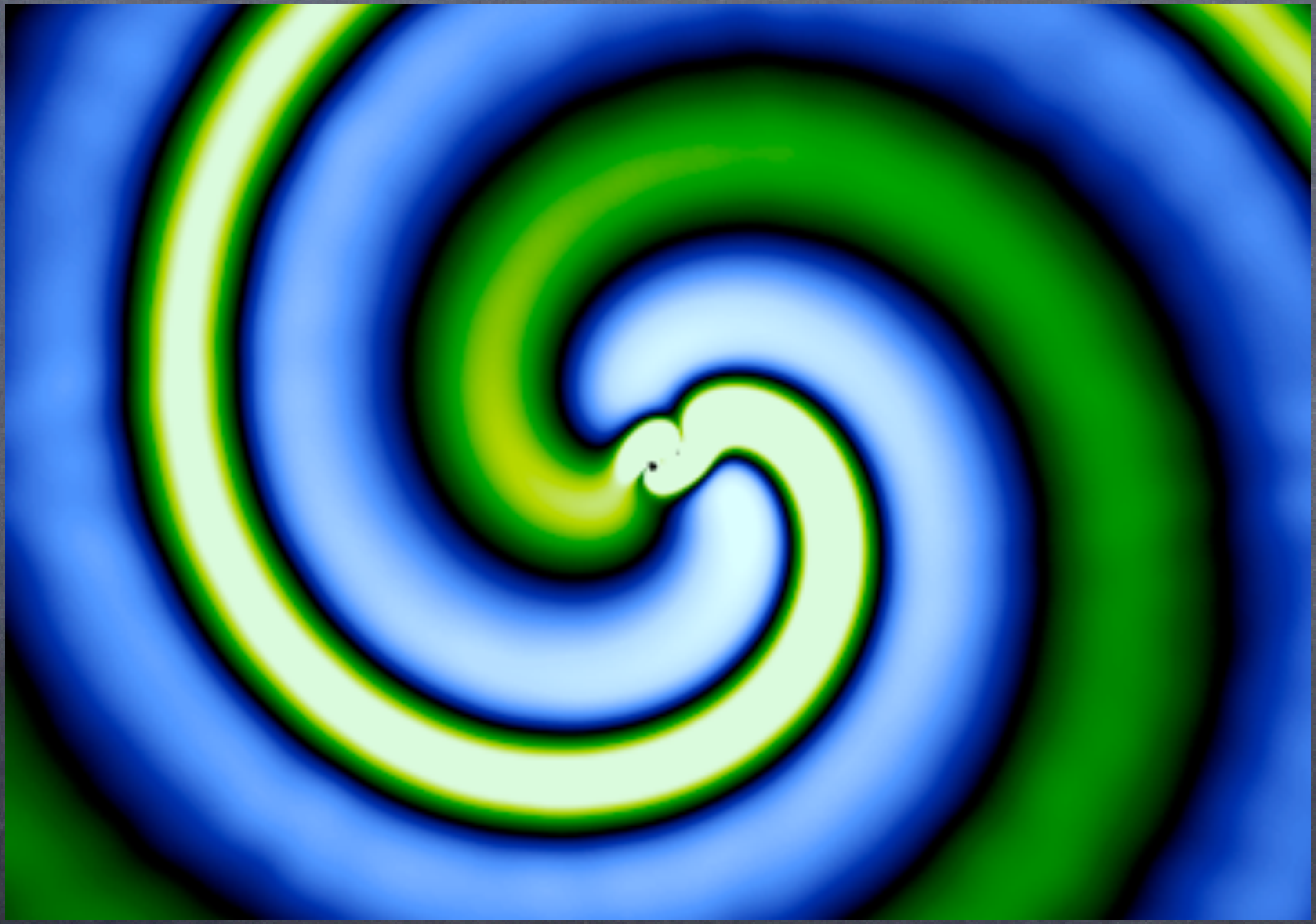
Introduction to Theory and Numerics of Partial Differential Equations II: Mathematical concepts of PDEs



ICTS Summer School on Numerical Relativity
Bangalore, June 2013

Lab goals for today

- Generalize your ODE code to handle systems of ODEs, solve the 2-body problem in GR for point particles in the "Newtonian orbits + quadrupole formula energy loss" approximation.
- Carry out a convergence test and evaluate the numerical error.

