$\mathbb{C}P^1$-structures, grafting and Teichmüller rays

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Groups, Geometry and Dynamics
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Definition

Let S be a closed oriented surface of genus $g \geq 2$.

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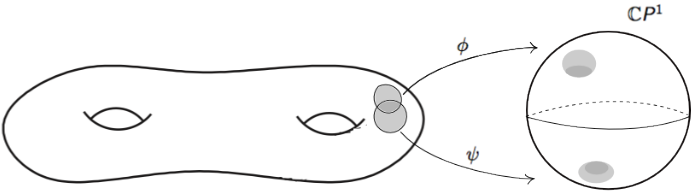
A *complex projective structure* on S is a maximal atlas of charts to $\mathbb{C}P^1$ with transition maps being restrictions of elements of $Aut(\mathbb{C}P^1) = PSL_2(\mathbb{C})$.



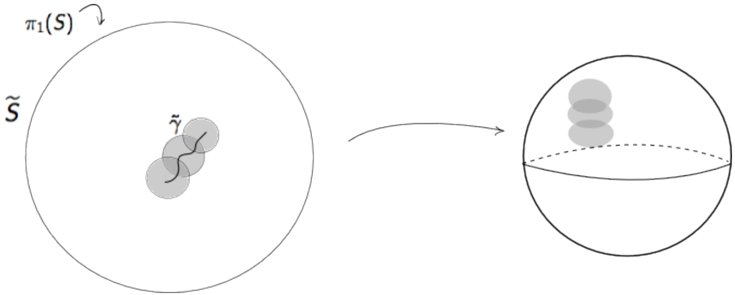
$\psi \circ \phi^{-1}$ *is a Möbius map* $z \mapsto \frac{az+b}{cz+d}$

$\mathbb{C}P^1$ structure: a "global" definition

A complex projective structure is specified by:

- A *developing map* that is an immersion from the universal cover $f : \widetilde{S} \to \mathbb{C}P^1$.
- A *holonomy representation* $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ that is compatible:

$$f \circ \gamma = \rho(\gamma) \circ f \text{ for all } \gamma \in \pi_1(S).$$



Examples: hyperbolic structures

Definition

A *hyperbolic structure on S* is a collection of charts to the hyperbolic plane $\mathbb{H}^2$ with transition maps in $PSL_2(\mathbb{R}) = Isom^+(\mathbb{H}^2)$.

This is a special case of a complex-projective structure:

- Can identify $\mathbb{H}^2$ with the *upper hemisphere* on $\mathbb{C}P^1$.
- The holonomy representation is *Fuchsian*, $PSL_2(\mathbb{R}) \hookrightarrow PSL_2(\mathbb{C})$.

Uniformization theorem: any Riemann surface has a hyperbolic structure.

A bundle picture

Conversely, a $\mathbb{C}P^1$-structure defines a complex structure on S since the transition maps are conformal.

$$\mathcal{P}_g = \{\text{space of marked } \mathbb{C}P^1\text{-structures on } S_g\}$$

$$\begin{array}{c} \mathcal{P}_g \\ \downarrow \text{p} \\ \mathcal{T}_g \\ \downarrow \pi \\ \mathcal{M}_g \end{array}$$

where $\mathcal{T}_g$ is Teichmüller space and $\mathcal{M}_g$ is the moduli space of Riemann surfaces.

Facts

$$\begin{aligned}\mathcal{T}_g &= \{\text{marked conformal/hyperbolic structures on } S\}/\sim \\ &= \{(f,\Sigma) | f : S_{g,n} \to \Sigma \text{ a homeomorphism}\}/\sim\end{aligned}$$

where

$$\begin{array}{ccc} S_{g,n} & & \\ {\scriptstyle f_1}\downarrow & \searrow {\scriptstyle f_2} & \\ \Sigma_1 & \xrightarrow[c]{} & \Sigma_2 \end{array} \iff (f,\Sigma_1) \sim (g,\Sigma_2)$$

- $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$
- $\mathcal{M}_g$ is the quotient by the action of the *mapping class group*

