Covering radius and mod-2 homology of hyperbolic manifolds

Peter Shalen

December 18, 2012

Covering radius and diameter

M a closed, orientable hyperbolic 3-manifold

Covering radius and diameter

M a closed, orientable hyperbolic 3-manifold

$$\operatorname{diam} M = \max_{P,Q \in M} \operatorname{dist}(P, Q)$$

Covering radius and diameter

M a closed, orientable hyperbolic 3-manifold

$$\operatorname{diam} M = \max_{P,Q \in M} \operatorname{dist}(P, Q)$$

For $P \in M$,

$$\operatorname{cov}_P M = \max_{Q \in M} \operatorname{dist}(P, Q)$$

Covering radius and diameter

M a closed, orientable hyperbolic 3-manifold

$$\operatorname{diam} M = \max_{P,Q \in M} \operatorname{dist}(P, Q)$$

For $P \in M$,

$$\operatorname{cov}_P M = \max_{Q \in M} \operatorname{dist}(P, Q)$$

So

$$\operatorname{diam} M = \max_{P \in M} \operatorname{cov}_P M.$$

Covering radius and diameter

M a closed, orientable hyperbolic 3-manifold

$$\operatorname{diam} M = \max_{P,Q \in M} \operatorname{dist}(P, Q)$$

For $P \in M$,

$$\operatorname{cov}_P M = \max_{Q \in M} \operatorname{dist}(P, Q)$$

So

$$\operatorname{diam} M = \max_{P \in M} \operatorname{cov}_P M.$$

In particular, a lower bound for $\min_{P \in M} \operatorname{cov}_P M$ is a lower bound for $\operatorname{diam} M$.

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\mathrm{thick}}(\epsilon) = \{x \in M : \mathrm{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\text{thick}}(\epsilon) = \{x \in M : \text{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\text{thick}}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$.

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\mathrm{thick}}(\epsilon) = \{x \in M : \mathrm{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\mathrm{thick}}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$. In particular, $\pi_1(M_{\mathrm{thick}}(\epsilon)) \to \pi_1(M)$ is surjective.

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\mathrm{thick}}(\epsilon) = \{x \in M : \mathrm{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\mathrm{thick}}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$. In particular, $\pi_1(M_{\mathrm{thick}}(\epsilon)) \to \pi_1(M)$ is surjective.

Let S be a maximal set of points separated pairwise by distance $\geq \epsilon/4$.

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\rm thick}(\epsilon) = \{x \in M : {\rm inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\rm thick}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$. In particular, $\pi_1(M_{\rm thick}(\epsilon)) \to \pi_1(M)$ is surjective.

Let S be a maximal set of points separated pairwise by distance $\geq \epsilon/4$. Elementary arguments show the balls of radius $\epsilon/4$ about points of $M_{\rm thick} = M_{\rm thick}(\epsilon)$ cover $M_{\rm thick}(\epsilon)$,

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\mathrm{thick}}(\epsilon) = \{x \in M : \mathrm{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\mathrm{thick}}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$. In particular, $\pi_1(M_{\mathrm{thick}}(\epsilon)) \to \pi_1(M)$ is surjective.

Let S be a maximal set of points separated pairwise by distance $\geq \epsilon/4$. Elementary arguments show the balls of radius $\epsilon/4$ about points of $M_{\mathrm{thick}} = M_{\mathrm{thick}}(\epsilon)$ cover $M_{\mathrm{thick}}(\epsilon)$, are contractible,

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\mathrm{thick}}(\epsilon) = \{x \in M : \mathrm{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\mathrm{thick}}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$. In particular, $\pi_1(M_{\mathrm{thick}}(\epsilon)) \to \pi_1(M)$ is surjective.

Let S be a maximal set of points separated pairwise by distance $\geq \epsilon/4$. Elementary arguments show the balls of radius $\epsilon/4$ about points of $M_{\mathrm{thick}} = M_{\mathrm{thick}}(\epsilon)$ cover $M_{\mathrm{thick}}(\epsilon)$, are contractible, and all their finite intersections are contractible.

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\mathrm{thick}}(\epsilon) = \{x \in M : \mathrm{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\mathrm{thick}}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$. In particular, $\pi_1(M_{\mathrm{thick}}(\epsilon)) \to \pi_1(M)$ is surjective.

Let S be a maximal set of points separated pairwise by distance $\geq \epsilon/4$. Elementary arguments show the balls of radius $\epsilon/4$ about points of $M_{\mathrm{thick}} = M_{\mathrm{thick}}(\epsilon)$ cover $M_{\mathrm{thick}}(\epsilon)$, are contractible, and all their finite intersections are contractible. So Leray $\Rightarrow$ nerve K of the covering

$$\{\underset{\epsilon/4}{\mathrm{ball}}(x) : x \in S\}$$

is homotopy-equivalent to $M_{\mathrm{thick}}(\epsilon)$.

Covering radius and rank of π_1

For any $\epsilon > 0$, and for M^n hyperbolic, define

$$M_{\mathrm{thick}}(\epsilon) = \{x \in M : \mathrm{inj}_x M \geq \frac{\epsilon}{2}\}.$$

Work of Zassenhaus, Margulis, Thurston shows there is a constant $\epsilon = \epsilon(n) > 0$ such that for any closed, orientable, hyperbolic M^n, each component of $\overline{M - M_{\mathrm{thick}}(\epsilon)}$ is diffeomorphic to $S^1 \times D^{n-1}$. In particular, $\pi_1(M_{\mathrm{thick}}(\epsilon)) \to \pi_1(M)$ is surjective.

Let S be a maximal set of points separated pairwise by distance $\geq \epsilon/4$. Elementary arguments show the balls of radius $\epsilon/4$ about points of $M_{\mathrm{thick}} = M_{\mathrm{thick}}(\epsilon)$ cover $M_{\mathrm{thick}}(\epsilon)$, are contractible, and all their finite intersections are contractible. So Leray $\Rightarrow$ nerve K of the covering

$$\{\underset{\epsilon/4}{\mathrm{ball}}(x) : x \in S\}$$

is homotopy-equivalent to $M_{\mathrm{thick}}(\epsilon)$.

