Martingales, Brownian Motion & Stochastic Calculus:

$(\Omega, \mathcal{F}, P)$ - prob. space.

Need a way to encode information

Done by means of a σ-algebra (field)

$\mathcal{G} \subseteq \mathcal{F}$ be a sigma algebra.

We wish to define the conditional expectation $E[X|\mathcal{G}]$.

- Given $\mathcal{G}$ can be thought of as "knowing" for each event in $\mathcal{G}$ whether it has happenned or not.
  - Another is as a "best" estimator of X given $\mathcal{G}$.

$\rightarrow$ Consider $\min_c E[(X-c)^2]$

$$L(c) = E[(X-c)^2] = E(X^2) - 2c\,E[X] + c^2$$

$$L'(c) = -2E[X] + 2c \quad = 0 \quad \Rightarrow \quad c = E[X]$$

So best estimator of $\cancel{X}$ when nothing is known is E

Similarly $E[X|\mathcal{G}]$ is the best estimator of X when $\mathcal{G}$ is given, ~~i.e~~

A r.v. Y is said to be $\mathcal{G}$-measurable if its value is known given $\mathcal{G}$.

So $E[X/\mathcal{G}]$ is minimizes $E[(X-Y)^2]$ over all Y that are $\mathcal{G}$-measurable.

Properties:-

0) $E[aX+bY/\mathcal{G}] = aE[X/\mathcal{G}] + bE[Y/\mathcal{G}]$.

1) $E[X] = E[E[X/\mathcal{G}]]$.

2) If X is $\mathcal{G}$-measurable then $E[X/\mathcal{G}] = X$.

3) If X is "independent" of $\mathcal{G}$ then $E[X/\mathcal{G}] = E[X]$.

4) If $\mathcal{G}_1 \subseteq \mathcal{G}_2$ then

$$E[X/\mathcal{G}_1] = E[E[X/\mathcal{G}_2]/\mathcal{G}_1].$$

σ-field generated by random variables

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}.$$

Note:- $\{X = x\} \in \sigma(X)$ => knowing $\sigma(X)$ we know value of X & vice versa. We will write

$$E[Y/\sigma(X)] = E[Y/X] \qquad \left[\begin{array}{l} E[Y/X=x] = \int y f_{Y/X}(y/x)dy = h(x) \\ E[Y/X] = h(X). \end{array}\right]$$

Ex. $E[N] = \lambda$ $\quad X_1, X_2, \ldots$ i.i.d. rvs indep of N

$E[X_i] = \mu$ $\quad N$- non-neg. integer-valued.

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[E\left[\sum_{i=1}^{N} X_i / N\right]\right] = E\left[\sum_{i=1}^{N} E[X_i/N]\right]$$

$$= E\left[\sum_{i=1}^{N} E[X_i]\right] = \mu E[N] = \mu\lambda.$$

5) $E[(X+Y)/\mathcal{G}] = E[X/\mathcal{G}] + E[Y/\mathcal{G}]$ (see 0)

Martingales

<u>Filtration</u>:- A filtration $\{\mathcal{F}_n\}_{n\geq 0}$ is an increasing seq. of σ-algebras. $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$.

<u>Def.</u> A stochastic process $\{M_n\}_{n\geq 0}$ is a $\{\mathcal{F}_n\}$ martingale if

- M_n is $\mathcal{F}_n$ measurable
- $E[|M_n|] < \infty$
- $E[M_{n+1} \mid \mathcal{F}_n] = M_n \equiv E[M_{n+k} \mid \mathcal{F}_n] = M_n$

— pure fluctuation.

