Kulkarni Limit Set of Subgroups of $PSL(3, \mathbb{C})$

joint work with Ángel Cano

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- Definition: A Complex Kleinian Group is a discrete subgroup of $PSL(n+1,\mathbb{C})$ acting on $\mathbb{P}^n_{\mathbb{C}}$ with non empty region of discontinuity.
- We restrict our attention to complex Kleinian subgroups of $PSL(3,\mathbb{C})$.
- There are two natural sources of complex Kleinian groups: Discrete subgroups of $PU(2,1)$ and discrete subgroups of $Aff(\mathbb{C}^2)$.
- It is difficult to decide when a discrete subgroup of $PSL(3,\mathbb{C})$ is complex Kleinian group, and Kulkarni limit set provides a valuable tool for the solution of this problem.

Definition [Groups with Domains of Discontinuity, R. Kulkarni, 1978]

Let X be a locally compact Hausdorff Space and G be a group acting by homeomorphisms on X.

- $L_0(G)$ is the closure of the set of points in X with infinite isotropy group
- $L_1(G)$ is the closure of the set of cluster points of $\{g(z) : g \in G\}$ where z runs over $X - L_0(G)$.
- $L_2(G)$ is the closure of the set of cluster points $\{g(K) : g \in G\}$ where K runs over compact subsets of $X - (L_0(G) \cup L_1(G))$.

The Kulkarni Limit Set is

$$\Lambda(G) = L_0(G) \cup L_1(G) \cup L_2(G)$$

Kulkarni Domain of Discontinuity

The domain of discontinuity of Γ is defined as the set

$$\Omega(G) = X - \Lambda(G).$$

G is said to have the Kleinian Property if $\Omega(G) \neq \emptyset$.

Kulkarni's Theorem

Definition. Let X be a locally compact Hausdorff space and G be a group acting on X by homeomorphisms. The action of G is properly discontinuous on a G-invariant subset Ω of X if for any two compact C and D of Ω, $g(C) \cap D \neq \emptyset$ only for finitely many $g \in G$.

Theorem[Kulkarni,1978] Let X and G be as above where G is equipped with the compact open topology. Then L_0, L_1, L_2, Λ, Ω are G-invariant and G acts properly discontinuously on Ω. If G has the Kleinian property then it is discrete. If X has a countable base for its topology then G is countable.

Remarks

- We remark that the usual limit set and Kulkarni limit set agree for classical Kleinian groups. In fact $L_0 = L_1 = L_2 = \Lambda$.
- On the other hand when working in complex projective geometry, the sets L_0, L_1, L_2 can be quite different amongst them.

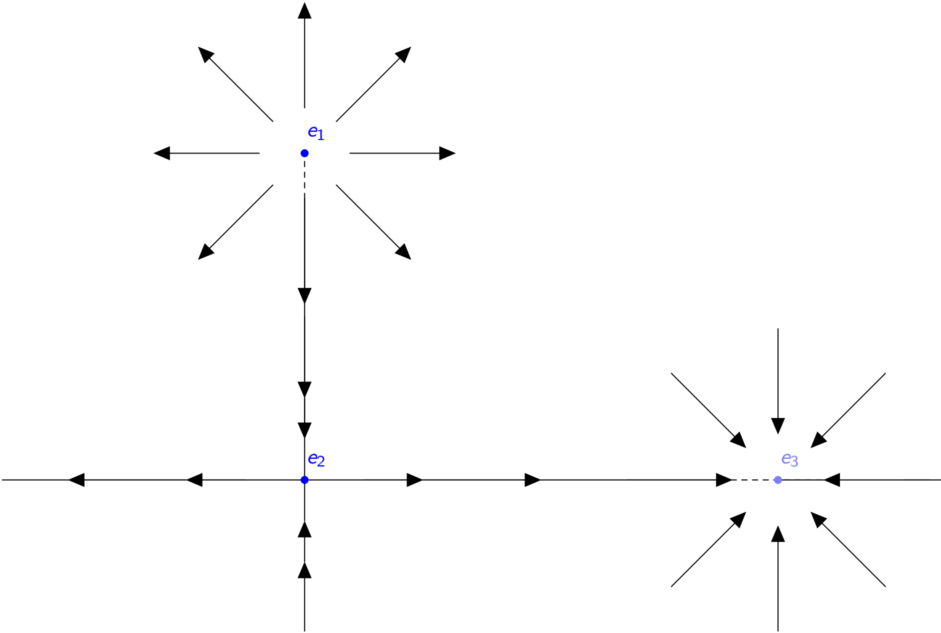
Example

$$\gamma = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad |\lambda_1| < |\lambda_2| < |\lambda_3|$$

If $e_1 = [1 : 0 : 0]$, $e_2 = [0 : 1 : 0]$ and $e_3 = [0 : 0 : 1]$, then

- $L_0(\gamma) = \{e_1, e_2, e_3\}$
- $L_1(\gamma) = \{e_1, e_2, e_3\}$
- $L_2(\gamma) = (e_1 e_2) \cup (e_2 e_3)$.