Motivation

- The Einstein equations are PDEs.
  - To learn GR, one needs to understand the basics of differential geometry.
  - To work in GR, one needs to understand the basics of PDEs, and how to solve them.

Motivation

- Classical physics is formulated in terms of PDEs for tensors.
- To understand a physical theory (GR, Maxwell, QCD, ...) requires to understand the space of solutions of the PDEs that describe it.
  - What predictions to these solutions make for observations?
  - Do solutions exist? Can the solutions become singular?
  - What degrees of freedom do these PDEs have? How can we specify a specific solution?
  - Time evolution problems: given initial data, does a unique time evolution exist? Does it depend continuously on the initial data?
- **Need a systematic way to find approximate solutions of PDEs: perturbation approaches, numerical analysis.**

Types of PDEs (linear for the moment)

- Can classify by the type of "problem" that can naturally be associated with a PDE: initial/initial boundary // boundary value problems.
- Standard types:
  - **hyperbolic**, generalize wave equation: information propagates with finite speed
    $$u(\vec{x},t)_{,tt} = \Delta u(\vec{x},t)$$
  - parabolic: generalize heat equation, well posed only forward in time, information propagates instantaneously
    $$u(\vec{x},t)_{,t} = \Delta u(\vec{x},t)$$
  - Schrödinger equation: information propagates instantaneously
    $$u(\vec{x},t)_{,t} = i\Delta u(\vec{x},t)$$
  - **elliptic**, e.g. Laplace equation:
    $$\Delta u(\vec{x}) = 0$$

Types of PDEs (linear for the moment)

- Time evolution problems can give rise to boundary value problems:
- Looking for a stationary (time independent) solution - an equilibrium state, e.g. for the wave equation we would get

$$u(\vec{x},t)_{,tt} = \Delta u(\vec{x},t) \longrightarrow \Delta u(\vec{x}) = 0$$

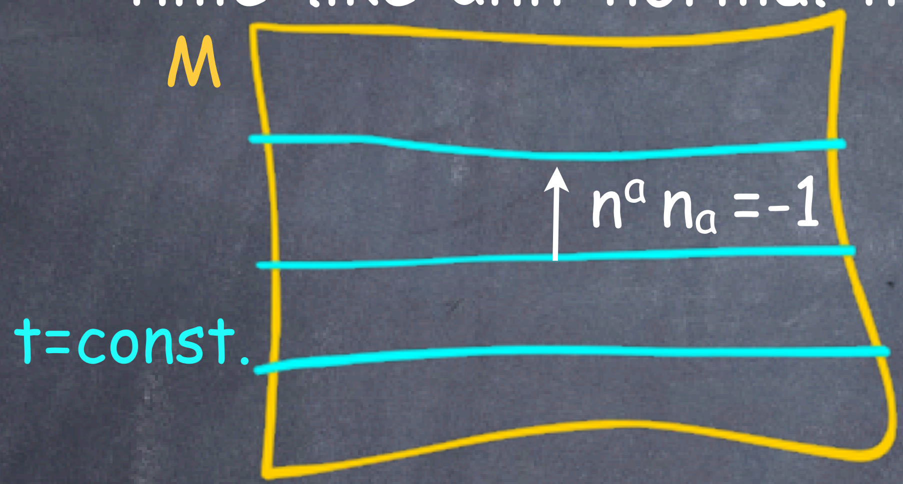
  - We may also ask for periodic solutions in time, and obtain an eigenvalue problem.

  - The known fundamental theories of nature (GR, elektro-weak theory, QCD) are gauge theories, the presence of gauge freedom leads to constraints - restrictions on the space of possible initial data for a time evolution problems, which typically take the form of elliptic boundary value problems.