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- $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$
- $\mathcal{M}_g$ is the quotient by the action of the *mapping class group*
- $dim(\mathcal{P}_g) = 2dim(\mathcal{T}_g)$
- $p^{-1}(X) \cong \mathcal{Q}(X) = \{\text{holomorphic quadratic differentials on } X\}$
  These are $(2,0)$*-tensors, locally* $q(z)dz^2$

Bundle picture

$$\begin{array}{ccc} (\tilde{f}, \rho) & & \rho \\ \mathcal{P}_g & \xrightarrow{\quad \text{hol} \quad} & \mathcal{R}ep(\pi_1(S), PSL_2(\mathbb{C})) \\ \downarrow \text{p} & & \\ \mathcal{T}_g & & \\ \downarrow \pi & & \\ \mathcal{M}_g & & \end{array}$$

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Question: *What is the structure of the level sets $P_\rho = hol^{-1}(\rho)$?*

$P_\rho \neq \emptyset$ for a generic representation. (Gallo-Kapovich-Marden)

Results known

- *hol* is a local homeomorphism, but not a covering map. (Hejhal, Earle, Hubbard)

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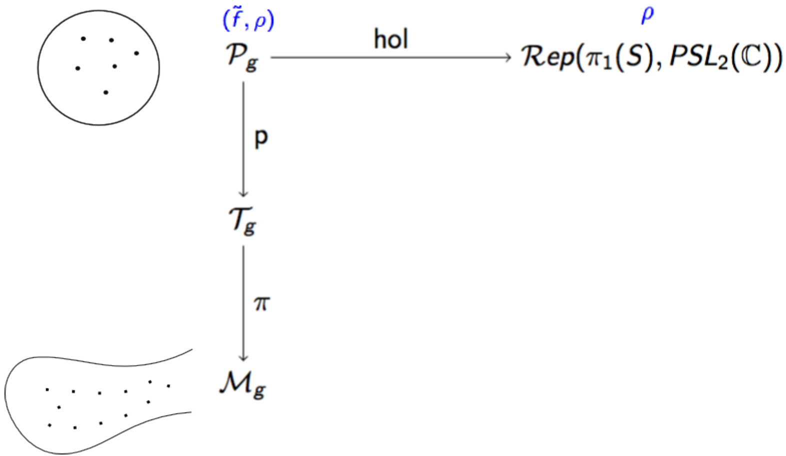
Theorem (G.)

For any Fuchsian representation ρ, the projection of P_ρ to $\mathcal{M}_g$ is dense.

The density result



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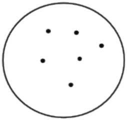

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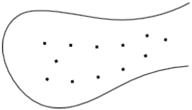
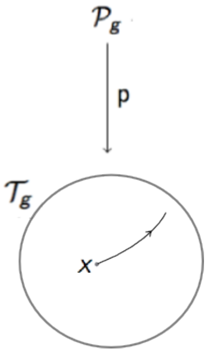


Obtained from a comparison of the dynamics of grafting rays and Teichmüller geodesic rays.

Grafting rays

Projective grafting gives deformations of $\mathbb{C}P^1$-structures from a Fuchsian one. Grafting rays are the shadows of these deformations in Teichmüller space:



Projective grafting

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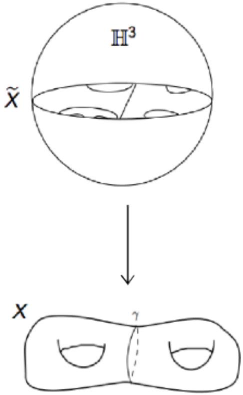

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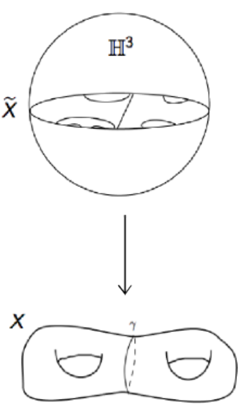