Loxodromic dynamic



More remarks

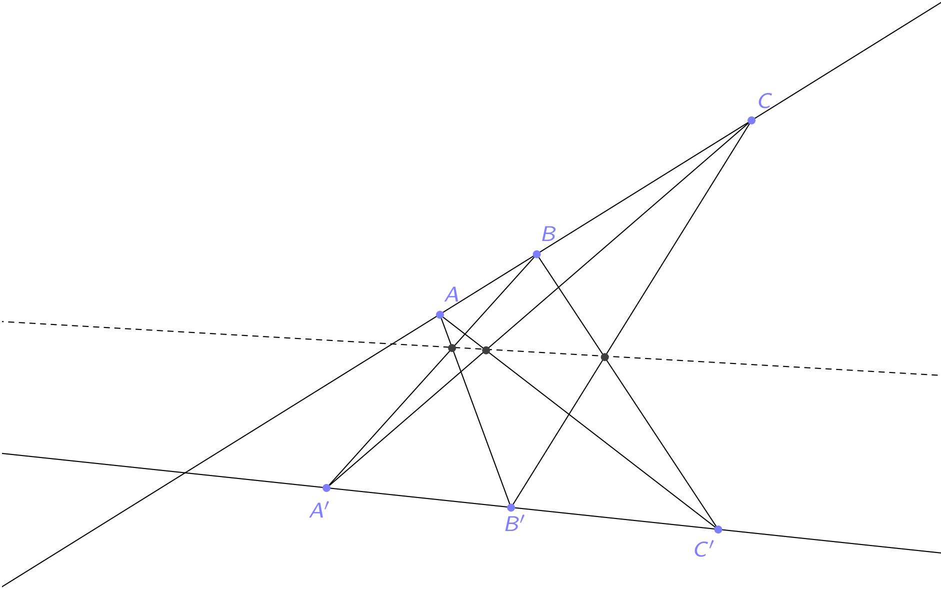
- $\Omega(G)$ is not always the maximal open set where the group acts proper and discontinuosly.
- If $H \leq G$ not necessarily $\Lambda(H) \subset \Lambda(G)$.
- If G is an infinite discrete subgroup of $PSL(3, \mathbb{C})$, then $\Lambda(G)$ consists of one complex projective line, one complex projective line and one point, two complex projective lines, three complex projective lines or a union of infinitely many complex projective lines.

- It is also important to know the maximum number of complex projectives lines in general position in Λ. It is known that this number is equal 1,2, 3 , 4 or ∞.
- Suspensions provide examples with many complex projectives lines in Λ but at most two in general position.
- There is a classification of those subgroups of $PSL(3, \mathbb{C})$ such that the maximum number of complex projectives lines in Λ is equal to four(Barrera-Cano-Navarrete).

PURPOSE

To construct complex Kleinian groups not conjugate to any subgroup of $PU(2,1)$ nor to any subgroup of $Aff(\mathbb{C}^2)$ with rich dynamics and with infinitely many projectives lines in general position in its Kulkarni limit set.

Pappus Theorem



Box operations

A *box* consists of four points p, q, r, s in general position in $\mathbb{P}^2_{\mathbb{C}}$, called the *vertices* of the box, plus two points t and b, in the complex lines pq and rs, respectively, such that:

- $p \neq t \neq q$, $t \neq (pq)(rs)$,
- $r \neq b \neq s$, $b \neq (pq)(rs)$.

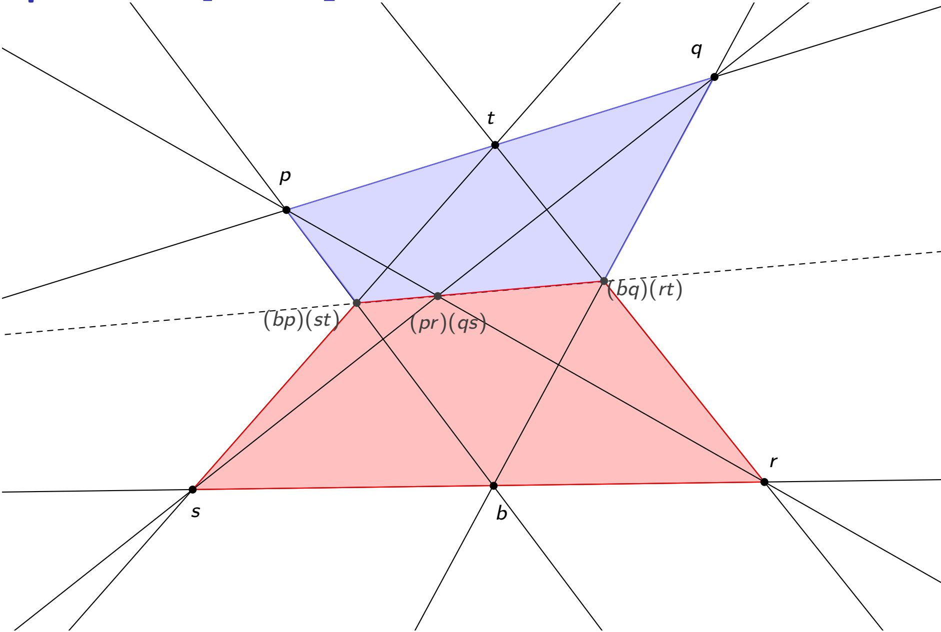
This box is denoted by $\Theta = (p, q, r, s; t, b)$ and the box operations are defined as follows:

$$i(\Theta) = (s, r, p, q; b, t),$$

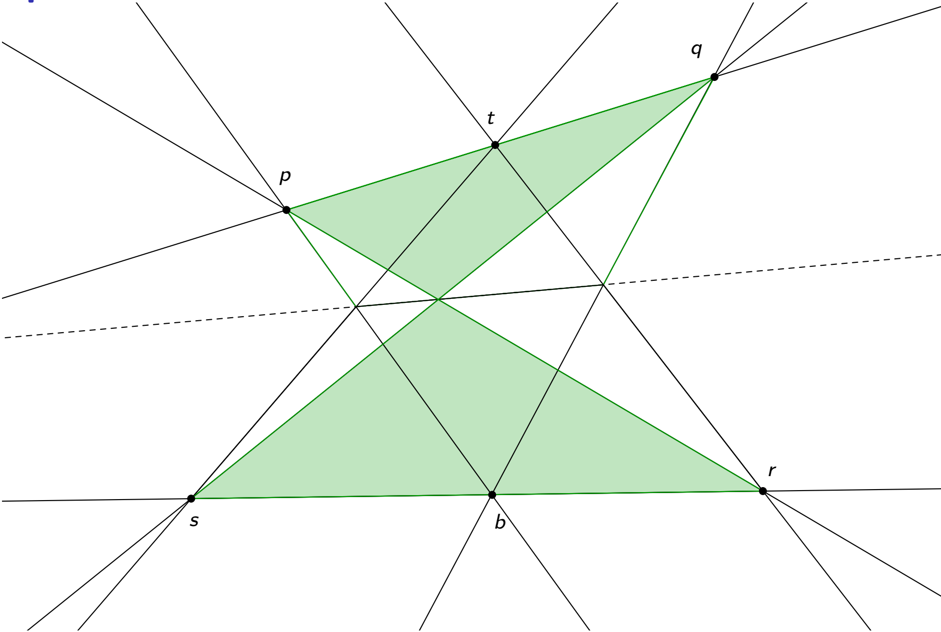
$$\tau_1(\Theta) = (p, q, (bq)(rt), (bp)(st); t, (qs)(pr)),$$

$$\tau_2(\Theta) = ((bq)(rt), (bp)(st), r, s; (qs)(pr), b).$$

Operation τ_1 and τ_2



Operation *i*



Box operations II

It is not hard to verify that the following relations are satisfied

$$i^4 = 1, \quad \tau_1 i^3 \tau_2 = i, \quad \tau_2 i \tau_1 = i^3, \quad \tau_1 i \tau_1 = i^2 \tau_2, \quad \tau_2 i^3 \tau_2 = \tau_1$$

These relations show that the operations on a box Θ form a group, and this group is generated by

$$\alpha = i \quad \text{and} \quad \beta = i\tau_1$$

Moreover

$$\alpha^4 = 1 = \beta^6$$

Box Operations III

- If $\mathbf{g} \in PSL(3, \mathbb{C})$ and $\Theta = (p, q, r, s; t, b)$ is a box, then

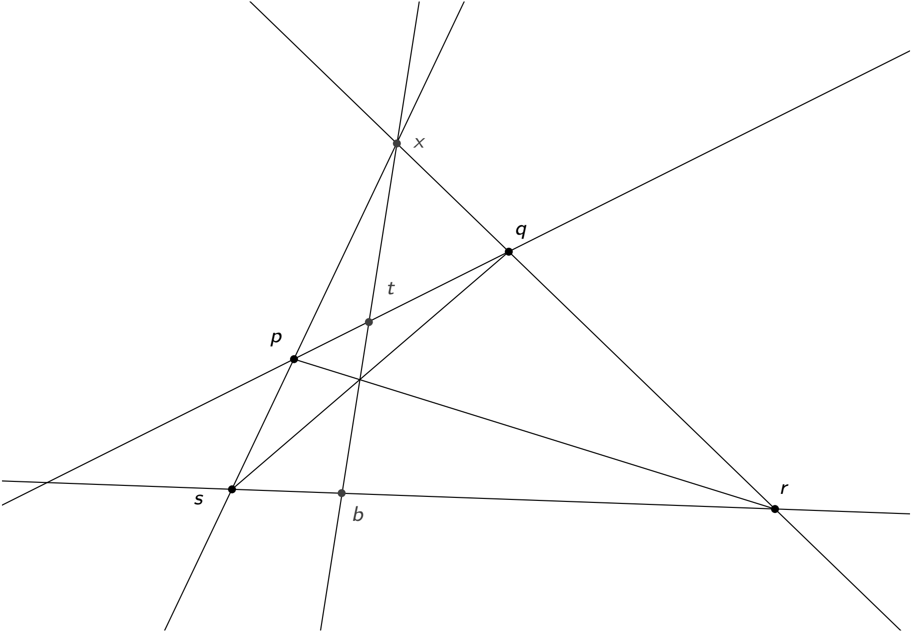
$$\mathbf{g}\,\Theta = (\mathbf{g}(p), \mathbf{g}(q), \mathbf{g}(r), \mathbf{g}(s); \mathbf{g}(t), \mathbf{g}(b)).$$