<u>Ex. 1</u> SRW : $X_1, X_2, \ldots$ be i.i.d. rvs with

$$P(X_i = +1) = P(X_i = -1) = \frac{1}{2} \Rightarrow E[X_i] = 0$$

$$M_0 = 0 ; \quad M_n = \sum_{i=1}^{n} X_i = M_{n-1} + X_n, \quad \mathcal{F}_n = \sigma(X_1, \ldots, X_n), \quad n \geq 1.$$

$$E[M_{n+1} \mid \mathcal{F}_n] = E\left[\sum_{i=1}^{n} X_i + X_{n+1} \,\middle|\, \mathcal{F}_n\right]$$

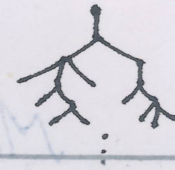
$$= \sum_{i=1}^{n} X_i + E[X_{n+1} \mid \mathcal{F}_n] = M_n .$$

<u>Def.</u> If $E[M_{n+1} \mid \mathcal{F}_n] \leq M_n$: ~~sub~~ super-martingale

& $E[M_{n+1} \mid \mathcal{F}_n] \geq M_n$ sub-martingale.

In above example if $P(X_i = +1) = p = 1 - P(X_i = -1)$
$E[X_i] = 2p - 1$

So if $p > \frac{1}{2}$ M_n – sub mart. & if $p < \frac{1}{2}$ it will be a super martingale.



$Z_0 = 1$
$Z_1 = 2$
$Z_2 = 4$
$Z_3 = 4$
$Z_4 = 5$
$\vdots$

Ex. 2 Galton-Watson Branching Process.

- Start with one individual at time 0 : $Z_0 = 1$.
- At time 1, this individual dies & gives birth to k offspring w.p. p_k, $k \geq 0$ who then behave in the same way indep. of everything else.
- Z_n = population size at time n, $n \geq 1$.

Let $\mu = \sum_{k=1}^{\infty} k p_k$: mean # of offspring produced by any individual.

Let X_{nj} = # of offspring produced by individual j in n^{th} gen. $\quad P(X_{nj} = k) = p_k, \; k \geq 0$

$$Z_{n+1} = \sum_{j=1}^{Z_n} X_{nj}$$

$$\mu_{n+1} = E[Z_{n+1}] = E\left[E\left[\sum_{j=1}^{Z_n} X_{nj} \,\Big|\, Z_n \right] \right]$$

$$= E[\mu \cdot Z_n] = \mu \cdot \mu_n = \mu^2 \mu_{n-1} = \cdots \mu^{n+1}$$

$$\therefore \boxed{E[Z_n] = \mu^n}$$

Let $\mathcal{F}_n = \sigma(Z_1, \ldots Z_n)$ $\quad M_n = \dfrac{Z_n}{\mu^n}$

$$E[M_{n+1} \mid \mathcal{F}_n] = \frac{1}{\mu^{n+1}} E\left[\sum_{j=1}^{Z_n} X_{nk} \,\Big|\, \mathcal{F}_n \right] = \frac{\mu Z_n}{\mu^{n+1}} = M_n .$$

- M_n martingale $\Rightarrow$ $E[M_{n+1} \mid \mathcal{F}_n] = M_n \Rightarrow E[M_{n+1}] = E[M_n]$
$= E[M_0]$

<u>Martingale</u> <u>Convergence</u> <u>Theorem</u> : $(M_n, \mathcal{F}_n)_{n\geq 0}$ be a martingale.

If $E[|M_n|] < K < \infty$ then M_n converges a.s.

<u>Cor.</u> A non-negative martingale converges.

<u>Ex.</u> Consider the martingale $M_n = \dfrac{Z_n}{\mu^n}$ from ex. 2.

By Cor. M_n converges.

- $\mu < 1 \Rightarrow \mu^n \to 0$ so if M_n has to converge Z_n must converge to 0.

Since Z_n is an integer it must happen that Z_n hits 0 at some stage. $\Rightarrow$ Extinction!!

- $\mu = 1 \Rightarrow M_n = Z_n$ converges.