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint,

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\operatorname{Vol} M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\mathrm{Vol}\, M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

where $\beta_n(r)$ is the volume of a ball of radius r in $\mathbb{H}^n$.

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\operatorname{Vol} M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

where $\beta_n(r)$ is the volume of a ball of radius r in $\mathbb{H}^n$. It grows like constant$\cdot e^{(n-1)r}$.

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\mathrm{Vol}\, M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

where $\beta_n(r)$ is the volume of a ball of radius r in $\mathbb{H}^n$. It grows like constant$\cdot e^{(n-1)r}$. (For example, $\beta_3(r) = \pi(\sinh(2r) - 2r)$.)

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\mathrm{Vol}\, M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

where $\beta_n(r)$ is the volume of a ball of radius r in $\mathbb{H}^n$. It grows like constant$\cdot e^{(n-1)r}$. (For example, $\beta_3(r) = \pi(\sinh(2r) - 2r)$.)

So

$$\#(S) \leq \frac{\mathrm{Vol}\, M}{\beta_n(\epsilon/8)}.$$

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\operatorname{Vol} M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

where $\beta_n(r)$ is the volume of a ball of radius r in $\mathbb{H}^n$. It grows like constant$\cdot e^{(n-1)r}$. (For example, $\beta_3(r) = \pi(\sinh(2r) - 2r)$.)

So

$$\#(S) \leq \frac{\operatorname{Vol} M}{\beta_n(\epsilon/8)}.$$

So the number of vertices of K is bounded linearly by $\operatorname{Vol} M$.

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\mathrm{Vol}\, M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

where $\beta_n(r)$ is the volume of a ball of radius r in $\mathbb{H}^n$. It grows like constant$\cdot e^{(n-1)r}$. (For example, $\beta_3(r) = \pi(\sinh(2r) - 2r)$.)

So

$$\#(S) \leq \frac{\mathrm{Vol}\, M}{\beta_n(\epsilon/8)}.$$

So the number of vertices of K is bounded linearly by $\mathrm{Vol}\, M$. It's also possible to bound the order of the link of a vertex in the 1-skeleton in terms of n (and $\epsilon = \epsilon(n)$).

Covering radius and rank of π_1, cont'd

On the other hand, the balls of radius $\epsilon/8$ about the points of S are pairwise disjoint, so

$$\mathrm{Vol}\, M \geq \#(S) \cdot \beta_n(\epsilon/8)$$

where $\beta_n(r)$ is the volume of a ball of radius r in $\mathbb{H}^n$. It grows like constant$\cdot e^{(n-1)r}$. (For example, $\beta_3(r) = \pi(\sinh(2r) - 2r)$.)

So

$$\#(S) \leq \frac{\mathrm{Vol}\, M}{\beta_n(\epsilon/8)}.$$

So the number of vertices of K is bounded linearly by $\mathrm{Vol}\, M$. It's also possible to bound the order of the link of a vertex in the 1-skeleton in terms of n (and $\epsilon = \epsilon(n)$). So we get a linear bound on the rank of $\pi_1(M)$ in terms of $\mathrm{Vol}\, M$.

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\mathrm{Vol}\, M \leq \beta_n(\min_{P \in M} \mathrm{cov}_P M).$$

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\mathrm{Vol}\, M \leq \beta_n(\min_{P \in M} \mathrm{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \mathrm{cov}_P(M)$,

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\mathrm{Vol}\, M \leq \beta_n(\min_{P \in M} \mathrm{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \mathrm{cov}_P(M)$, and let $\pi : \mathbb{H}^n \to M$ be a locally isometric covering map.

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\mathrm{Vol}\, M \leq \beta_n(\min_{P \in M} \mathrm{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \mathrm{cov}_P(M)$, and let $\pi : \mathbb{H}^n \to M$ be a locally isometric covering map.

Choose $P \in M$ with $\mathrm{cov}_P(M) = R$;

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\operatorname{Vol} M \leq \beta_n(\min_{P \in M} \operatorname{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \operatorname{cov}_P(M)$, and let $\pi : \mathbb{H}^n \to M$ be a locally isometric covering map.

Choose $P \in M$ with $\operatorname{cov}_P(M) = R$; choose $p \in \pi^{-1}(P) \subset \mathbb{H}^3$.

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\operatorname{Vol} M \leq \beta_n(\min_{P \in M} \operatorname{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \operatorname{cov}_P(M)$, and let $\pi : \mathbb{H}^n \to M$ be a locally isometric covering map.

Choose $P \in M$ with $\operatorname{cov}_P(M) = R$; choose $p \in \pi^{-1}(P) \subset \mathbb{H}^3$.

Set $B =$ closed ball of radius R centered at p.

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\mathrm{Vol}\, M \leq \beta_n(\min_{P \in M} \mathrm{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \mathrm{cov}_P(M)$, and let $\pi : \mathbb{H}^n \to M$ be a locally isometric covering map.

Choose $P \in M$ with $\mathrm{cov}_P(M) = R$; choose $p \in \pi^{-1}(P) \subset \mathbb{H}^3$.

Set $B =$ closed ball of radius R centered at p. Then π maps B onto M.

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\operatorname{Vol} M \leq \beta_n(\min_{P \in M} \operatorname{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \operatorname{cov}_P(M)$, and let $\pi : \mathbb{H}^n \to M$ be a locally isometric covering map.

Choose $P \in M$ with $\operatorname{cov}_P(M) = R$; choose $p \in \pi^{-1}(P) \subset \mathbb{H}^3$.

Set $B =$ closed ball of radius R centered at p. Then π maps B onto M.

