order in space systems have high frequency damping built in!

Some simple incarnations of the scalar wave equation

- Scalar WEQ defined with metric g_{ab}, may consider fixed metric, or couple scalar field to Einstein equations:

$$g^{ab}\nabla_a \nabla_b \phi = 0 \qquad\qquad G_{ab}[g] = 8\pi G T_{ab}[\phi]$$

- e.g. WEQ on Minkowski space. 1+1 dimensional problems are obtained by considering plane waves

$$g^{\mu\nu} = \eta_{\mu\nu}, \quad \phi(x,t) \to \phi_{tt} = \phi_{xx}$$

- or spherically symmetric waves

$$g^{\mu\nu} = \eta_{\mu\nu}, \quad \phi(r,t) \to \phi_{tt} = \phi_{rr} + \frac{2}{r}\phi_{rr}$$

- Scaling of variables can do miracles:

$$\tilde{\phi}(r,t) := r\phi(r,t) \to \tilde{\phi}_{tt} = \tilde{\phi}_{rr}$$

Scalar field energy in flat space

- Energy density ρ gives rise to a conserved energy E:

$$E = \int_{R^3} \rho \, d^3x \qquad \rho = \left(\frac{\partial \phi}{\partial t}\right)^2 + |\nabla \phi|^2$$

- For plane waves we get

$$\rho = \left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\frac{\partial \phi}{\partial x}\right)^2$$

- Because of energy conservation, for plane waves the field strength can't decay.

- In spherical symmetry we expect decay with 1/r

$$\phi(r,t) := \frac{\tilde{\phi}(r,t)}{r}$$

Boundary conditions

- Can consider 3 distinct cases:
  - finite grid without boundaries, use periodic boundary conditions = identify end points, DONE
  - finite grid with boundaries, need to impose boundary conditions (reflecting, incoming signal, outgoing=no incoming signal)
  - infinite grid. need to "pull in" infinity with a coordinate transformation, will lead to singular equations -> investigate tomorrow

Spice up the wave equation with moving coordinates

- Restrict to plane waves in 1 space dimension:

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2$$

  - redefine x coordinate using shift (vector)

$$dt = d\tilde{t}, \quad dx = d\tilde{x} - \beta d\tilde{t}$$

  - The metric becomes

$$g_{\mu\nu} = \begin{pmatrix} (-1+\beta^2) & \beta \\ \beta & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & \beta \\ \beta & (1-\beta^2) \end{pmatrix}$$

  - Rewrite the WEQ using e.g.

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi] = 0$$

Shifted wave equation

$$
\begin{aligned}
\Box\phi &= \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi] \\
&= \partial_t[g^{t\nu}\partial_\nu\phi] + \partial_x[g^{x\nu}\partial_\nu\phi] \\
&= \partial_t[g^{tt}\partial_t\phi + g^{tx}\partial_x\phi] + \partial_x[g^{xt}\partial_t\phi + g^{xx}\partial_x\phi] \\
&= \partial_t[-\partial_t\phi + \beta\partial_x\phi] + \partial_x[\beta\partial_t\phi + (1-\beta^2)\partial_x\phi] \\
&= 0
\end{aligned}
$$

- Suggests definition of new variables:

  $\psi := \partial_x\phi \quad \pi := \partial_t\phi - \beta\partial_x\phi$

- Evolution equations: $\partial_t\phi = \pi + \beta\psi$

  $\partial_t\psi = \partial_x(\pi + \beta\psi) \qquad \partial_t\pi = \partial_x(\psi + \beta\pi)$

characteristic variables

- matrix formulation: $$\mathbf{u} = (\pi, \psi)^T$$

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{u}\partial_x \beta \qquad \mathbf{A} = -\begin{pmatrix} \beta & 1 \\ 1 & \beta \end{pmatrix}$$

- A is diagonalizable with eigenvalues = characteristic speeds $\lambda_1 = -\beta + \alpha$, $\lambda_2 = -\beta - \alpha$, and eigenvectors

$$v_1 = (1, -1)^T, \quad v_2 = (1, 1)^T$$

- characteristic variables, propagating with characteristic speeds:

$$u_1 = u_R = \frac{1}{2}(\pi - \partial_x \phi) \qquad u_2 = u_L = \frac{1}{2}(\pi + \partial_x \phi)$$

- Plot the characteristic variables in your code! Observe that these quantities propagate as expected.

Boundary conditions

- Putting boundary conditions on outgoing characteristic fields is not logically consistent - initial boundary value problem will not be well-posed.
- Can only put boundary conditions on incoming characteristic fields!
- Examples:
  - reflecting boundary conditions

$$\phi = \partial_t \phi = \pi = \partial_t \pi = \partial_x \psi = 0$$

  - outgoing boundary conditions: incoming signal set to zero, e.g. at left boundary:

$$u_R = 0 = \pi - \psi \quad \Rightarrow \pi = \psi$$

Scalar field coupled to gravity

- Simple form of the metric in spherical symmetry with zero shift $\partial_t \phi = \frac{\alpha}{a}\pi$

$$ds^2 = -\alpha(r,t)^2 dt^2 + a(r,t)^2 dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

- Definitions:

$$\psi = \partial_r \phi, \quad \pi = \frac{a}{\alpha}\partial_t \phi$$

- Einstein equations:

$$\frac{\partial_r \alpha}{\alpha} = \frac{\partial_r a}{a}\frac{a^2-1}{r} \qquad \frac{\partial_r a}{a} = \frac{1-a^2}{2r} + \frac{r}{4}\left(\psi^2 + \pi^2\right) \qquad \partial_t a = \frac{1}{2} r a \alpha \phi \pi$$

- Scalar field equations:

$$\partial_t \pi = \frac{1}{r^2}\left(\frac{r^2 \alpha \psi}{a}\right) \qquad \partial_t \psi = \partial_r\left(\frac{\alpha \pi}{a}\right)$$