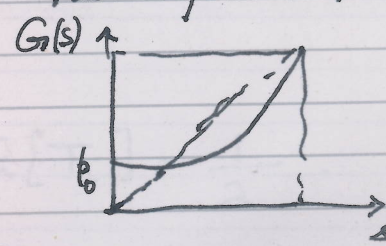
But if $Z_n(\omega) \to Z_\infty(\omega)$ then since Z_n is an integer $Z_n(\omega) = Z_\infty(\omega)$ for all $n \geq N(\omega)$. But this cannot happen unless $Z_\infty(\omega) \equiv 0$. Extinction!!
(OR $p_1 = 1$)

- If $\mu > 1$ then with positive prob. $Z_n \to \infty$.

Let $G(s) = \sum_{k=0}^{\infty} p_k s^k$: Prob. gen. fn.

If η is the extinction prob. then η satisfies

$$\eta = G(\eta) \quad \text{-- } (*)$$



η = smallest non-ng sol. of (*)

A survival set, $P(A) = 1 - \eta$. On A, $\dfrac{Z_n}{\mu^n} \to W > 0$

Erdos-Renyi Random Graphs

$G_n(p)$: graph with n vertices with edge between any two pair of points w.p. p independent of all else.

$p = p_n = \frac{c}{n}$.

expected vertex degree $= c$. deg. dist. $\approx$ Poisson(c)
For large n graph is locally tree like.

- If $c < 1$ then graph has only small connected components Largest component of size $O(\log n)$.

: If $c > 1$ there is a giant component covering roughly $(1-\eta)$ fraction of the nodes.

All other components are small; $O(\log n)$ is size of second largest component.

Poisson Process

The Poisson process is the simplest model for a counting process.

eg. # of accidents claims, # of defaults etc.

Def. 1 A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process if

i) $N(0) = 0$

ii) $P[N(t+h) - N(t) = 1] = \lambda h + o(h)$

iii) $P[N(t+h) - N(t) \geq 2] = o(h)$

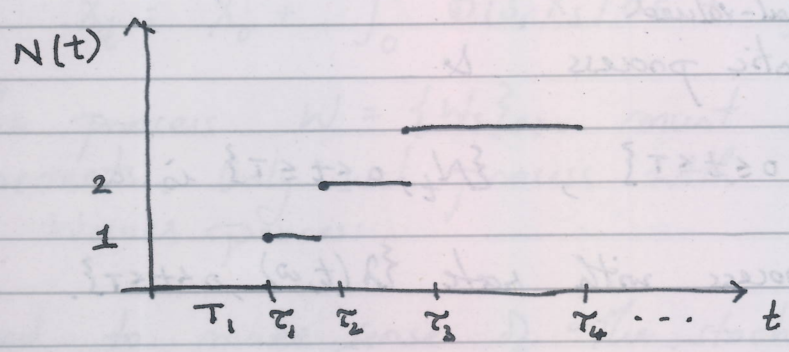
λ - arrival rate.

$\equiv$ **Def. 2** (i) $N(0) = 0$

(ii) stationary & Independent Increments

iii) $P[N(t+s) - N(s) = k] = \dfrac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots$

indep. of s.!!



τ_i = time of i^{th} arrival

$T_i = \tau_i - \tau_{i-1} \sim \exp(\lambda)$. $\quad E[T_i] = \dfrac{1}{\lambda}$.

~~λdon~~ $E[N(t)] = \lambda t$

$\mathcal{F}_s = \sigma(N_u,\ 0 \le u \le s)$.

$E[N(t+s) \mid \mathcal{F}_s] = E[N(t+s) - N(s) + N(s) \mid \mathcal{F}_s]$

$= E[(N(t+s) - N(s) \mid \mathcal{F}_s] + N(s)$

$= \lambda t + N(s) \quad = \lambda(t+s) + (N_s - \lambda s)$

$\Rightarrow M_t = N_t - \lambda t$ is a martingale.