- This action commutes with the box operations i, τ_1, τ_2.

- We want to find projective transformations A, T_1, $T_2 \in PSL(3, \mathbb{C})$ (depending on Θ) such that:

a) $\tau_1(\Theta) = T_1\,\Theta$,
b) $\tau_2(\Theta) = T_2\,\Theta$,
c) $i(\Theta) = A\,\Theta$.

Good box



Lemma

- There exists $T_1 \in PSL(3, \mathbb{C})$ such that $\tau_1(\Theta) = T_1\,\Theta$, if and only if, Θ is a good box.
- There exists $T_2 \in PSL(3, \mathbb{C})$ such that $\tau_2(\Theta) = T_2\,\Theta$, if and only if, Θ is a good box.
- There exists $A \in PSL(3, \mathbb{C})$ such that $i(\Theta) = A\,\Theta$, if and only if, Θ is a good box.

Remark

- Let $\mathbf{g} \in PSL(3, \mathbb{C})$ be a projective transformation. It is not hard to see that Θ is a good box, if and only if, $\mathbf{g}\,\Theta$ is a good box. It follows that the orbit of a good box Θ by box operations consists of good boxes.

Theorem

- If p, q, r, s in $\mathbb{P}^2_{\mathbb{C}}$ are four points in general position, then there exist two unique points t and b such that the group generated by the operations in the box $\Theta = (p, q, r, s; t, b)$ can be represented as a subgroup of $PSL(3, \mathbb{C})$.
- This group is conjugate in $PSL(3, \mathbb{C})$ to a group such that every element has a lift to $SL(3, \mathbb{C})$ of the form
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix},$$
where
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
is an element in $SL(2, \mathbb{Z})$.
- Moreover, its Kulkarni limit set is the union of those concurrent complex lines determined by the common intersection point $(pq)(rs)$, and the points in the real projective line tb.

The group of a good box

Now we show that $PSL(3,\mathbb{Z})$ is a discrete subgroup of $PSL(3,\mathbb{C})$ acting on $P^2_{\mathbb{C}}$ in such way that its Kulkarni limit set is all of $\mathbb{P}^2_{\mathbb{C}}$.
The plan is as follows:

- We introduce a new operation on good boxes that together with τ_1, τ_2 and i generate a group of operations on good boxes which may be represented as a group of projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ called the *group of the a good box*, and denoted $\mathcal{P}$.
- It is shown that the group of the good box contains a discrete subgroup, denoted $\mathcal{P}_2$, which is also a subgroup of $PSL(3,\mathbb{Z})$ and whose Kulkarni limit set is all of $\mathbb{P}^2_{\mathbb{C}}$.
- It is proved that the $\mathcal{P}_2$ orbit of any point in $\mathbb{P}^2_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{C}}$ has a dense orbit in $\mathbb{P}^2_{\mathbb{R}}$. In other words, the action of $\mathcal{P}_2$ on $\mathbb{P}^2_{\mathbb{R}}$ is minimal.

Theorem

[Barrera-Cano-Navarrete] Let $\Gamma \subset PSL(3, \mathbb{C})$ be an infinite discrete subgroup, without fixed points nor invariant complex lines. Let $\mathcal{E}(\Gamma)$ be the subset of $(\mathbb{P}^2_{\mathbb{C}})^*$ consisting of all the complex lines l for which there exists an element $\gamma \in \Gamma$ such that $l \subset \Lambda(\gamma)$.

a)
$$\Lambda(\Gamma) = \overline{\bigcup_{l \in \mathcal{E}(\Gamma)} l} = \bigcup_{l \in \overline{\mathcal{E}(\Gamma)}} l = \overline{\bigcup_{\gamma \in \Gamma} \Lambda(\gamma)}$$

is

b) If $\mathcal{E}(\Gamma)$ contains more than three complex lines, then $\overline{\mathcal{E}(\Gamma)} \subset (\mathbb{P}^2_{\mathbb{C}})^*$ is a perfect set. Also, it is the minimal closed Γ-invariant subset of $(\mathbb{P}^2_{\mathbb{C}})^*$.