  - Details about elliptic problems -> Mark Hannam's lectures

Initial Value formulation of a simple gauge theory: Maxwell

- 4-dimensional formulation: $\nabla_{[a}F_{bc]} = 0, \quad \nabla_b F^{ab} = j^a$
- Introduce a space-time split, define hypersurfaces of constant time by time-like unit normal n^a, and electric and magnetic fields E^a, B



$$E^a = F_{ab} n^b, \qquad B^c = \frac{1}{2} F_{ab}\, {}^3\epsilon^{abc}$$

- Get 2 evolution equations (contain time derivs.), in flat space:

$$\partial_t E^a = \epsilon_{abc}\partial^b B^c - 4\pi j_a, \qquad \partial_t B^a = -\epsilon_{abc}\partial^b E^c$$

- Get 2 constraint equations (contain no time derivs.):

$$\partial_a E^a = 4\pi\rho, \qquad \partial_a B^a = 0$$

- Maxwell equations need to be solved consistently with equations for j^a, ρ

Maxwell II

- **Exercise:** show that constraints propagate (always satisfied by virtue of the evolution equations, if satisfied at t=0)
- Initial value problem makes sense: constraints are preserved, fpr given initial data a unique time evolution exists, which depends continuously on initial data = **well-posed initial value problem**
- Information propagates at the speed of light. We will soon understand connection between propagation speeds and the property of an IVP to be well-posed!

Maxwell III

- Using vect. pot. A additional gauge issues appear!

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \;\Rightarrow\; \nabla^a \left(\nabla_a A_b - \nabla_b A_a\right) = j_b$$

- Lorentz gauge -> Wave equation:

$$\nabla^a A_a = 0 \;\Rightarrow \nabla^a \nabla_a A_b = j_b$$

- Numerical ED is difficult (preserve constraints!), but well understood: analytical formulation, numerical algorithms, comparison with experiment!

- curved background:

$$\mathcal{L}_n D_i E^i = -K D_i E^i, \qquad \mathcal{L}_n D_i B^i = -K D_i B^i$$

- In collapsing case (K < 0)) instability of constraints!
    - Well-posedness is necessary but not sufficient to accurately approximate the continuum problem with finite precision!
- Solution for Maxwell: use $\sqrt{g}E^a, \sqrt{g}B^a$. GR ??

Existence of analytic solutions

- For a given PDE, do any solutions exist?
- Consider an initial value problem for the wave equation as an example:
  - wave equation: $$\frac{\partial^2 \phi(t,\vec{x})}{\partial t^2} = \Delta \phi(t,\vec{x})$$
  - set initial data at $t=t_0$: $\phi(t_0,\vec{x}), \frac{\partial}{\partial t}\phi(t,\vec{x})|_{t=t_0}$
  - Initial data & WE tells us about first and second time derivatives, differentiating the WE in time we can construct all higher time derivatives:

$$\frac{\partial^3 \phi(t,\vec{x})}{\partial t^3}|_{t=t_0} = \Delta \frac{\partial \phi(t,\vec{x})}{\partial t}\phi(t,\vec{x})|_{t=t_0}$$

  - Does this formal power series converge? Yes, for analytic initial data! - Theorem of Cauchy-Kowalevskaya!

Theorem of Cauchy-Kowalevskaya

- Let $t, x_1, \ldots, x_{n-1}$ be coordinates of R^n .

  Consider a system of m PDEs for m unknowns $\Phi_i(t, x_\mu)$, $i=1,\ldots,m$, where each RHS function F_i is an analytic function of its variables:

$$\frac{\partial^2 \phi_i(t,\vec{x})}{\partial t^2} = F_i(t, \vec{x}, \phi_j, \frac{\partial \phi_j}{\partial t}, \frac{\partial \phi_j}{\partial x^\mu}, \frac{\partial^2 \phi_j}{\partial t \partial x^\mu}, \frac{\partial^2 \phi_j}{\partial x^\mu \partial x^\nu})$$

  - Let $f_i(x_\mu)$ and $g_i(x_\mu)$ be analytic functions.
  - => ∃ open neighborhood O of the hypersurface $t=t_0$: within O ∃! analytic solution of the PDE system with initial data $\Phi_i(t_0, x_j) = f_i$, $\partial_t\Phi_i(t_0, x_j) = g_i$.