· <u>Non-Homogenous</u> <u>Poisson Process</u>

Arrival rate at time $t = \lambda(t)$

$$P[N(t+s) - N(s) = k] = e^{-\int_s^{t+s} \lambda(u)\,du} \frac{\left(\int_s^{t+s} \lambda(u)\,du\right)^k}{k!}$$

$$N(t+s) - N(s) \sim \mathrm{Poi}\left(\int_s^{t+s} \lambda(u)\,du\right).$$

· Cox process or doubly stochastic Poisson process

λ itself is a positive real-valued stochastic process &

given a path $\{\lambda(t,\omega),\ 0 \le t \le T\}$, $\{N_t,\ 0 \le t \le T\}$ is a non-homogeneous Poisson process with rate $\{\lambda(t,\omega),\ 0 \le t \le T\}$.

Brownian Motion. & Stochastic Calculus.

<u>Motivation</u>:- In many applications we need to consid noisy versions of differential equations, for example, the evolution of the price of a stock.

Need a mathematical interpretation of a equation such as

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{"noise"} \qquad -(1)$$

Would like the noise process η_t to satisfy

i) η_{t_1} & η_{t_2} independent. if $t_1 \neq t_2$

ii) η_t - stationary (distribution remains same)

iii) $E[\eta_t] = 0$.

No such process!! So we write (1) as

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \text{"}\int_0^t \sigma(s, X_s)\, dW_s\text{"} \qquad -(2)$$

The process $W = \{W_s\}_{s \geq 0}$ must have stationary, independent increments. Only such process with continuous paths is the BM or Wiener process.

Need to make sense of the stochastic integral in (2).

Brownian Motion or Wiener Process

$W = \{W_t, t \geq 0\}$ is a real-valued process satisfying

i) $W_0 = 0$

ii) $E[W_t] = 0$, $E[W_t W_s] = s \wedge t = \min\{s, t\}$.

iii) W is Gaussian, i.e distribution of $(W_{t_1}, \dots, W_{t_k})$ is a multivariate normal.

iv) W has continuous paths.

Remarks:- i) $W_t \sim N(0, t)$; ii) $W_t - W_s \sim N(0, t-s)$ - stationary increments.

iii) Let $0 \leq s < t \leq u < v$

$$E[(W_t - W_s)(W_v - W_u)] = 0$$

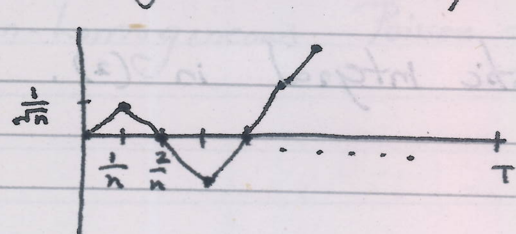
$\Rightarrow$ W has independent increments.

Does such a process exist?

BM as limit of scaled RWs (Donsker's Invariance)

Let $\{\xi_k\}_{k \geq 1}$ be a seq. of iid r.vs with $P(\xi_k = +1) = P(\xi_k = -1) = \frac{1}{2}$.

$$X_t^{(n)} = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{\sqrt{n}} \sum_{k=1}^{nt} \xi_k & \text{if } t = \frac{1}{n}, \frac{2}{n}, \dots, \lfloor nT \rfloor \\ \text{by linear interpolation for other values of } t \end{cases}$$



The seq. of processes $\{X_t^{(n)}\}_{t \in [0,T]}$ converges in "distribution" to a process that satisfies all the conditions in def. of the BM.

Recall: We wish to define $\int_0^t \sigma(s, X_s)\, dW_s$.

Problem 1: The BM has paths of unbounded variation. So cannot define

$$\int_0^t \cancel{\sigma(s,X_s)} f(s,\omega)\, dW_s = \lim_{n\to\infty} \sum_{j=1}^{k(n)} f(t_j^{(n)}, \omega)(W_{t_{j+1}} - W_{t_j})$$

Limit has to be taken in the $\mathcal{L}^2$-sense.

$t_1^{(n)} \; \cdots\cdots \; t_{k(n)}^{(n)}$

$\max |t_j^{(n)} - t_{j-1}^{(n)}| \to 0$.