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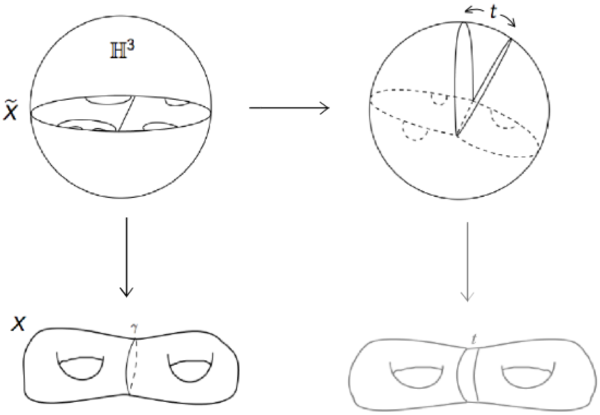

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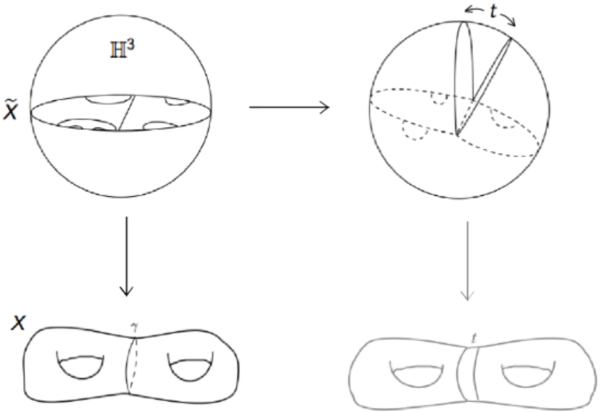
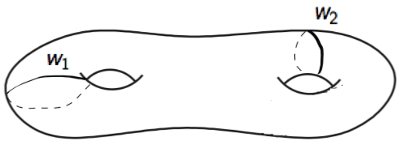


2π-grafting along a multicurve preserves Fuchsian holonomy *(Goldman)*

Measured geodesic laminations

A *measured geodesic lamination* λ is a closed set on a hyperbolic surface which is a union of a disjoint collection of simple geodesics, equipped with a transverse measure μ.



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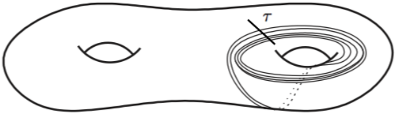

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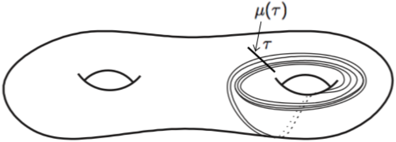

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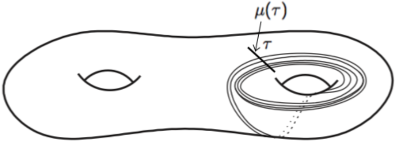


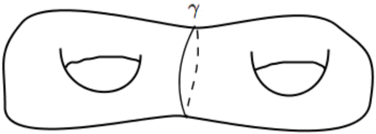
- Weighted s.c.c. are dense in $\mathcal{ML}$.
- *(Thurston)* Projective grafting along measured laminations parametrize $\mathcal{P}_g$:

$$\mathcal{T}_g \times \mathcal{ML} \cong \mathcal{P}_g$$

Conformal grafting

$gr : \mathcal{T}_g \times \mathcal{ML} \to \mathcal{T}_g$

X a hyperbolic surface, $t\gamma$ a simple closed geodesic of weight t



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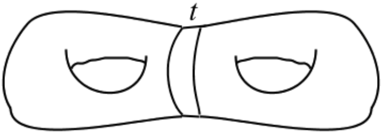

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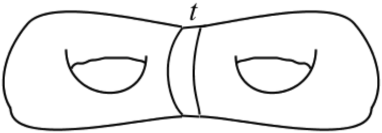

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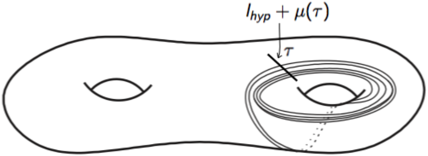
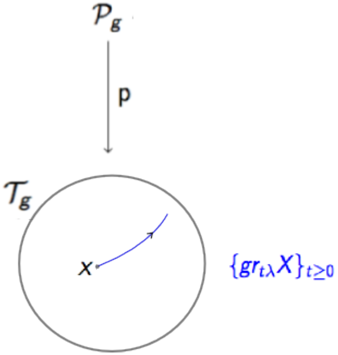


Grafting introduces a euclidean region of width equal to the transverse measure.