- CK-theorem shows that:
  - the WE and similar equations have an initial value formulation for analytic intial data.
  - There is a large class of solutions (as many as there are pairs of analytic functions of the spatial coordinates x_μ).

Non-analytic equations: example of Lewy

Even linear PDEs with non-analytic coefficients do not in general have solutions!

On $\mathbb{R}\times\mathbb{C}$, suppose that $u(t,z)$ is a function satisfying, in a neighborhood of the origin,

$$\frac{\partial u}{\partial \bar{z}} - iz\frac{\partial u}{\partial t} = \varphi'(t)$$

for some C^1 function ϕ. Then ϕ must be real-analytic in a (possibly smaller) neighborhood of the origin.

http://en.wikipedia.org/wiki/Lewy's_example

Analytic solutions are not enough!

- For analytic solutions, any finite neighborhood determines the whole solution - makes no sense for relativistic theories, where information propagates at finite speed.
- We can only require C^k, or C^∞ (smooth is sufficient for us).
- C-K does not distinguish between wave and Laplace equations:
  - Let's see the difference between wave and Laplace equations in an example ...

Example (Hadamard)

- Functions U_n satisfy WE, V_n satisfy Laplace eq.:

$$U_n(t,x) = \frac{1}{n^2}\sin(nt)\sin(nx), \quad V_n(t,x) = \frac{1}{n^2}\sinh(nt)\sin(nx)$$

$$\ddot{U} = U_n^{''}, \quad \ddot{U} + U_n^{''} = 0$$

- At t=0 we have

$$U_n(0,x) = V_n(0,x) = 0, \quad \partial_t U_n(0,x) = \frac{1}{n}\sin(nx)$$

- The Cauchy data converge to 0 as n-> ∞. For WE, solutions converge to 0, For the Laplace Eq the V_n blow up for any t>0.

- Key idea of 'hyperbolic' eqs: have stable solutions for the initial value problem.

Domain of dependence

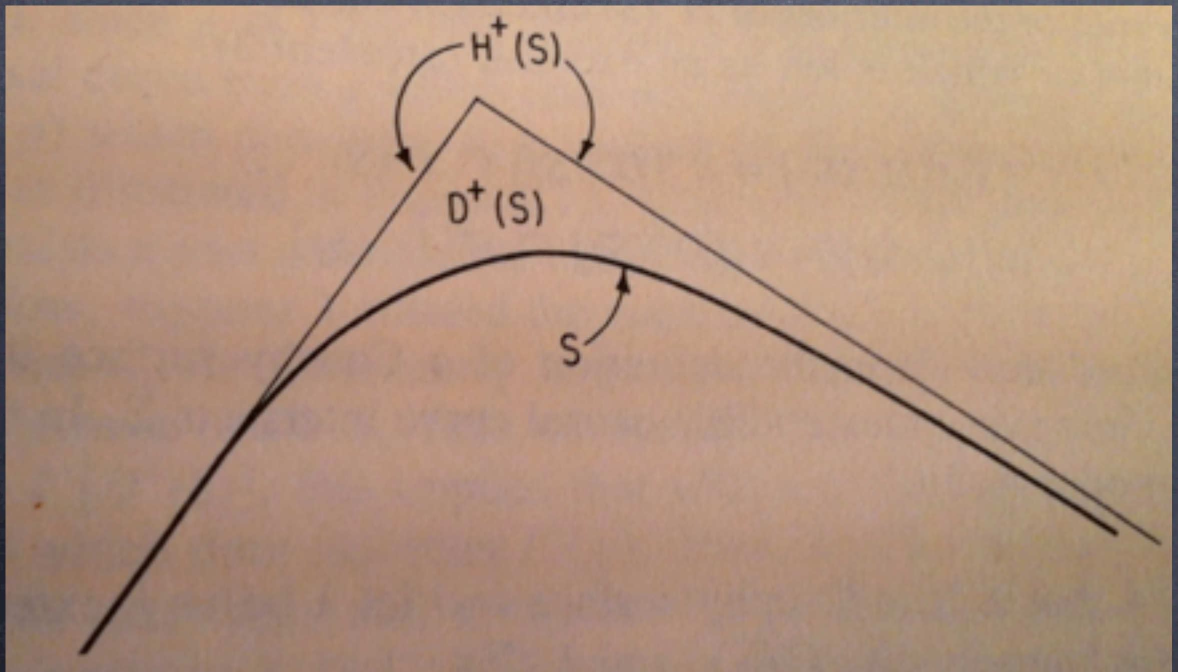
- Let S be a 3-D "hypersurface of constant time" [an achronal (non-timelike) embedded submanifold of a manifold M (points of S can not communicate causally].