So

$$\operatorname{Vol} M \leq \operatorname{Vol} B = \beta_n(R),$$

and the proposition is proved.

Covering radius and rank of π_1, cont'd

The volume can in turn be related to the minimum covering radius:

Proposition

If M is a closed hyperbolic n-manifold, we have

$$\mathrm{Vol}\, M \leq \beta_n(\min_{P \in M} \mathrm{cov}_P M).$$

To prove this, set $R = \min_{P \in M} \mathrm{cov}_P(M)$, and let $\pi : \mathbb{H}^n \to M$ be a locally isometric covering map.

Choose $P \in M$ with $\mathrm{cov}_P(M) = R$; choose $p \in \pi^{-1}(P) \subset \mathbb{H}^3$.

Set $B =$ closed ball of radius R centered at p. Then π maps B onto M.

So

$$\mathrm{Vol}\, M \leq \mathrm{Vol}\, B = \beta_n(R),$$

and the proposition is proved.

Covering radius and rank of π_1, cont'd

Since $\beta_n(R)$ grows like constant$\cdot e^{(n-1)R}$, it follows that

$$\operatorname{rank} \pi_1(M) \leq C_n \exp((n-1)R),$$

where $R = \min_{P \in M} \operatorname{cov}_P M$, and C_n is a constant for each dimension n.

Covering radius and rank of π_1, cont'd

Since $\beta_n(R)$ grows like constant$\cdot e^{(n-1)R}$, it follows that

$$\operatorname{rank} \pi_1(M) \leq C_n \exp((n-1)R),$$

where $R = \min_{P \in M} \operatorname{cov}_P M$, and C_n is a constant for each dimension n.

This result implies that the lim sup of the quantity $(\operatorname{rank} \pi_1(M)) \exp(-(n-1)R)$, as M varies over the closed hyperbolic manifolds of a fixed dimension n, is finite. The result appears to be "qualitatively sharp" in the sense that this lim sup is strictly positive for any n.

Covering radius and rank of π_1, cont'd

Since $\beta_n(R)$ grows like constant$\cdot e^{(n-1)R}$, it follows that

$$\operatorname{rank} \pi_1(M) \leq C_n \exp((n-1)R),$$

where $R = \min_{P \in M} \operatorname{cov}_P M$, and C_n is a constant for each dimension n.

This result implies that the lim sup of the quantity $(\operatorname{rank} \pi_1(M)) \exp(-(n-1)R)$, as M varies over the closed hyperbolic manifolds of a fixed dimension n, is finite. The result appears to be "qualitatively sharp" in the sense that this lim sup is strictly positive for any n.

For $n = 3$, using Meyerhoff's explicit Margulis constant $\epsilon = 0.104$, one can show that

$$\operatorname{rank} \pi_1(M) \leq C \exp(2R),$$

where C is about 10^6.

Covering radius and rank of π_1, cont'd

Since $\beta_n(R)$ grows like constant$\cdot e^{(n-1)R}$, it follows that

$$\operatorname{rank}\pi_1(M) \leq C_n \exp((n-1)R),$$

where $R = \min_{P\in M} \operatorname{cov}_P M$, and C_n is a constant for each dimension n.

This result implies that the lim sup of the quantity $(\operatorname{rank}\pi_1(M))\exp(-(n-1)R)$, as M varies over the closed hyperbolic manifolds of a fixed dimension n, is finite. The result appears to be "qualitatively sharp" in the sense that this lim sup is strictly positive for any n.

For $n = 3$, using Meyerhoff's explicit Margulis constant $\epsilon = 0.104$, one can show that

$$\operatorname{rank}\pi_1(M) \leq C\exp(2R),$$

where C is about 10^6.

Covering radius and rank of π_1, cont'd

If M^3 is orientable, and if $H_1(M;\mathbb{Z}_p)$ has rank at least 4 for some prime p (or more generally if $\pi_1(M)$ has no two-generator subgroup of finite index), one can use the value $\epsilon = \log 3$ by the $\log 3$ theorem, and obtain

$$\mathrm{rank}\,\pi_1(M) \leq C' \exp(2R),$$

where C' is about 10^3.

Covering radius and rank of π_1, cont'd

If M^3 is orientable, and if $H_1(M;\mathbb{Z}_p)$ has rank at least 4 for some prime p (or more generally if $\pi_1(M)$ has no two-generator subgroup of finite index), one can use the value $\epsilon = \log 3$ by the $\log 3$ theorem, and obtain

$$\mathrm{rank}\,\pi_1(M) \leq C' \exp(2R),$$

where C' is about 10^3.

In particular this implies that for any closed, orientable hyperbolic 3-manifold M we have

$$\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \leq C' \exp(2R),$$

where C' is about 10^3.

Covering radius and rank of π_1, cont'd

If M^3 is orientable, and if $H_1(M;\mathbb{Z}_p)$ has rank at least 4 for some prime p (or more generally if $\pi_1(M)$ has no two-generator subgroup of finite index), one can use the value $\epsilon = \log 3$ by the $\log 3$ theorem, and obtain

$$\mathrm{rank}\,\pi_1(M) \leq C' \exp(2R),$$

where C' is about 10^3.

In particular this implies that for any closed, orientable hyperbolic 3-manifold M we have

$$\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \leq C' \exp(2R),$$

where C' is about 10^3.

The main result I'll be discussing is an improvement of this, and uses some difficult topology.

The main geometrical result

Recall that $V_8 = 3.66\ldots$ denotes the volume of a regular ideal hyperbolic octahedron in $\mathbb{H}^3$.

The main geometrical result

Recall that $V_8 = 3.66\ldots$ denotes the volume of a regular ideal hyperbolic octahedron in $\mathbb{H}^3$.

