OMITTED VALUES AND HERMAN RINGS WITH SMALL PERIODS

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Plan of Presentation

1. Introduction
2. Omitted Values
3. Motivation
4. Results: f is general meromorphic and has at least an omitted value.

- All the poles are multiple $\Longrightarrow$ No Herman ring.
- Herman ring of period one or two does not exist.
- At least two poles, one of which is omitted $\Longrightarrow$ No Herman ring.
- 3-periodic Herman rings have only ONE configuration.
- For $p = 3, 4$, at most one p-cycle of Herman rings.

The Fatou and Julia set

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function.
Normal Family: Each sequence contains a subsequence that converges uniformly on every compact subset of the domain, the limit allowed to be ∞.

- Fatou Set of $f(z)$
  $\mathfrak{F}(f) = \{z \in \widehat{\mathbb{C}}$: The sequence of iterates $\{f^n\}$ is defined and normal in some neighborhood of $z\}$

- Julia Set/ Chaotic set of $f(z)$, $\mathfrak{J}(f) = \widehat{\mathbb{C}} \setminus \mathfrak{F}(f)$

Fatou component: A maximal open connected subset U of $\mathfrak{F}(f)$.

Periodic: U is periodic if $f^p(U) \subset U$. The smallest p is called the *period*.

Periodic Fatou components are of five kinds

U p-periodic and $S = f^p$.

1. **Attracting domain:** $\exists$ an attracting p-periodic point z_0 in U and $\lim_{k\to\infty} S^k(z) = z_0 \; \forall \; z \in U$.
2. **Parabolic domain** or **Leau domain:** $\exists$ a neutral (parabolic) p-periodic point z_0 in ∂U and $\lim_{k\to\infty} S^k(z) = z_0 \; \forall \; z \in U$.
3. **Siegel disk:** $\exists$ an analytic homeomorphism $\phi : U \to \{z \in \mathbb{C} : |z| = 1\}$ such that $\phi(S(\phi^{-1}(z))) = e^{2\pi i \, \alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
4. **Herman ring or Arnold's ring:** $\exists$ an analytic homeomorphism $\phi : U \to \{z : 1 < |z| < r\}$ such that $\phi(S(\phi^{-1}(z))) = e^{2\pi i \, \alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
5. **Baker domain** or **Domain at infinity:** $\exists$ $z_0 \in \partial U$ such that $\lim_{k\to\infty} S^k(z) = z_0 \; \forall \; z \in U$, but $S(z_0)$ is not defined.

In the cases 3-4, under a conformal change of co-ordinates $S|_U$ is a rotation: Non-constant limit functions.

Singular values dominate dynamics

$a \in \widehat{\mathbb{C}}$, $r > 0$, $D_r(a)$ is a disc (in spherical metric $\frac{dz}{1+|z|^2}$), Choose a component U_r of $f^{-1}(D_r(a))$ for $r > 0$ in such a way that $U_{r_1} \subset U_{r_2}$ for $r_1 < r_2$.

1. $\bigcap_{r>0} U_r = \{z\}$ for $z \in \mathbb{C}$: $f(z) = a$.
   - z is an *ordinary point* if $a \in \mathbb{C}$ and $f'(z) \neq 0$, or $a = \infty$ and z is a simple pole.
   - z is a *critical point* if $a \in \mathbb{C}$ and $f'(z) = 0$, or $a = \infty$ and z is a multiple pole, a is a critical value. An algebraic singularity lies over a.

2. $\bigcap_{r>0} U_r = \emptyset$: The choice $r \to U_r$ defines a transcendental singularity of f^{-1}. We say U lies over a. a is an asymptotic value $\iff \exists$ a singularity lying over a.

a is omitted $\iff$ it is not an ordinary point and each singularity lying over it, is transcendental.

Omitted values: transcendental yet simple

Possibly the simplest kind of transcendental singularities

- **Rational functions:** No omitted value.

- **Transcendental functions:** Essential singularity at ∞
  - Entire : ∞ is omitted. At most another one e.g. e^z.
  - Meromorphic with exactly one pole w which is an omitted value : w and ∞ are only omitted.
  - General meromorphic(M): Meromorphic functions with atleast two poles or has exactly one pole which is not an omitted value. ∞ is not omitted. At most two omitted values e.g. $\tan z$

STANDING ASSUMPTION:

$f \in M_o = \{f \in M \ : \ f$ has at least one omitted value$\}$

Omitted values and Herman rings

- **Herman rings:**
  - Doubly connected.
  - Consists of disjoint Jordan curves that are f^p-invariant.
  - Always have a bounded complementary component.
  - The map is one-one on each Herman ring.
  - May contain a singular value but NEVER an omitted value.

Omitted value controls Herman rings.

Motivation

♣ Polynomials and transcendental entire maps have no Herman ring.

♣ Analytic self maps of $\mathbb{C}^*$ have no Herman ring.

♣ $f \in M_o$

– No invariant Herman ring

– f has only one pole $\Longrightarrow$ No Herman ring of period 2.