Definition

The *side operation* λ is defined as follows: If $\Theta = (p, q, r, s; t, b)$ is a box, then

$$\lambda(\Theta) = (q, r, s, p\,; L(t), L(b)),$$

where L is the only projective transformation satisfying

$$L(p) = q, \quad L(q) = r, \quad L(r) = s, \quad L(s) = p.$$

It follows that $L\,\Theta = \lambda(\Theta)$, and $\Theta = (p, q, r, s; t, b)$ is a good box, if and only if, $\lambda(\Theta)$ is a good box.

Let $p = [1 : 0 : 0]$, $q = [0 : 1 : 0]$, $r = [0 : 0 : 1]$, $s = [1 : 1 : 1]$, and $\Theta = (p, q, r, s; t, b)$ a good box, then

$$L = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

- $\mathcal{P} = \langle A, T_1, L \rangle$ (the group of a good box).
- $\mathcal{P}_2 = \langle A, T_1^2, T_2^2, L \rangle \subset PSL(3, \mathbb{Z})$.

Lemma

Let M_2 be the group $\langle A^2, T_1^2, T_2^2\rangle$, acting on $\mathbb{P}^2_{\mathbb{C}}$, then M_2 is conjugate (in $PSL(3,\mathbb{C})$) to a double covering group of the classical Kleinian group $Mod(2)$ and its limit set according to Kulkarni $\Lambda(M_2) \subset \Lambda(\mathcal{P}_2)$, is the union of those concurrent complex projective lines determined by the common point $y = (pq)(rs)$ and those points on the real projective line bt. In other words,

$$\Lambda(M_2) = \bigcup_{w \in (bt)_{\mathbb{R}}} (wy)_{\mathbb{C}}.$$

Theorem

The groups $\mathcal{P}$ and $\mathcal{P}_2$ acting on $\mathbb{P}^2_{\mathbb{C}}$ are not complex Kleinian. In fact, $\Lambda(\mathcal{P}) = \Lambda(\mathcal{P}_2) = \mathbb{P}^2_{\mathbb{C}}$.

- The complete pencil of projective real lines with vertex y is contained in the Kulkarni limit set $\Lambda(\mathcal{P}_2)$.
- Since $L \in \mathcal{P}_2$, it follows that the complete pencil of projective real lines with vertex x is contained in $\Lambda(\mathcal{P}_2)$.
- Moving this pencil along the real projective tb with the action of $Mod(2)$, we obtain that every real projective line is contained in $\Lambda(\mathcal{P}_2)$.
- And finally it can be proved that $\Lambda(\mathcal{P}_2) = \mathbb{P}^2_{\mathbb{C}}$.

Corollaries

- The Kulkarni limit set of the group $PSL(3, \mathbb{Z})$ acting on $\mathbb{P}^2_{\mathbb{C}}$ is all of $\mathbb{P}^2_{\mathbb{C}}$.
- The group $\mathcal{P}_2$ acts minimally on the space of real projective lines of $\mathbb{P}^2_{\mathbb{R}}$. In other words, the $\mathcal{P}_2$-orbit of every real projective line in $\mathbb{P}^2_{\mathbb{R}}$ is dense in the space of real projective lines in $\mathbb{P}^2_{\mathbb{R}}$.
- The $\mathcal{P}_2$-orbit of any point in $\mathbb{P}^2_{\mathbb{R}}$ is dense in $\mathbb{P}^2_{\mathbb{R}}$.

Groups with more than five lines: Construction

Now we construct a family of examples of groups $\Gamma \subset SL(3,\mathbb{R})$, such that:

i) Γ is a free group not conjugate, in $PSL(3,\mathbb{C})$, to any subgroup of $PU(2,1)$ nor conjugate to any subgroup of $\mathit{Aff}(\mathbb{C}^2)$.

ii) Γ, acting on $\mathbb{P}^2_{\mathbb{C}}$, is a complex Kleinian group. In other words, its Kulkarni discontinuity region, $\Omega(\Gamma) \subset \mathbb{P}^2_{\mathbb{C}}$, is not empty.

iii) $\Gamma^* = \{(\gamma^t)^{-1} : \gamma \in \Gamma\}$ can be realized as a group of operations on good boxes and it is a group of Schottky type.

iv) $\Lambda(\Gamma)$ contains at least five lines in general position.

Group of Schottky type (Conze and Guivarch, Tits)

Definition.

- Let (X, δ) be a complete metric space.
- p a point in X.
- Σ a finite set of homeomorphisms of X which is symmetric (namely, $a^{-1} \in \Sigma$ for all $a \in \Sigma$).
- Let $\{C_a\}_{a\in\Sigma}$ be a family of compact subsets of X such that $p \notin \cup_{a\in\Sigma} C_a$ and $a(p) \in C_a$ for all $a \in \Sigma$.