Geometrical Theorem

Let M be a closed, orientable hyperbolic 3-manifold, and let R denote the minimum covering radius of M. Then

$$\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \leq B\, e^{2R},$$

The main geometrical result

Recall that $V_8 = 3.66\ldots$ denotes the volume of a regular ideal hyperbolic octahedron in $\mathbb{H}^3$.

Geometrical Theorem

Let M be a closed, orientable hyperbolic 3-manifold, and let R denote the minimum covering radius of M. Then

$$\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \le B\, e^{2R},$$

where

$$B = \frac{94\pi}{V_8} + 27 = 107.600\ldots.$$

Some topological definitions

The geometrical theorem stated above will be proved using a purely topological result.

Some topological definitions

The geometrical theorem stated above will be proved using a purely topological result. The following definition is needed for the statement of the topological result.

Some topological definitions

The geometrical theorem stated above will be proved using a purely topological result. The following definition is needed for the statement of the topological result.

Let G be a finitely generated group. I will say that elements $x_1, \ldots, x_k$ of G are *independent* if $x_1, \ldots, x_k$ freely generate a free subgroup of G.

Some topological definitions

The geometrical theorem stated above will be proved using a purely topological result. The following definition is needed for the statement of the topological result.

Let G be a finitely generated group. I will say that elements $x_1, \ldots, x_k$ of G are *independent* if $x_1, \ldots, x_k$ freely generate a free subgroup of G. If S is a finite generating set for G, I will define the *index of freedom* of S, denoted $\mathrm{I}_f(S)$, to be the largest integer k such that S contains k independent elements.

Some topological definitions

The geometrical theorem stated above will be proved using a purely topological result. The following definition is needed for the statement of the topological result.

Let G be a finitely generated group. I will say that elements $x_1, \ldots, x_k$ of G are *independent* if $x_1, \ldots, x_k$ freely generate a free subgroup of G. If S is a finite generating set for G, I will define the *index of freedom* of S, denoted $\mathrm{I}_f(S)$, to be the largest integer k such that S contains k independent elements. I will define the *index of freedom* of G, denoted $\mathrm{I}_f(G)$, by

$$\mathrm{I}_f(G) = \min_S \mathrm{I}_f(S),$$

where S ranges over all finite generating sets for G.

Some topological definitions

The geometrical theorem stated above will be proved using a purely topological result. The following definition is needed for the statement of the topological result.

Let G be a finitely generated group. I will say that elements $x_1, \ldots, x_k$ of G are *independent* if $x_1, \ldots, x_k$ freely generate a free subgroup of G. If S is a finite generating set for G, I will define the *index of freedom* of S, denoted $\mathrm{I}_f(S)$, to be the largest integer k such that S contains k independent elements. I will define the *index of freedom* of G, denoted $\mathrm{I}_f(G)$, by

$$\mathrm{I}_f(G) = \min_S \mathrm{I}_f(S),$$

where S ranges over all finite generating sets for G.

In addition, the statement of the topological result uses the notations $\overline{\chi}(X)$ and $\mathrm{kish}(M, F)$, which were defined in my second talk.

The main topological result

Topological Theorem

Let M be a closed, orientable hyperbolic 3-manifold. Then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that

$$188 \cdot \overline{\chi}(\mathrm{kish}(M, F)) + 54 \cdot \mathrm{I}_f(\pi_1(M)) \geq \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2).$$

Deducing the geometrical theorem from the topological theorem

The geometrical theorem is proved by combining the topological theorem with the following two results.

Deducing the geometrical theorem from the topological theorem

The geometrical theorem is proved by combining the topological theorem with the following two results.

Proposition A

For any complete, orientable hyperbolic 3-manifold M, we have

$$\min_{x \in M} \operatorname{cov}_x(M) \geq \frac{1}{2} \log(2\mathrm{I}_f(\pi_1(M)) - 1).$$

Deducing the geometrical theorem from the topological theorem

The geometrical theorem is proved by combining the topological theorem with the following two results.

Proposition A

For any complete, orientable hyperbolic 3-manifold M, we have

$$\min_{x \in M} \mathrm{cov}_x(M) \geq \frac{1}{2} \log(2\mathrm{I}_f(\pi_1(M)) - 1).$$

I'll discuss the proof in a moment. This immediately implies:

Deducing the geometrical theorem from the topological theorem

The geometrical theorem is proved by combining the topological theorem with the following two results.

Proposition A

For any complete, orientable hyperbolic 3*-manifold* M*, we have*

$$\min_{x \in M} \mathrm{cov}_x(M) \geq \frac{1}{2} \log(2\mathrm{I}_f(\pi_1(M)) - 1).$$

I'll discuss the proof in a moment. This immediately implies:

Corollary

Suppose that M *is a complete, orientable hyperbolic* 3*-manifold. Set* $R = \min_{x \in M} \mathrm{cov}_x(M)$*. Then*

$$\mathrm{I}_f(\pi_1(M) \leq \frac{1}{2} e^{2R} + 1.$$

Deducing the geometrical theorem from the topological theorem, cont'd

Here is the second result needed to pass from the topological theorem to the geometrical theorem:

Deducing the geometrical theorem from the topological theorem, cont'd

Here is the second result needed to pass from the topological theorem to the geometrical theorem:

Lemma B

Suppose that M is a complete, orientable hyperbolic 3-manifold. Set $R = \min_{x \in M} \operatorname{cov}_x(M)$. Then for any incompressible surface $F \subset M$, we have

$$\overline{\chi}(\operatorname{kish}(M, F)) \leq \frac{\pi}{V_8}(\sinh(2R) - 2R).$$