Thurston metric

Grafting rays

Starting from a hyperbolic surface X, can graft (for time t) along a measured geodesic lamination. Grafting rays are the shadows of these deformations in Teichmüller space:



Teichmüller metric

For $X, Y \in \mathcal{T}_g$ we can define the *Teichmüller distance*

$$d_{\mathcal{T}}(X, Y) = \frac{1}{2} \inf_f \ln K$$

where K is the dilatation of a quasiconformal map

$$f : X \to Y$$

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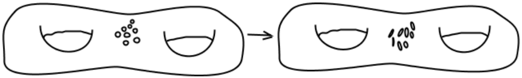
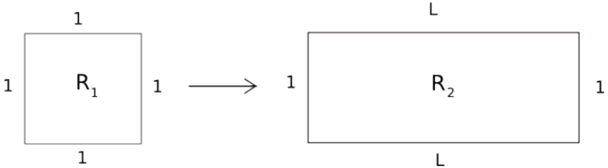
A *K-quasiconformal map* is a homeomorphism that takes infinitesimal circles to ellipses of eccentricity $\leq K$.



Teichmüller metric

Toy example: Rectangles R_1 and R_2 of different *moduli*.



$d_{\mathcal{T}}(R_1, R_2) = \frac{1}{2} \ln L$, realized by the *stretch map*.

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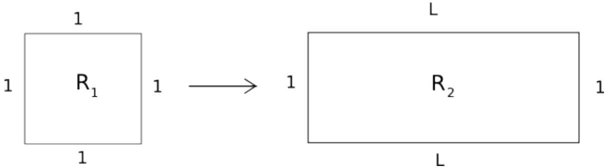

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- $d_{\mathcal{T}}$ is a complete metric.
- $d_{\mathcal{T}}$ is the Finsler metric given by the L^1-norm on $\mathcal{Q}(X)$.
- Co-tangent space $T_X^* \mathcal{T}_g \cong \mathcal{Q}(X) = \{$ holomorphic quadratic differentials$\}$

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A *Teichmüller ray* with *basepoint* X in the *direction* of $q \in \mathcal{Q}(X)$:

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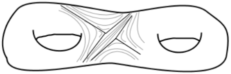
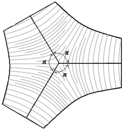
q determines a singular flat metric $|q(z)||dz|^2$ on X together with a *vertical* measured foliation $\mathcal{F}_v$ and a *horizontal* measured foliation $\mathcal{F}_h$.





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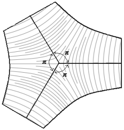

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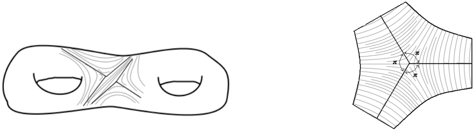

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This ray is geodesic in the Teichmüller metric.

Facts

A Teichmüller ray can be thought of as determined by the pair $(X, \mathcal{F}_v)$:

Theorem (Hubbard-Masur)

$Q(X) \cong \mathcal{MF}$ via the map $q \mapsto \mathcal{F}_v(q)$.

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Comparison of grafting and Teichmüller rays: Diaz-Kim, Choi-Dumas-Rafi

The asymptoticity result

Theorem (G.)

Let $(X, \lambda) \in \mathcal{T}_g \times \mathcal{ML}$. *Then there exists a* $Y \in \mathcal{T}_g$ *such that the grafting ray determined by* (X, λ) *is strongly asymptotic to the Teichmüller ray determined by* (Y, λ), *that is,*

$$d_{\mathcal{T}}(gr_{e^t\lambda}X, Teich_{t\lambda}Y) \to 0$$

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$$d_{\mathcal{T}}(gr_{e^t\lambda}X, Teich_{t\lambda}Y) \to 0$$

as $t \to \infty$.