Assume the following conditions are sastified

(1) If $a, b \in \Sigma$ are two distinct elements, then $C_a \cap C_b = \varnothing$.

(2) If $a, b \in \Sigma$ and $a \neq b^{-1}$, then $a(C_b) \subset Int(C_a)$.

(3) For all sequences $\{a_n\}$ such that $a_n \neq a_{n+1}^{-1}$ for all $n \geq 1$, the diameter of $a_1 \ldots a_n C_{a_{n+1}}$ goes to 0 as $n \to \infty$.

Remarks I

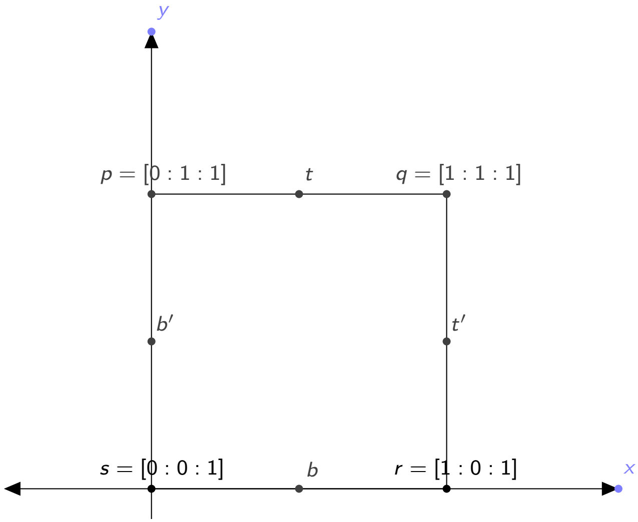
- The compact sets C_a are not necessarily circles.
- In the case of a classical Schottky group acting on the Riemann sphere, the common exterior of the circumferences is a fundamental domain. However, this is no longer valid for a group of Schottky type.
- In the construction of a classical Schottky group one requires that the circles (the compact sets) bound a domain D and $g_m(D) \cap D = \varnothing$ for all $m = 1, \ldots, n$ which implies that the group is free and discrete. Analogously, in the case of a group of Schottky type one requires the existence of a point $p \notin \bigcup_{a \in \Sigma} C_a$ such that $a(p) \in C_a$ for all $a \in \Sigma$, and this condition together with condition (2) assures that the group is free and discrete.
- The condition (3) is not deduced from (1) and (2), since the conformal properties of Möbius transformations are no longer valid. However, in the real projective space $\mathbb{P}^n_{\mathbb{R}}$, if the sets C_a are convex then conditions (1) and (2) do imply condition (3)

Remarks II

- A closed subset C of $\mathbb{P}^2_{\mathbb{R}}$ is called *convex* if it is contained in the complement of a real projective line ℓ and it is convex as a subset of $\mathbb{P}^2_{\mathbb{R}} - \ell$.
- A matrix $a \in GL(3, \mathbb{R})$ is called *loxodromic* if it has an eigenvalue λ_0 such that $|\lambda_0| > |\lambda|$ for all the others eigenvalues λ of γ (whether real or complex). For such a matrix a, an eigenvector $a^+ \in \mathbb{R}^3$ corresponding to the eigenvalue λ_0 is called a dominant eigenvector of γ.
- The subset $K(a)$ is defined as the projectivization of the set $\{w \in \mathbb{R}^3 : \lambda_0^{-n} a^n(w) \to (0,0,0) \text{ as } n \to \infty\}$.

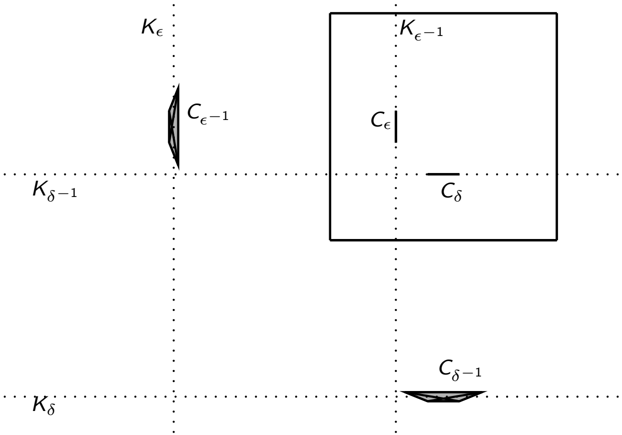
Proposition

- $\hat{\Sigma} = \{(a, C_a) \mid a \in \Sigma\}$ a system, where Σ is a set of projective transformations and C_a are compact convex sets. If $\hat{\Sigma}$ satisfies conditions (1) and (2) of definition of group of Schottky type, then every element in Σ is loxodromic, with $a^+ \in C_a$ and $K(b) \cap C_a = \varnothing$ for $b \neq a^{-1}$.
- $\hat{\Sigma} = \{(a, C_a) \mid a \in \Sigma\}$ a system, where C_a are disjoint compact convex sets and Σ is a set of loxodromic projective transformations with $a^+ \in C_a$ and $K(b) \cap C_a = \varnothing$ whenever $b \neq a^{-1}$, then for all sufficiently large n the system $\hat{\Sigma}_n = \{(a^n, C_a) | a \in \Sigma\}$ satisfies conditions (1) and (2) of definition group of Schottky type.