Proof of Proposition A

Given $M = \mathbb{H}^3/\Gamma$, $P \in M$, $k = \mathrm{I}_f(\pi_1(M))$

Proof of Proposition A

Given $M = \mathbb{H}^3/\Gamma$, $P \in M$, $k = \mathrm{I}_f(\pi_1(M))$

Must show

$$\mathrm{cov}_P(M) \geq \frac{1}{2}\log(2k-1).$$

Proof of Proposition A

Given $M = \mathbb{H}^3/\Gamma$, $P \in M$, $k = \mathrm{I}_f(\pi_1(M))$

Must show

$$\mathrm{cov}_P(M) \geq \frac{1}{2}\log(2k-1).$$

Choose $p \in \mathbb{H}^3$ lying above P

Proof of Proposition A

Given $M = \mathbb{H}^3/\Gamma$, $P \in M$, $k = \mathrm{I}_f(\pi_1(M))$

Must show

$$\mathrm{cov}_P(M) \geq \frac{1}{2}\log(2k-1).$$

Choose $p \in \mathbb{H}^3$ lying above P

$D \subset \mathbb{H}^3 =$ Dirichlet domain centered at p

Proof of Proposition A

Given $M = \mathbb{H}^3/\Gamma$, $P \in M$, $k = \mathrm{I}_f(\pi_1(M))$

Must show
$$\mathrm{cov}_P(M) \geq \frac{1}{2}\log(2k-1).$$

Choose $p \in \mathbb{H}^3$ lying above P

$D \subset \mathbb{H}^3 =$ Dirichlet domain centered at p

$S = \{$ face-pairings of $D\}$ a generating set for Γ

Proof of Proposition A

Given $M = \mathbb{H}^3/\Gamma$, $P \in M$, $k = \mathrm{I}_f(\pi_1(M))$

Must show

$$\mathrm{cov}_P(M) \geq \frac{1}{2}\log(2k-1).$$

Choose $p \in \mathbb{H}^3$ lying above P

$D \subset \mathbb{H}^3 =$ Dirichlet domain centered at p

$S = \{$ face-pairings of $D\}$ a generating set for Γ

So S contains k independent elements $x_1, \ldots, x_k$.

Proof of Proposition A

Given $M = \mathbb{H}^3/\Gamma$, $P \in M$, $k = \mathrm{I}_f(\pi_1(M))$

Must show

$$\mathrm{cov}_P(M) \geq \frac{1}{2}\log(2k-1).$$

Choose $p \in \mathbb{H}^3$ lying above P

$D \subset \mathbb{H}^3 =$ Dirichlet domain centered at p

$S = \{$ face-pairings of $D\}$ a generating set for Γ

So S contains k independent elements $x_1, \ldots, x_k$.

By the $\log(2k-1)$ Theorem we have

$$\mathrm{dist}(p, x_j \cdot p) \geq \log(2k-1) \text{ for some } j \in \{1, \ldots, k\}.$$

Proof of Proposition A, concluded

Set $H =$ plane containing one of the faces of D paired by x_j

Proof of Proposition A, concluded

Set $H =$ plane containing one of the faces of D paired by x_j

Set $q =$ point of intersection of H with the line joining p to x_j

Proof of Proposition A, concluded

Set $H =$ plane containing one of the faces of D paired by x_j

Set $q =$ point of intersection of H with the line joining p to x_j

Set $Q =$ image of q in M

Proof of Proposition A, concluded

Set $H =$ plane containing one of the faces of D paired by x_j

Set $q =$ point of intersection of H with the line joining p to x_j

Set $Q =$ image of q in M

Definition of Dirichlet domain now implies that

$$\mathrm{dist}(P, Q) = \mathrm{dist}(p, q)$$

Proof of Proposition A, concluded

Set H = plane containing one of the faces of D paired by x_j

Set q = point of intersection of H with the line joining p to x_j

Set Q = image of q in M

Definition of Dirichlet domain now implies that

$$\mathrm{dist}(P, Q) = \mathrm{dist}(p, q) = \frac{1}{2}\mathrm{dist}(p, x_j \cdot p)$$

Proof of Proposition A, concluded

Set $H =$ plane containing one of the faces of D paired by x_j

Set $q =$ point of intersection of H with the line joining p to x_j

Set $Q =$ image of q in M

Definition of Dirichlet domain now implies that

$$\mathrm{dist}(P, Q) = \mathrm{dist}(p, q) = \frac{1}{2}\mathrm{dist}(p, x_j \cdot p) \geq \frac{1}{2}\log(2k-1)$$

Proof of Proposition A, concluded

Set $H =$ plane containing one of the faces of D paired by x_j

Set $q =$ point of intersection of H with the line joining p to x_j

Set $Q =$ image of q in M

Definition of Dirichlet domain now implies that

$$\mathrm{dist}(P, Q) = \mathrm{dist}(p, q) = \frac{1}{2}\mathrm{dist}(p, x_j \cdot p) \geq \frac{1}{2}\log(2k-1)$$

and hence

$$\mathrm{cov}_P(M) \geq \frac{1}{2}\log(2k-1).$$

Proof of Lemma B

By the proposition from the beginning of the talk we have

$$\mathrm{Vol}\, M \leq \mathrm{Vol}\, B = \pi(\sinh(2R) - 2R).$$

Proof of Lemma B

By the proposition from the beginning of the talk we have

$$\mathrm{Vol}\, M \leq \mathrm{Vol}\, B = \pi(\sinh(2R) - 2R).$$

But by Agol-Storm-Thurston we have

$$\mathrm{Vol}\, M \geq V_8 \overline{\chi}(\mathrm{kish}(M, F).$$

Proof of Lemma B

By the proposition from the beginning of the talk we have

$$\mathrm{Vol}\, M \leq \mathrm{Vol}\, B = \pi(\sinh(2R) - 2R).$$

But by Agol-Storm-Thurston we have

$$\mathrm{Vol}\, M \geq V_8 \overline{\chi}(\mathrm{kish}(M, F).$$

So

$$V_8 \overline{\chi}(\mathrm{kish}(M, F) \leq \pi(\sinh(2R) - 2R),$$

which gives the conclusion.