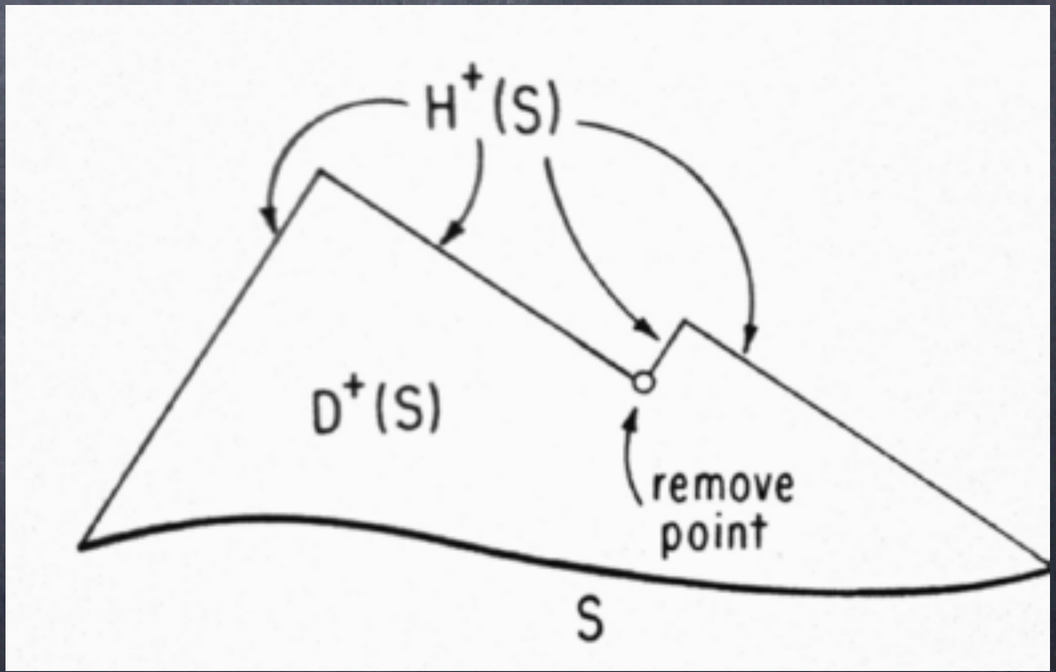
- Future domain of dependence $D^+(S)$:

$$D^+(S) = \left\{ p \in M \middle| \begin{array}{l} \text{every past inextendible causal curve;} \\ \text{through } p \text{ intersects } S. \end{array} \right.$$

- analogous for $D^-(S)$

- If nothing can travel faster than light, any signal sent to $p \in D^+(S)$ must have registered on S. Thus, given initial conditions on S, we should be able to predict what happens at p.





Global hyperbolicity

- $D(S) = D^+(S) \cup D^-(S)$
- A set such that $D(\Sigma) = M$ is called a Cauchy hypersurface, is a snapshot of the universe a spacetime which possesses a Cauchy hypersurface is called globally hyperbolic.
- Theorem (see e.g. Wald, chapter 8): Let (M, g_{ab}) be a globally hyperbolic spacetime. Then (M, g_{ab}) allows a global time function t, such that each surface of constant t is a Cauchy surface, and the topology of M is $R \times \Sigma$, where Σ denotes any Cauchy surface.
- Globally hyperbolic spacetimes are those which can be constructed as an initial value problem.
- Globally hyperbolic spacetimes do not allow closed timelike curves (time machines).
- Spacetimes with time machines are not “predictable”.