y
p = [0 : 1 : 1]
t
q = [1 : 1 : 1]
b′
t′
s = [0 : 0 : 1]
b
r = [1 : 0 : 1]
x

i	τ_1^2	τ_2^2	λ
A	T_1^2	T_2^2	L
$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$


K_ϵ
$K_{\epsilon-1}$
$C_{\epsilon-1}$
C_ϵ
$K_{\delta-1}$
C_δ
$C_{\delta-1}$
K_δ

- Let $\Gamma_1 = \langle \delta, \epsilon \rangle$, where $\delta = T_2^2 T_1^2 T_2^2$ and $\epsilon = L \delta L^{-1}$.
- Let $V = \{p, q, r, s\}$ be the set of vertices of the box Θ_0.
- The sets $V, \delta(V), \delta^{-1}(V), \epsilon(V)$ and $\epsilon^{-1}(V)$ are contained in $\mathbb{R}^2 = \{[x : y : 1] \in \mathbb{P}^2_{\mathbb{R}}\}$
- Define C_δ, $C_{\delta^{-1}}$, C_ϵ, $C_{\epsilon^{-1}}$ as the convex hull of the sets $\delta(V)$, $\delta^{-1}(V)$, $\epsilon(V)$, $\epsilon^{-1}(V)$, respectively.
- The point $[0 : 1 : 1] = p \notin C_\delta \cup C_{\delta^{-1}} \cup C_\epsilon \cup C_{\epsilon^{-1}}$, but $\delta(p), \delta^{-1}(p), \epsilon(p), \epsilon^{-1}(p)$ do belong to $C_\delta \cup C_{\delta^{-1}} \cup C_\epsilon \cup C_{\epsilon^{-1}}$.
- It is not hard to check that the compact convex sets $C_\delta, C_{\delta^{-1}}, C_\epsilon, C_{\epsilon^{-1}}$ are disjoint.

- Each one of the elements $\delta, \delta^{-1}, \epsilon, \epsilon^{-1}$ is loxodromic, and $\delta^+ \in C_\delta$, $(\delta^{-1})^+ \in C_{\delta^{-1}}$, $\epsilon^+ \in C_\epsilon$, $(\epsilon^{-1})^+ \in C_{\epsilon^{-1}}$.
- Moreover,

$$K(\delta) \cap C_\delta = \varnothing, \quad K(\delta) \cap C_\epsilon = \varnothing, \quad K(\delta) \cap C_{\epsilon^{-1}} = \varnothing;$$

$$K(\delta^{-1}) \cap C_{\delta^{-1}} = \varnothing, \quad K(\delta^{-1}) \cap C_\epsilon = \varnothing, \quad K(\delta^{-1}) \cap C_{\epsilon^{-1}} = \varnothing;$$

$$K(\epsilon) \cap C_\delta = \varnothing, \quad K(\epsilon) \cap C_{\delta^{-1}} = \varnothing, \quad K(\epsilon) \cap C_\epsilon = \varnothing;$$

$$K(\epsilon^{-1}) \cap C_\delta = \varnothing, \quad K(\epsilon^{-1}) \cap C_{\delta^{-1}} = \varnothing, \quad K(\epsilon^{-1}) \cap C_{\epsilon^{-1}} = \varnothing.$$

- For all sufficiently large n, the system

$$\Sigma_n = \{(\delta^n, C_\delta), (\delta^{-n}, C_{\delta^{-1}}), (\epsilon^n, C_\epsilon), (\epsilon^{-n}, C_{\epsilon^{-1}})\}$$

satisfies conditions (1) , (2) and (3). Therefore, for all sufficiently large n, the group $\Gamma_n = \langle \delta^n, \epsilon^n \rangle$ is of Schottky type.

- The closure of the set of attracting fixed points of loxodromic transformations in Γ_n is a closed minimal set for the action of Γ_n on $\mathbb{P}^2_{\mathbb{R}}$.
- Given four arbitrary open sets $U_\delta, U_{\delta^{-1}}, U_\epsilon, U_{\epsilon^{-1}}$ neighborhoods of δ^+, $(\delta^{-1})^+$, ϵ^+, $(\epsilon^{-1})^+$, respectively. We can choose n large enough in such way that the closed minimal set of Γ_n is contained in the union of these four arbitrary neighborhoods.