Corollary

Almost every grafting ray projects to a dense set in moduli space $\mathcal{M}_g$.

Density of integer graftings

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Let $X \in \mathcal{T}_g$. Then the set

$$\mathcal{S} = \{gr_{2\pi\gamma} X \mid \gamma \text{ is a multicurve }\}$$

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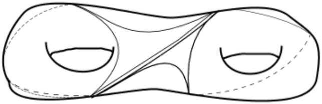
Corollary

Complex projective surfaces with any fixed Fuchsian holonomy are dense in moduli space.

Idea of the proof (Arational case)

Let λ be an *arational* (maximal and minimal) measured lamination.



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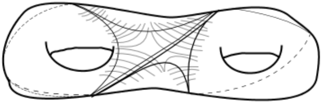

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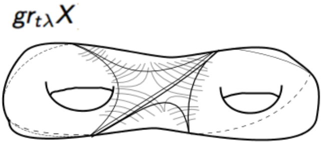

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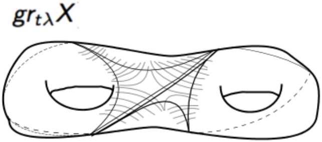

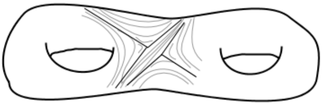

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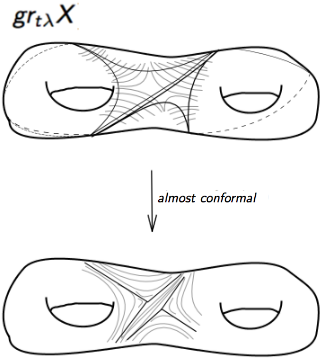

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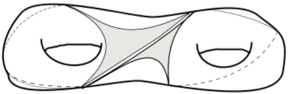
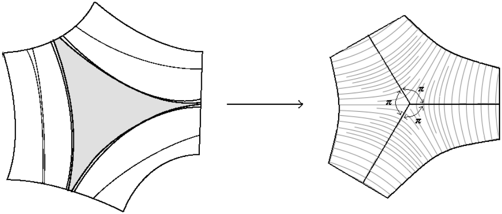
Mapping the surface

Decompose the surface into *truncated ideal triangles* and *long, thin rectangles*:



To construct the map



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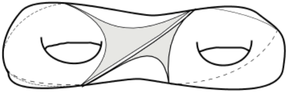
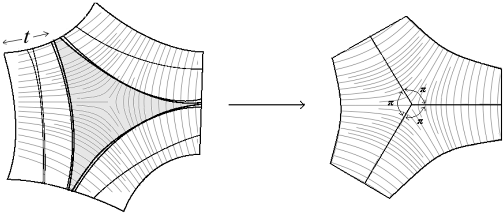


To construct the map we use the transverse foliation:



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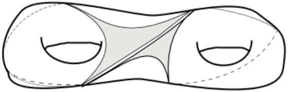

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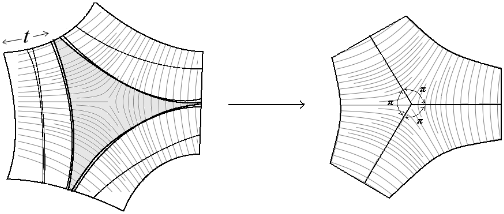
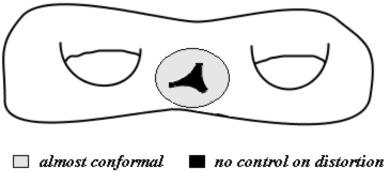


This is almost-conformal wherever the hyperbolic part is thin.

Mapping the surface

For t sufficiently large, the map is almost-conformal for *most* of the time-t grafted surface. (*Problem: central region of the ideal triangles*)



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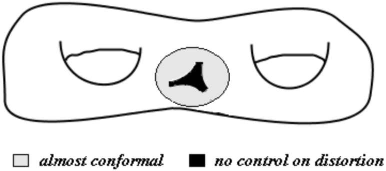
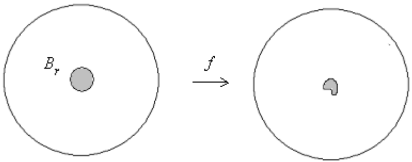


It remains to adjust this to a map that is almost-conformal *everywhere*.