Problem 2:- It matters whether we take in approx. $f(t_j^{(n)}, \omega)$ or $f(t_{j+1}^{(n)}, \omega)$ or something in between.

We will take $f(t_j^{(n)}, \omega)$ since this will be $\mathcal{F}_{t_j}$ measurable & as a consequence the integral will be $\mathcal{F}_t$-measurable, if $f(t,\omega)$ is an adapted process.

Adapted: ex. $\{W_t\}$, $\{W_{t/2}\}$ are adapted with respect to $\mathcal{F}_t = \sigma(W_s, 0 \le s \le t)$ but not $\{W_{2t}\}$.

The above integral can be defined for all adapted processes $f(t,\omega)$ satisfying

$$E\left[\int_0^T f^2(t,\omega)\, dt\right] < \infty.$$

Properties:- i) Can be realised as a cont. process in t, i.e

$$I_t(f) = \int_0^t f(s,\omega)\, dW_s \quad \text{continuous in } t.$$

ii) $\int_0^T f\, dW = \int_0^S f\, dW + \int_S^T f\, dW$

iii) $\int (cf+g)\, dW = c\int f\, dW + \int g\, dW$

iv) $E\left[\int_0^T f\, dW\right] = 0$

... $\mathcal{F}_T$ - measurable.) In fact $M_t = \int_0^t f\, dW$ is

Ito Formula & Applications.

Ito process : $X_t = X_0 + \int_0^t u(s,\omega)\,ds + \int_0^t v(s,\omega)\,dW_s$. — (1)

u, v adapted "nice".

We will write (1) as $dX_t = u\,dt + v\,dW_t$.

Ito Formula:- Let $Y_t = g(t, X_t)$.

$$Y_t = g(0, X_0) + \int_0^t \frac{\partial}{\partial t} g(s, X_s)\,ds + \int_0^t \frac{\partial g(s,X_s)}{\partial x}\,dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s, X_s)\,v^2\,ds$$

$$dY_t = \frac{\partial g(t,X_t)}{\partial t}\,dt + \frac{\partial g(t,X_t)}{\partial x}\,dX_t + \underbrace{\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot v^2\,dt}_{\text{Ito correction term.}}$$

$v^2 dt$ "=" $dX_t\, dX_t$

$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0$; $dW_t \cdot dW_t = dt$

$$Y_t = g(0, X_0) + \int_0^t \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}u + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}v^2\right) ds + \int_0^t \frac{\partial g}{\partial x}\,dW.$$

Ex. 1 Evaluate $\int_0^t W_t\,dW_t$.

$X = W \quad (u = 0,\ v = 1) \qquad g(t,x) = \frac{1}{2}x^2$

$$\frac{1}{2}W_t^2 = 0 + \int_0^t \left(0 + 0 + \tfrac{1}{2}\right) ds + \int_0^t W_s\,dW_s$$

$$\Rightarrow \int_0^t W_s\,dW_s = \frac{1}{2}\left(W_t^2 - t\right).$$

Ex. 2 $\quad Y_t = \cancel{e^{ct+\frac{1}{2}\sigma^2}}\, Y_0 \exp(ct + \sigma W_t)$

$X = W$; $\quad g(t,x) = \exp(ct + \sigma x)$

$$\frac{\partial g}{\partial t} = c\, g(t,x)\,; \quad \frac{\partial g}{\partial x} = \sigma g(t,x) \quad \frac{\partial^2 g}{\partial x^2} = \sigma^2 g(t,x).$$

$$Y_t = g(t, W_t) = Y_0 + \int_0^t c\, Y_s\, ds + \int_0^t \sigma Y_s\, dW_s + \frac{1}{2}\int_0^t \sigma^2 Y_s\, ds.$$

$$dY_t = \left(c + \tfrac{1}{2}\sigma^2\right) Y_t\, dt + \sigma Y_t\, dW_t.$$

(i) $\quad c = -\frac{1}{2}\sigma^2 \Rightarrow dY_t = \sigma Y_t\, dW_t$

$\Rightarrow Y_t = Y_0 \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$ is a martingale

ii) $\quad c = \cancel{\mu} \;\mu - \frac{1}{2}\sigma^2$

$$\frac{dY_t}{Y_t} = \mu\, dt + \sigma\, dW_t$$

$Y_t = Y_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma dW_t\right)$: Geometric BM

- model for stock price.