Well-posedness and stability for evolution equations

Continuum problem:

- WP: A unique solution exists (when gauge is chosen), depends continuously on initial data. Can formulate continuity as

$$\exists K, a \in R : \quad ||u(t)|| \leq K e^{a\,t} ||u(0)|| \quad \forall u(0)$$

- Exponential growth (instability) ok, arbitrarily fast growth not.
  - "mode stability": can't have modes which grow arbitrarily fast
  - typical ill-posed problems: Higher frequencies correspond to larger a, K -> better resolution, worse solution.

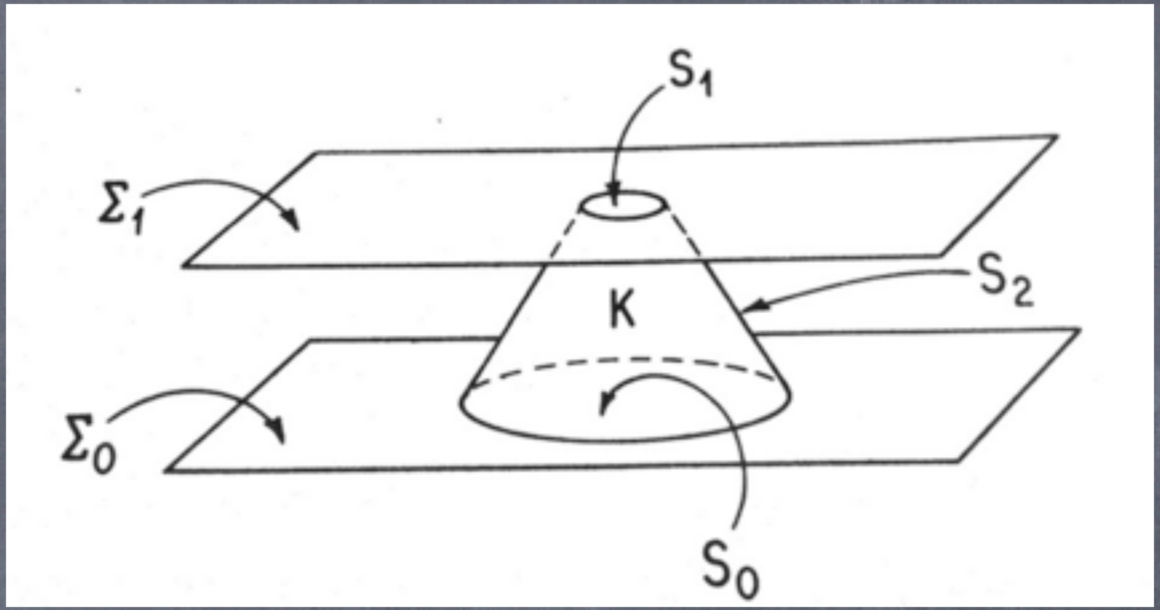
Discrete problem:

- WP (stable) in numerical context for iterative problem ($e^{\lambda t}$ ok, $e^{\lambda n}$ not)

$$v^{n+1} = Q(t_n, v^n, v^{n+1}) v^n : \quad ||v^n|| \leq K e^{\alpha t_n} ||v^0|| \quad \forall v^0$$

- Lax equivalence theorem: "a consistent (formally convergent) finite difference scheme for a linear PDE for which the initial value problem is **well posed** is convergent iff it is stable."