A quasiconformal extension lemma

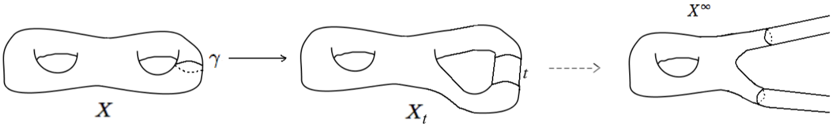
Lemma

For any $\epsilon > 0$ sufficiently small and any $0 \leq r \leq \epsilon$ if $f : \mathbb{D} \to \mathbb{D}$ satisfies
(1) f is a quasiconformal map
(2) The quasiconformal distortion is $(1+\epsilon)$ on $\mathbb{D} \setminus B_r$
then there exists a $(1 + C\epsilon)$-quasiconformal map $g : \mathbb{D} \to \mathbb{D}$ such that $f_{|\partial\mathbb{D}} = g_{|\partial\mathbb{D}}$. (Here C is a universal constant)



Idea of proof (multicurve case)

Consider the *conformal limit* of the grafting ray.



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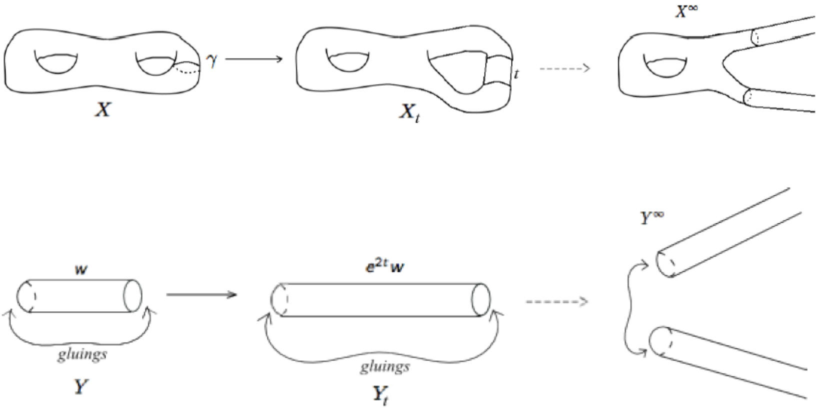
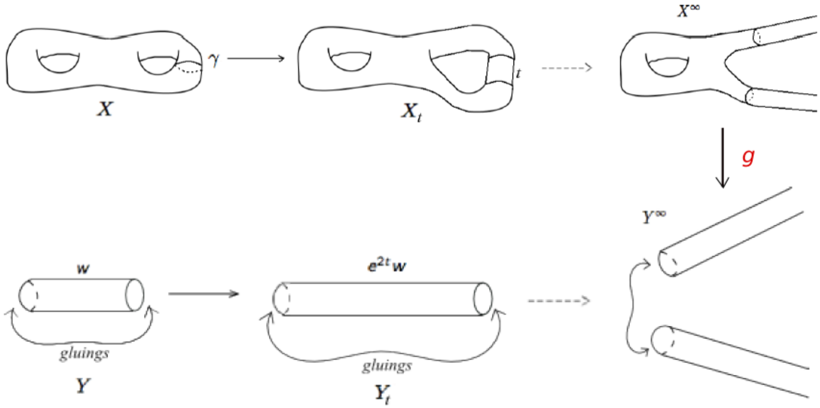