Some ingredients in the proof of the topological theorem

Recall the statement:

Topological Theorem

Let M be a closed, orientable hyperbolic 3-manifold. Then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that

$$188 \cdot \overline{\chi}(\mathrm{kish}(M, F)) + 54 \cdot \mathrm{I}_f(\pi_1(M)) \geq \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2).$$

Some ingredients in the proof of the topological theorem

Recall the statement:

Topological Theorem

Let M be a closed, orientable hyperbolic 3-manifold. Then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that

$$188 \cdot \overline{\chi}(\mathrm{kish}(M, F)) + 54 \cdot \mathrm{I}_f(\pi_1(M)) \geq \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2).$$

One ingredient is the following result:

Proposition C

For any compact, orientable, irreducible, atoroidal 3-manifold N, we have $\overline{\chi}(N) < \mathrm{I}_f(\pi_1(N))$.

Some ingredients in the proof of the topological theorem, cont'd

Proposition C has the

Corollary

Let M be a closed, orientable hyperbolic 3-manifold, and let G be a finitely generated subgroup of $\pi_1(M)$. Then $\overline{\chi}(G) < \mathrm{I}_f(G)$.

Some ingredients in the proof of the topological theorem, cont'd

Proposition C has the

Corollary

Let M be a closed, orientable hyperbolic 3-manifold, and let G be a finitely generated subgroup of $\pi_1(M)$. Then $\overline{\chi}(G) < \mathrm{I}_f(G)$.

This Corollary follows from Proposition C via the compact core theorem (Scott-S.):

Some ingredients in the proof of the topological theorem, cont'd

Proposition C has the

Corollary

Let M be a closed, orientable hyperbolic 3-manifold, and let G be a finitely generated subgroup of $\pi_1(M)$. Then $\overline{\chi}(G) < \mathrm{I}_f(G)$.

This Corollary follows from Proposition C via the compact core theorem (Scott-S.): if $\widetilde{M}$ denotes the covering space of M defined by the subgroup G, there is a compact, irreducible submanifold N of $\widetilde{M}$ such that the inclusion homomorphism $\pi_1(N) \to \pi_1(\widetilde{M})$ is an isomorphism.

Some ingredients in the proof of the topological theorem, cont'd

Proposition C has the

Corollary

Let M be a closed, orientable hyperbolic 3-manifold, and let G be a finitely generated subgroup of $\pi_1(M)$. Then $\overline{\chi}(G) < \mathrm{I}_f(G)$.

This Corollary follows from Proposition C via the compact core theorem (Scott-S.): if $\widetilde{M}$ denotes the covering space of M defined by the subgroup G, there is a compact, irreducible submanifold N of $\widetilde{M}$ such that the inclusion homomorphism $\pi_1(N) \to \pi_1(\widetilde{M})$ is an isomorphism. Apply Proposition C to this N.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Set $c = \overline{\chi}(N)$. Suppose $x_1, \ldots, x_m$ generate $G \doteq \pi_1(N)$. Need to show at least $c + 1$ of the x_i are independent.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Set $c = \overline{\chi}(N)$. Suppose $x_1, \ldots, x_m$ generate $G \doteq \pi_1(N)$. Need to show at least $c + 1$ of the x_i are independent. Use induction on N; easy for $N = 1$.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Set $c = \overline{\chi}(N)$. Suppose $x_1, \ldots, x_m$ generate $G \doteq \pi_1(N)$. Need to show at least $c+1$ of the x_i are independent. Use induction on N; easy for $N = 1$. Suppose $m > 1$, set $G' = \langle x_1, \ldots, x_m \rangle$.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Set $c = \overline{\chi}(N)$. Suppose $x_1, \dots, x_m$ generate $G \doteq \pi_1(N)$. Need to show at least $c+1$ of the x_i are independent. Use induction on N; easy for $N = 1$. Suppose $m > 1$, set $G' = \langle x_1, \dots, x_m \rangle$. Compact core theorem $\Rightarrow G' = \pi_1(N')$ for some N'.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Set $c = \overline{\chi}(N)$. Suppose $x_1, \ldots, x_m$ generate $G \doteq \pi_1(N)$. Need to show at least $c + 1$ of the x_i are independent. Use induction on N; easy for $N = 1$. Suppose $m > 1$, set $G' = \langle x_1, \ldots, x_m \rangle$. Compact core theorem $\Rightarrow G' = \pi_1(N')$ for some N'. If G is a free product $G' \star \langle x_m \rangle$ then $\overline{\chi}(N') = c - 1$.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Set $c = \overline{\chi}(N)$. Suppose $x_1, \ldots, x_m$ generate $G \doteq \pi_1(N)$. Need to show at least $c+1$ of the x_i are independent. Use induction on N; easy for $N = 1$. Suppose $m > 1$, set $G' = \langle x_1, \ldots, x_m \rangle$. Compact core theorem $\Rightarrow G' = \pi_1(N')$ for some N'. If G is a free product $G' \star \langle x_m \rangle$ then $\overline{\chi}(N') = c - 1$. Induction hypothesis $\Rightarrow x_1, \ldots, x_c$ independent after re-indexing. So $x_1, \ldots, x_c, x_N$ independent and done.

Some ingredients in the proof of the topological theorem, cont'd

Idea of proof of Proposition C:

The characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(N)$ are identified with points of a complex affine algebraic set $X(N)$, the *character variety* of N. The minimum (complex) dimension of any component of $X(N)$ is $3\chi(N)$.

Set $c = \overline{\chi}(N)$. Suppose $x_1, \ldots, x_m$ generate $G \doteq \pi_1(N)$. Need to show at least $c + 1$ of the x_i are independent. Use induction on N; easy for $N = 1$. Suppose $m > 1$, set $G' = \langle x_1, \ldots, x_m \rangle$. Compact core theorem $\Rightarrow G' = \pi_1(N')$ for some N'. If G is a free product $G' \star \langle x_m \rangle$ then $\overline{\chi}(N') = c - 1$. Induction hypothesis $\Rightarrow x_1, \ldots, x_c$ independent after re-indexing. So $x_1, \ldots, x_c, x_N$ independent and done.