- For every $n \in \mathbb{N}$, we define $\Gamma_n^* = \{(\gamma^t)^{-1} : \gamma \in \Gamma_n\} \leq SL(3, \mathbb{R})$ acting on $\mathbb{P}^2_{\mathbb{C}}$ (the groups Γ_n^* and Γ_n are isomorphic, but not necessarily equal).
- Γ_n^* acts on $\mathbb{P}^2_{\mathbb{C}}$ without globally fixed points (because $(\delta^t)^{-1}$ and $(\epsilon^t)^{-1}$ are loxodromic elements and they have no common fixed point).
- Γ_n^* acts on $\mathbb{P}^2_{\mathbb{C}}$ without invariant complex projective lines (because δ and ϵ are loxodromic elements and have no common fixed point).

- The Kulkarni limit set, $\Lambda(\Gamma_n^*)$, contains at least four complex projective lines in general position, because

$$\Lambda(\Gamma_n^*) = \overline{\bigcup_{\gamma \in \Gamma_n} \Lambda((\gamma^t)^{-1})} \supset \Lambda((\delta^t)^{-n}) \cup \Lambda((\epsilon^t)^{-n}) = \Lambda((\delta^t)^{-1}) \cup \Lambda((\epsilon^t)^{-1}),$$

  and $\Lambda((\delta^t)^{-1}) \cup \Lambda((\epsilon^t)^{-1})$ is equal to four complex projective lines in general position.
- It follows that $\Lambda(\Gamma_n^*)$ contains at least five complex projective lines in general position. Therefore it contains infinitely many complex projective lines in general position.

- Let $\mathcal{E}(\Gamma_n^*)$ be the set consisting of those complex projective lines ℓ for which there exists $\gamma \in \Gamma_n$ such that $\ell \subset \Lambda((\gamma^t)^{-1})$.
- (Barrera-Cano-Navarrete) $\overline{\mathcal{E}(\Gamma_n^*)}$ is the minimal Γ_n^*-invariant closed set for the action of Γ_n^* on $(\mathbb{P}^2_{\mathbb{C}})^*$.
- But this action is precisely the natural action Γ_n on $\mathbb{P}^2_{\mathbb{C}}$
- Thus, for any n large enough, $\overline{\mathcal{E}(\Gamma_n^*)}$ is identified with the minimal Γ_n-invariant closed set of the group of Schottky type Γ_n acting on $\mathbb{P}^2_{\mathbb{R}}$, because $\mathbb{P}^2_{\mathbb{R}}$ is a closed invariant set for the action of Γ_n on $\mathbb{P}^2_{\mathbb{C}}$.

- For sufficiently large n, this minimal closed set is contained in the union of arbitrary neighborhoods (in $(\mathbb{P}^2_{\mathbb{C}})^*$) of the complex projective lines δ^+, $(\delta^{-1})^+$, ϵ^+, $(\epsilon^{-1})^+$.
- We denote these neighborhoods by $U_\delta, U_{\delta^{-1}}, U_\epsilon, U_{\epsilon^{-1}}$, and we can choose them in such way that

$$\left(\bigcup_{\ell \in U_\delta \cup U_{\delta^{-1}} \cup U_\epsilon \cup U_{\epsilon^{-1}}} \ell \right) \subsetneq \mathbb{P}^2_{\mathbb{C}},$$

then

$$\Lambda(\Gamma_n^*) = \bigcup_{\ell \in \overline{\mathcal{E}(\Gamma_n^*)}} \ell \subset \left(\bigcup_{\ell \in U_\delta \cup U_{\delta^{-1}} \cup U_\epsilon \cup U_{\epsilon^{-1}}} \ell \right) \subsetneq \mathbb{P}^2_{\mathbb{C}}.$$

- In consequence, the region of discontinuity $\Omega(\Gamma_n^*) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda(\Gamma_n^*)$ is not empty for all sufficiently large n.

- **Lemma.** If $g \in PSL(3,\mathbb{C})$ is conjugate to a loxodromic element of $PU(2,1)$ and $g^n \in PU(2,1)$ for some $n \in \mathbb{Z} \setminus \{0\}$, then $g \in PU(2,1)$.
- **Proposition**. The group $\Gamma_1^* = \langle (\delta^{-1})^t, (\epsilon^{-1})^t \rangle$ is not conjugate to any subgroup of $PU(2,1)$.
- **Proposition**
  For every $n \in \mathbb{N}$, the group $\Gamma_n^* = \langle (\delta^{-n})^t, (\epsilon^{-n})^t \rangle$ is not conjugate to any subgroup of $PU(2,1)$.
  Proof
  Let C be an element in $PSL(3,\mathbb{C})$ such that

$$(C(\delta^{-1})^t C^{-1})^n = C(\delta^{-n})^t C^{-1} \in PU(2,1),$$

$$(C(\epsilon^{-1})^t C^{-1})^n = C(\epsilon^{-n})^t C^{-1} \in PU(2,1),$$

then by lemma, $C(\delta^{-1})^t C^{-1}$ and $C(\epsilon^{-1})^t C^{-1}$ are elements in $PU(2,1)$, so Γ_1^* is conjugate to a subgroup of $PU(2,1)$, which contradicts above proposition.