<u>Girsanov's Theorem</u>:- Suppose $\{W_t\}_{t\ge0}$ is a P-BM.

$\mathcal{F}_t = \sigma(W_s,\ 0\le s\le t)$ $\quad \{\theta_t\}_{t\ge0}$ - adapted process.

$$L_t = \exp\left(-\int_0^t \theta_s\, dW_s - \frac{1}{2}\int_0^t \theta_s^2\, ds\right), \quad 0\le t\le T.$$

- martingale if $E\left[\exp\left(\frac{1}{2}\int_0^t \theta_s^2 ds\right)\right] < \infty$.

Let $\tilde{W}_t = W_t + \int_0^t \theta_s\, ds$ & define

$$Q(A) = \int_A L_T(\omega)\, \mathbb{P}(d\omega). \quad \left(Q(d\omega) = L_T(\omega)\,\mathbb{P}(d\omega)\right)$$

Then $\tilde{W}_t$ is a Q - BM.

<u>Application</u>:- Let $\{W_t\}$ be a $\mathbb{P}$-BM

$(X_t := \mu t + \sigma W_t$: BM with drift μ & vol. σ^2.)

$$S_t = S_0 \exp\left((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right)$$

- stock price process under $\mathbb{P}$. $\quad$ r-short rate.

$$S_t = S_0 \exp\left((r-\tfrac{1}{2}\sigma^2)t + \sigma\left(W_t + \frac{\mu - r}{\sigma}t\right)\right)$$

$$= S_0 \exp\left(rt + \left(\sigma\tilde{W}_t - \tfrac{1}{2}\sigma^2 t\right)\right) \quad - \quad (1)$$

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma}t$$

So if we take $\theta_s = \frac{\mu - r}{\sigma}$, i.e $L_t = e^{-\frac{\mu-r}{\sigma}W_t - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)t}$

then under Q $\tilde{W}_t$ is a BM. Hence from (1)

$$e^{-rt}S_t = S_0 \exp\left(\cancel{rt} + \sigma\tilde{W}_t - \tfrac{1}{2}\sigma^2 t\right) \quad \text{is a}$$

martingale.

In other words, if we set $\tilde{S}_t = e^{-rt} S_t$, the discounted stock price, it satisfies the SDE

$$d\tilde{S}_t = \sigma \tilde{S}_t \, d\tilde{W}_t \qquad \text{under } \mathbb{Q}.$$

OR $\quad dS_t = r S_t \, dt + \sigma S_t \, d\tilde{W}_t \quad$ —"— .

Ex. European Call Option : $C_T(S_T, K) = (S_T - K)^+$ ← strike price.

Price of option at time t:

$$C_t(x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} (S_T - K)^+ \,\middle|\, S_t = x \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} C_T(S_T) \,\middle|\, S_t = x \right].$$

Apply Ito's formula to $e^{-rT} C_T(S_T)$:

$$e^{-rT} C_T(S_T) = e^{-rt} C_t(S_t) + \int_t^T -r e^{-ru} C_u(S_u) \, du + \int_t^T e^{-ru} \frac{\partial}{\partial u} C_u(S_u) \, du$$

$$+ \int_t^T e^{-ru} \frac{\partial}{\partial x} C_u(S_u) \, dS_u + \frac{1}{2} \int_t^T e^{-ru} \frac{\partial^2}{\partial x^2} C_u(S_u) \, \sigma^2 S_u^2 \, du$$

$$= e^{-rt} C_t(S_t) + \int_t^T e^{-ru} \left[\frac{\partial}{\partial u} C_u(S_u) - r C_u(S_u) + r S_u \frac{\partial}{\partial x} C_u(S_u) + \frac{1}{2} \sigma^2 S_u^2 \frac{\partial^2}{\partial x^2} C_u(S_u) \right] du$$

$$+ \int_t^T e^{-ru} \frac{\partial}{\partial x} C_u(S_u) \cdot \sigma S_u \, d\tilde{W}_u$$