Some ingredients in the proof of the topological theorem, cont'd

If G not a free product, can use a relation to show lowest dimension of a component of $X(N)$ exceeds lowest dimension of a component of $X(N')$ by < 3,

Some ingredients in the proof of the topological theorem, cont'd

If G not a free product, can use a relation to show lowest dimension of a component of $X(N)$ exceeds lowest dimension of a component of $X(N')$ by < 3, so $3c = 3\chi(N) < 3\chi(N') + 3$, hence $\chi(N') \geq c$.

Some ingredients in the proof of the topological theorem, cont'd

If G not a free product, can use a relation to show lowest dimension of a component of $X(N)$ exceeds lowest dimension of a component of $X(N')$ by < 3, so $3c = 3\chi(N) < 3\chi(N') + 3$, hence $\chi(N') \geq c$. Induction hypothesis $\Rightarrow$ at least $c + 1$ of the x_i are independent, so done in this case too.

Some ingredients in the proof of the topological theorem, cont'd

If G not a free product, can use a relation to show lowest dimension of a component of $X(N)$ exceeds lowest dimension of a component of $X(N')$ by < 3, so $3c = 3\chi(N) < 3\chi(N') + 3$, hence $\chi(N') \geq c$. Induction hypothesis $\Rightarrow$ at least $c + 1$ of the x_i are independent, so done in this case too.

The idea of using the character variety for this kind of argument is due to Agol, and seems to give stronger results of this kind than homological arguments used earlier by Jaco-S. and others.

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Here is a very fuzzy sketch of the proof, under a strong simplifying assumption that will emerge in the course of the sketch.

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Here is a very fuzzy sketch of the proof, under a strong simplifying assumption that will emerge in the course of the sketch. Let S be a generating set for $\pi_1(M)$ such that $\mathrm{I}_f(S) = \mathrm{I}_f(\pi_1(M))$.

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Here is a very fuzzy sketch of the proof, under a strong simplifying assumption that will emerge in the course of the sketch. Let S be a generating set for $\pi_1(M)$ such that $\mathrm{I}_f(S) = \mathrm{I}_f(\pi_1(M))$. Set $d = \lfloor n/2 \rfloor$,

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Here is a very fuzzy sketch of the proof, under a strong simplifying assumption that will emerge in the course of the sketch. Let S be a generating set for $\pi_1(M)$ such that $\mathrm{I}_f(S) = \mathrm{I}_f(\pi_1(M))$. Set $d = \lfloor n/2 \rfloor$, choose elements $x^{(1)}, \dots, x^{(d)}$ of S whose images in $V = H_1(M; \mathbb{Z}_2)$ are linearly independent,

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Here is a very fuzzy sketch of the proof, under a strong simplifying assumption that will emerge in the course of the sketch. Let S be a generating set for $\pi_1(M)$ such that $\mathrm{I}_f(S) = \mathrm{I}_f(\pi_1(M))$. Set $d = \lfloor n/2 \rfloor$, choose elements $x^{(1)}, \ldots, x^{(d)}$ of S whose images in $V = H_1(M; \mathbb{Z}_2)$ are linearly independent, and set $G = \langle x^{(1)}, \ldots, x^{(d)} \rangle \leq \pi_1(M)$.

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Here is a very fuzzy sketch of the proof, under a strong simplifying assumption that will emerge in the course of the sketch. Let S be a generating set for $\pi_1(M)$ such that $\mathrm{I}_f(S) = \mathrm{I}_f(\pi_1(M))$. Set $d = \lfloor n/2 \rfloor$, choose elements $x^{(1)}, \ldots, x^{(d)}$ of S whose images in $V = H_1(M; \mathbb{Z}_2)$ are linearly independent, and set $G = \langle x^{(1)}, \ldots, x^{(d)} \rangle \leq \pi_1(M)$. The definitions imply that $\mathrm{I}_f(G)$ is $\leq \mathrm{I}_f(M)$ and is therefore "small" compared with n (and hence compared with d).

Some ingredients in the proof of the topological theorem, cont'd

Roughly speaking, the theorem says that if $n = \dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$ "big" (in a multiplicative sense) in comparison with $\mathrm{I}_f(\pi_1(M))$, then there is a (possibly empty and possibly disconnected) incompressible surface $F \subset M$ such that $\overline{\chi}(\mathrm{kish}(M, F))$ is not "very small" in comparison with $\dim_{\mathbb{Z}_2} H_1(M, \mathbb{Z}_2)$.

Here is a very fuzzy sketch of the proof, under a strong simplifying assumption that will emerge in the course of the sketch. Let S be a generating set for $\pi_1(M)$ such that $\mathrm{I}_f(S) = \mathrm{I}_f(\pi_1(M))$. Set $d = \lfloor n/2 \rfloor$, choose elements $x^{(1)}, \dots, x^{(d)}$ of S whose images in $V = H_1(M; \mathbb{Z}_2)$ are linearly independent, and set $G = \langle x^{(1)}, \dots, x^{(d)} \rangle \leq \pi_1(M)$. The definitions imply that $\mathrm{I}_f(G)$ is $\leq \mathrm{I}_f(M)$ and is therefore "small" compared with n (and hence compared with d). So by Proposition C, the compact core N of the covering corresponding to G has $\overline{\chi}(N)$ very small compared with d.

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map.

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible,

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible, $\overline{\chi}(\mathrm{kish}(M, F_0)) \geq \overline{\chi}(N')$, and $\overline{\chi}(N')$ is small compared with $\mathrm{rank}\, H_1(N'; \mathbb{Z}_2)$.