♣ **Def.** V **is SCH** if $c(V) > 1 \Longrightarrow c(V_n) > 1$ for all n and $V_{\bar{n}}$ is an Herman ring for some $\bar{n} \in \mathbb{N}$. Clearly, U is SCH and $c(U) > 1 \Longrightarrow U_1$ is SCH.

SCH means simply connected or ultimately Herman ring

$f \in M_o \Longrightarrow$ **Most Fatou components are SCH**

QUESTION

Does $\exists$ a function in $f \in M_o$ having a Herman ring?

No Herman rings (in certain situations)

If $f \in M_o$ then f has no Herman ring which is nested or strictly non-nested.

Corollary

1. It has no Herman ring of period one or two.
2. If a pole of f is an omitted value then it has no Herman ring of any period.

Fatou components surround omitted values

Lemma 1

Let $f \in M$
V be a multiply connected Fatou component
γ is a closed curve in V with $B(\gamma) \bigcap \mathcal{J}(f) \neq \emptyset$
$\Longrightarrow \exists\ n \in \mathbb{N} \bigcup \{0\}$ and $\gamma_n \subseteq f^n(\gamma) \subset V_n$ s.t. $B(\gamma_n)$ contains a pole.
If $O_f \neq \emptyset$ then $O_f \subset B(\gamma_{n+1})$ for some closed curve $\gamma_{n+1} \subset f(\gamma_n)$.

Lemma 2 H is a Herman ring of $f \in M_o \Longrightarrow f : B(H) \to \widehat{\mathbb{C}}$ is one-one.

Lemma 2 $\Longrightarrow$ If all poles are multiple then no Herman rings exist.

Arrangement of Herman rings

H is called

- **Nested** if $\exists j$ such that $H_i \subset B(H_j)$ for all i
- **Strictly Nested** if for each $i \neq j$, either $H_i \subset B(H_j)$ or $H_j \subset B(H_i)$.
- **strictly non-nested** if $B(H_i) \bigcap B(H_j) = \emptyset$ for all $i \neq j$.

Each Herman ring of period 2 is nested or strictly non-nested

Proof

- **Nested:** Suppose that $H_i \subset B(H_0)$ for all $i > 0$. Then
  1. $\exists$ a pole in $B(H_0)$ & $O_f \subset B(H_1)$.
  2. $f(B(H_0))$ is the unbounded component of $\widehat{\mathbb{C}} \setminus H_1$

$\bigcup_{i=1}^{p-1} H_i \subset B(H_0) \Longrightarrow \bigcup_{i=2}^{p-1} H_i \bigcup H_0 \subset f(B(H_0))$: H_1 is inner.
No component of $f^{-1}(H_j)$ in $B(H_1) \Longrightarrow f(B(H_1))$ contains no $H_j,\ j \geq 0$.

– f has a pole in $B(H_1) \Longrightarrow H_2 = H_0$.
– f is analytic in $B(H_1) \Longrightarrow H_2$ is inner.

Note: $f(H_{inner})$ is inner or $= H_0$, $f(H_0)$ is inner.

$j^* = \min\{j > 0 \ : \ \ H_j = H_0\}$
$\Longrightarrow H_j, 0 < j < j*$ is inner, H_{j*-1} encloses a pole and $H_{j^*} = H_0$

Proof (contd.)

- Take an f^p-invariant Jordan curve γ_0 in H_0 &
- Consider the region A bounded by $\gamma = \bigcup_{i=0}^{j^*-1} f^i(\gamma_0)$.
- No pole in A $\because f : B(H_0) \to \mathbb{C}$ is univalent and $B(H_{j^*-1})$ contains a pole.

Thus f is conformal in A & $f(A) = A$ because $f(\partial A) = f(\gamma) = \gamma$
$\implies f^n(A) = A$ for all n: Not possible as A intersects the Julia set.

- **Strictly Nested:** Suppose H_0 is strictly non-nested
  $B(H_0)$ contains a pole of f.
  $\implies O_f \subset B(H_1)$
  There is a ring H_i, $i > 1$ containing a pole.
  Therefore $O_f \subset B(H_{i+1})$.
  $B(H_1) \subseteq B(H_{i+1})$ or $B(H_{i+1}) \subset B(H_1)$: Not possible as H_0 is strictly non-nested.

Herman rings of period 3 and 4

Similar arguments proves:

- 3-periodic Herman rings have only ONE configuration.
- For $p = 3, 4$, at most one p-cycle of Herman rings.

Reference

[1] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 151–188.

[2] L. Keen and J. Kotus, *Dynamics of the family λtanz*, Conform. Geom.. Dyn **1** (1997), 28–57.

[3] T. Nayak and Z. Jian Hua, *Omitted values and dynamics of meromorphic functions*, J. London Math. Society **83** 1 (2011), 121–136.

[4] S. Morosowa and Y. Nishimura and M. Taniguchi and T. Ueda, *Holomorphic Dynamics*, Cambridge University Press (2000).

[5] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer (1993).

[6] J. Milnor, *Dynamics in One Complex Variable*, Friedrick Vieweg and Son (2000).

[7] A. F. Beardon, *Iteration of Rational Functions*, Springer-Verlag (1991).

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Thank You
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