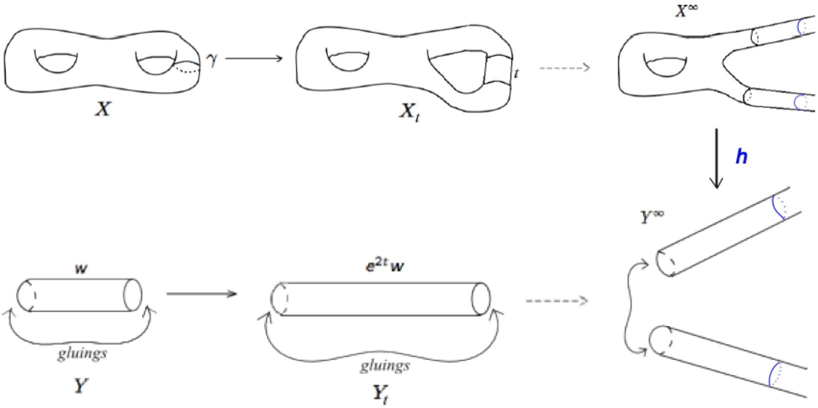
Idea of proof (multicurve case)

Obtain Y^∞ by specifying a **meromorphic** quadratic differential. *(Strebel)*



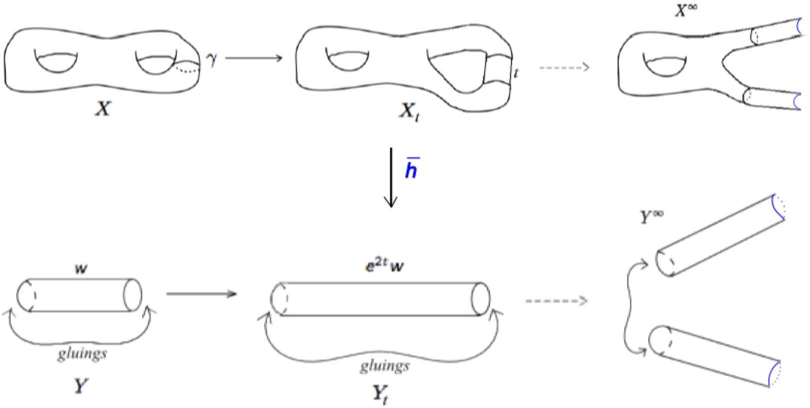
Idea of proof (multicurve case)

Adjust g to an *almost-conformal map* that takes circles to circles.



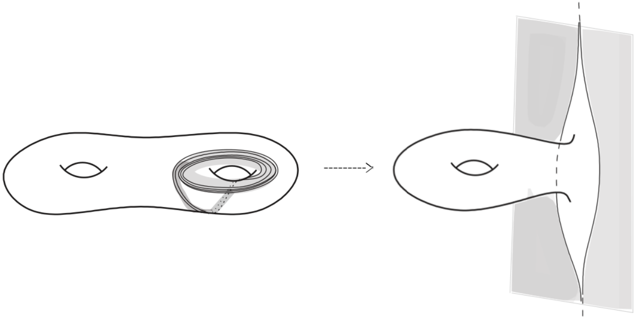
Idea of proof (multicurve case)

Truncating along those circles and *gluing* gives the required map.



Idea of proof (general case)

"Minimal, non-filling" case:



Corresponding limit of the Teichmüller ray is given by a half-plane differential, a meromorphic quadratic differential with higher order poles and a "half-plane structure".

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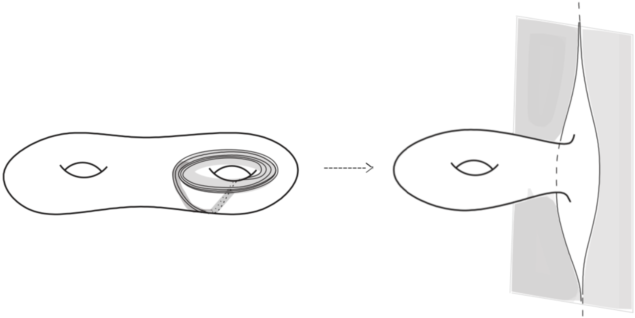

Corresponding limit of the Teichmüller ray is given by a half-plane differential, a meromorphic quadratic differential with higher order poles and a "half-plane structure". (Generalization of Strebel's result)

Further questions

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- Do the integer graftings *equidistribute* in $\mathcal{M}_g$?

- For a generic $\rho \in \mathcal{R}ep(\pi_1(S), PSL_2(\mathbb{C}))$, does the holonomy level set $\mathcal{P}_\rho$ project to a dense set in $\mathcal{M}_g$?