Outline of well-posedness proof for KG



- Klein-Gordon equation in flat spacetime:

$$\partial_a \partial^a \phi = -\frac{\partial^2 \phi}{\partial t^2} + \Delta \phi = m^2 \phi$$

  - energy momentum tensor divergence free: $\partial^a T_{ab} = 0$

$$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} \eta_{ab} \left(\partial_c \phi \partial^c \phi + m^2 \phi^2 \right)$$

- satisfies dominant energy condition: if v^a is a future directed timelike vector, then $-T^a{}_b v^b$ is a future directed timelike or null vector (mass energy can not be observed to flow faster than light)

- Using the Gauss law we can rewrite as:

$$\int_{S_1} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 + |\nabla \phi|^2 + m^2 \phi^2 \right] \leq \int_{S_0} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 + |\nabla \phi|^2 + m^2 \phi^2 \right]$$

Well-posedness proof for KG - II

$$\int_{S_1}\left[\left(\frac{\partial\phi}{\partial t}\right)^2+|\nabla\phi|^2+m^2\phi^2\right]\leq\int_{S_0}\left[\left(\frac{\partial\phi}{\partial t}\right)^2+|\nabla\phi|^2+m^2\phi^2\right]$$

- There can at most be 1 solution in $D^+(S_0)$ with given initial data $(\Phi, \partial_t\Phi)$ on S_0:
  If Φ_1, Φ_2 are both C^2 solutions with the same initial data, then $\psi = \Phi_1 - \Phi_2$ would be a solution with vanishing initial data (using linearity!).
  The RHS of the above inequality thus vanishes, implying $\psi=0$ at S_1, S_1 was arbitrary, so ψ vanishes on $D^+(S_0)$ and $D^-(S_0)$.

- -> A variation of the initial data outside of S_0 can not affect the solution within $D^+(S_0)$ and $D^-(S_0)$.

- Solutions depend continuously on initial data in the above "energy norm".

- Other norms (Sobolev) can be constructed to bound the solution and its partial derivatives directly, see e.g. Wald, GR, p. 249).

Well-posedness proof for KG - III

Outline of existence proof for smooth solutions Φ for arbitrary initial data $\Phi_i(t_0, x_j)$, $\partial_t \Phi_i(t_0, x_j)$ on Σ_0

- Smooth functions can be approximated (with uniform convergence) by analytical functions.
- By C-K theorem, these give rise to analytical solutions of the KG equation. Using the energy norm (and derived Sobolev norms) one can show that these analytical solutions have to converge to a solution of KG.
- As seen before, the limiting solution has to be unique.
- Unlike C-K, this proof uses specific properties of the wave equation: linearity, conserved T_{ab}, dominant energy condition, "wave equation character" - proof would not work for Laplace equation!
- Can we obtain a proof of well-posedness for a general class of equations?

Nonlinear PDEs

- Non-linear PDEs in general have to be discussed on a case-by-case basis.
- Quasi-linear: linear in highest derivatives (principal part), coefficients depend on the independent variables and their lower order derivatives.
- Quasi-linear PDEs allow statements on well-posedness based on properties of the principal part, EEs are quasi-linear.
- Classes of systems of hyperbolic equations which admit a well-posed intial value problem:
  - generalized wave equations (g_{ab} a smooth Lorentz metric)

$$g^{ab}(x, \phi_j, \nabla_c \phi) \nabla_a \nabla_b \phi_i = F_i(x, \phi_j, \nabla_c \phi)$$

  - strongly hyperbolic systems -> investigate in more detail ...

example: advection equation

- Construct general solution via Fourier transform in space:

$$\frac{\partial}{\partial t}u(\vec{x},t) + v^j \frac{\partial}{\partial x^j}u(\vec{x},t) = 0$$

$$\hat{u}(\vec{k},t) := \frac{1}{(2\pi)^{n/2}} \int e^{-i\vec{k}\cdot\vec{x}} u(\vec{x},t) d^n x \quad \Rightarrow \quad \widehat{\partial_{\vec{x}} u} = i\vec{k}\hat{u}$$

$$\partial_t \hat{u}(\vec{k},t) = -i v^j k_j \hat{u}(\vec{k},t) \quad \Rightarrow \quad \hat{u}(\vec{k},t) = e^{-i\,\vec{v}\cdot\vec{k}\,t}\hat{u}_0(\vec{k})$$

- Solution moves with speed $\vec{v}$ without changing profile:

$$u(\vec{x},t) = \frac{1}{(2\pi)^{n/2}} \int \hat{u}(\vec{k},0) e^{i\vec{k}(\vec{x}-\vec{v}t)} d^n k = u_0(\vec{x}-\vec{v}t)$$

- Fourier method works for general constant coefficient PDEs!
- Norm remains constant -> equation is well posed!
- key idea: can solve constant coeff. case explicitly
- exercise: well-posedness for heat/wave/Schrödinger eq.