Let ~~$C_t(x)$~~ $C_u(x)$ be ~~a~~ the solution of the PDE

$$\frac{\partial}{\partial u} C_u(x) - r C_u(x) + r x \frac{\partial}{\partial x} C_u(x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} C_u(x) = 0, \quad 0 \le t \le T$$

Then, $e^{-rT} C_T(S_T) = e^{-rt} C_t(S_t) + \frac{1}{2}\int_t^T e^{-ru} \frac{\partial}{\partial x} C_u(S_u)\, \sigma S_u \, d\tilde{W}_u$

Take expectations on both sides given $S_t = x$.

$$E^{Q}\left[e^{-rT} C_T(S_T) \mid S_t = x\right] = e^{-rt} C_t(x) + 0$$

$$\Rightarrow \quad C_t(x) = E^{Q}\left[e^{-r(T-t)} (S_T - K)^+\right]$$

is the price of the option, where $C_t(x)$ ~~sol~~ is the sol of the Black-Scholes PDE

<u>Ex.</u> Mean Reversing or Ornstein-Uhlenbeck process.

$$dX_t = \alpha(m - X_t)dt + \sigma dW_t \quad \text{(Vasichek model for interest rates)}$$

Apply Ito's formula to $e^{+\alpha t} X_t$ $\quad (g(t,x) = e^{\alpha t} x)$

$$d(e^{+\alpha t} X_t) = \alpha e^{\alpha t} X_t \, dt + e^{\alpha t} dX_t + 0$$

$$= \alpha e^{\alpha t} X_t \, dt + e^{\alpha t} \cdot \alpha(m - X_t)\, dt + \sigma e^{\alpha t} dW_t$$

$$= \alpha m e^{\alpha t} dt + \sigma e^{\alpha t} dW_t$$

$$X_t = e^{-\alpha t} x_0 + \alpha m \int_0^t e^{-\alpha(t-s)} ds + \sigma \int_0^t e^{-\alpha(t-s)} dW_s$$

$$E\left[e^{\theta X_t}\right] = \cancel{e^{-\alpha t}} \exp\left(\left(e^{-\alpha t} x_0 + m(1 - e^{-\alpha t})\right)\theta\right) E\left[e^{\theta\sigma \int_0^t e^{-\alpha(t-s)} dW_s - \frac{1}{2}\sigma^2\theta^2 \int_0^t e^{-2\alpha(t-s)} ds}\right]$$

$$\cdot \exp\left(\frac{1}{4}\frac{\sigma^2\theta^2}{\alpha}\right)$$

$$\to \exp\left(m\theta + \frac{\sigma^2\theta^2}{4\alpha}\right) \quad \text{as } t \to \infty$$

$\to$ (i.e. the dist of X is $N(m, \frac{\sigma^2}{\ldots})$

$$Y_s = \exp\Big(\underbrace{c\int_0^s e^{\alpha u}\,dW_u - \frac{1}{2}c^2\int_0^s e^{2\alpha u}\,du}_{Z}\Big) \quad , 0 \le s \le t$$

$$dZ_s = c e^{\alpha s}\,dW_s - \frac{1}{2}c^2 e^{2\alpha s}\,ds$$

$$g(x) = e^{x}$$

$$Y_s = 1 + \int_0^s e^{Z_u}\,dZ_u + \frac{1}{2}\int_0^s e^{Z_u}\,c^2 e^{2\alpha u}\,du$$

$$= 1 + \int_0^s c e^{\alpha u} e^{Z_u}\,dW_u \quad : \text{ martingale.}$$

$$EY_t = EY_0 = 1.$$