Constant coefficient hyperbolic systems

- First order differential systems:

$$\partial_t u^a(\vec{x},t) = A_b{}^{aj}\partial_j u^b(\vec{x},t)$$

$$\partial_t \hat{u}^a(\vec{k},t) = iA_b{}^{aj}k_j\hat{u}^b(\vec{k},t) \quad \Rightarrow \quad \hat{u}^a(\vec{k},t) = e^{i\,A_b{}^{aj}k_j\,t}\hat{u}^a_0(\vec{k})$$

- Choose direction n:

$$\vec{n}\cdot\vec{n} = 1,\ k = |k| \quad \Rightarrow \quad \hat{u}^a(k,\vec{n},t) = e^{i\,A_{n\,b}{}^a k\,t}\hat{u}^a_0$$

- Compute matrix exponential by transforming A to Jordan form:

$$PAP^{-1} = D + N,\ N^n = 0 \quad \Rightarrow \quad e^{iAkt} = e^{iDkt}e^{iNkt} = e^{iDkt}\sum_{l=0}^{l=n-1} N^l\frac{k^l t^l}{l!}$$

- A diagonalizable & real eigenvalues: each component of u in the diagonal basis is advected with speed corresponding to (-)eigenvalue of A.

- P.u are called "characteristic variables".

- Fourier domain solution is oscillatory and preserves norm.

- Lower order terms (u_t = A ∂u + Bu + C) can result in exponential growth (frequency independent), propagation speeds and WP only depend on A (principal part = highest derivatives).

Constant coefficient hyperbolic systems

- First order differential systems:

$$\partial_t u^a(\vec{x},t) = A_b{}^{aj}\partial_j u^b(\vec{x},t)$$

$$\partial_t \hat{u}^a(\vec{k},t) = iA_b{}^{aj}k_j\hat{u}^b(\vec{k},t) \quad \Rightarrow \quad \hat{u}^a(\vec{k},t) = e^{i\,A_b{}^{aj}k_j\,t}\hat{u}^a_0(\vec{k})$$

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- Jordan blocks (N≠0) cause frequency (k) dependent polynomial growth - obstruction to WP!

$$\mathrm{JordanForm}(A) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- complex eigenvalues -> exponential growth (in future or past) -> ~~WP~~.

classification of hyperbolic systems

- **weakly hyperbolic:** Speeds (eigenvalues of A) all real (well posed in absence of l.o.t. in appropriate norm)
- **strongly hyperbolic:** weakly hyperbolic with complete set of eigenvectors (characteristic variables span solution space), **well posed initial value problem**
- **symmetric/symmetrizable hyperbolic:** strongly hyperbolic, and A can be diagonalized with the **same** similarity transformation P for **all** space-directions.
  - strongly hyperbolic implies symm. hyperbolic in 1D
  - admits a conserved energy: can be used to prove well-posed initial boundary value problem with appropriate BCs
- **strictly hyperbolic:** all eigenvalues are distinct

hyperbolic systems: remarks

- Quasi-linear = nonlinearities only lower order terms (e.g. Einstein equations): well-posedness carries over from equations linearized around some background solution.
- Solutions may become singular in finite time -> well-posedness only guarantees existence of solution for some small time
- local/global in time existence problem.
- first order in time system was convenient for solution procedure in Fourier domain - what happens with higher differential order systems, e.g. wave equation? -> next lecture
- Clarification of hyperbolicity of ADM, BSSN etc. has taken until 1999 -2006 [Frittelli, Reula, Sarbach, Beyer, Tiglio, Calabrese, Gundlach, Martín-García, ...]