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible, $\overline{\chi}(\mathrm{kish}(M, F_0)) \geq \overline{\chi}(N')$, and $\overline{\chi}(N')$ is small compared with $\mathrm{rank}\, H_1(N'; \mathbb{Z}_2)$. If $\overline{\chi}(\mathrm{kish}(M, F_0))$ is not very small in comparison with n, take $F = F_0$.

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible, $\overline{\chi}(\mathrm{kish}(M, F_0)) \geq \overline{\chi}(N')$, and $\overline{\chi}(N')$ is small compared with $\mathrm{rank}\, H_1(N'; \mathbb{Z}_2)$. If $\overline{\chi}(\mathrm{kish}(M, F_0))$ is not very small in comparison with n, take $F = F_0$. Now suppose $\overline{\chi}(\mathrm{kish}(M, F_0))$ is very small in comparison with n,

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible, $\overline{\chi}(\mathrm{kish}(M, F_0)) \geq \overline{\chi}(N')$, and $\overline{\chi}(N')$ is small compared with $\mathrm{rank}\, H_1(N'; \mathbb{Z}_2)$. If $\overline{\chi}(\mathrm{kish}(M, F_0))$ is not very small in comparison with n, take $F = F_0$. Now suppose $\overline{\chi}(\mathrm{kish}(M, F_0))$ is very small in comparison with n, and for simplicity suppose F_0 is connected and separates M.

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible, $\overline{\chi}(\mathrm{kish}(M, F_0)) \geq \overline{\chi}(N')$, and $\overline{\chi}(N')$ is small compared with $\mathrm{rank}\, H_1(N'; \mathbb{Z}_2)$. If $\overline{\chi}(\mathrm{kish}(M, F_0))$ is not very small in comparison with n, take $F = F_0$. Now suppose $\overline{\chi}(\mathrm{kish}(M, F_0))$ is very small in comparison with n, and for simplicity suppose F_0 is connected and separates M. Let A and B denote the images of the homology of the components of $M - F_0$ in V.

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible, $\overline{\chi}(\mathrm{kish}(M, F_0)) \geq \overline{\chi}(N')$, and $\overline{\chi}(N')$ is small compared with $\mathrm{rank}\, H_1(N'; \mathbb{Z}_2)$. If $\overline{\chi}(\mathrm{kish}(M, F_0))$ is not very small in comparison with n, take $F = F_0$. Now suppose $\overline{\chi}(\mathrm{kish}(M, F_0))$ is very small in comparison with n, and for simplicity suppose F_0 is connected and separates M. Let A and B denote the images of the homology of the components of $M - F_0$ in V. After doing a little linear algebra, can find an integer d' close to $n/2$ and elements $x_1^{(1)}, \ldots, x_1^{(d')}$ of S whose images in V are linearly independent,

Some ingredients in the proof of the topological theorem, cont'd

ASSUME that N embeds in M via the covering map. Using that $\overline{\chi}(N)$ is small compared with $\mathrm{rank}\, H_1(N; \mathbb{Z}_2)$, can cut N along disks and annuli to get a submanifold N' such that $F_0 = \partial N'$ is incompressible, $\overline{\chi}(\mathrm{kish}(M, F_0)) \geq \overline{\chi}(N')$, and $\overline{\chi}(N')$ is small compared with $\mathrm{rank}\, H_1(N'; \mathbb{Z}_2)$. If $\overline{\chi}(\mathrm{kish}(M, F_0))$ is not very small in comparison with n, take $F = F_0$. Now suppose $\overline{\chi}(\mathrm{kish}(M, F_0))$ is very small in comparison with n, and for simplicity suppose F_0 is connected and separates M. Let A and B denote the images of the homology of the components of $M - F_0$ in V. After doing a little linear algebra, can find an integer d' close to $n/2$ and elements $x_1^{(1)}, \ldots, x_1^{(d')}$ of S whose images in V are linearly independent, and such that the subspace of V spanned by these images meets A and B in subspaces having at most half the dimensions of A and B respectively.

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1 which we again assume embeds in M.

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1 which we again assume embeds in M. Isotop N_1 so ∂N_1 meets F_0 transversally in a minimal number of curves.

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1 which we again assume embeds in M. Isotop N_1 so ∂N_1 meets F_0 transversally in a minimal number of curves. Set $N_1^* = N_1 \setminus X$ where X is a standard neighborhood of F_0.

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1 which we again assume embeds in M. Isotop N_1 so ∂N_1 meets F_0 transversally in a minimal number of curves. Set $N_1^* = N_1 \setminus X$ where X is a standard neighborhood of F_0. (Note N_1^* is disconnected.)

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1 which we again assume embeds in M. Isotop N_1 so ∂N_1 meets F_0 transversally in a minimal number of curves. Set $N_1^* = N_1 \setminus X$ where X is a standard neighborhood of F_0. (Note N_1^* is disconnected.) Repeat cut-and-paste construction with N_1^* in place of N to get N_1', and set $F_1 = F_0 \cup \partial N_1'$.

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1 which we again assume embeds in M. Isotop N_1 so ∂N_1 meets F_0 transversally in a minimal number of curves. Set $N_1^* = N_1 \setminus X$ where X is a standard neighborhood of F_0. (Note N_1^* is disconnected.) Repeat cut-and-paste construction with N_1^* in place of N to get N_1', and set $F_1 = F_0 \cup \partial N_1'$. If $\overline{\chi}(\mathrm{kish}(M, F_1))$ is not very small in comparison with n, take $F = F_0$.

Some ingredients in the proof of the topological theorem, cont'd

Repeat the above construction with the $x_1^{(i)}$ in place of the $x^{(i)}$ to get a manifold N_1 which we again assume embeds in M. Isotop N_1 so ∂N_1 meets F_0 transversally in a minimal number of curves. Set $N_1^* = N_1 \setminus X$ where X is a standard neighborhood of F_0. (Note N_1^* is disconnected.) Repeat cut-and-paste construction with N_1^* in place of N to get N_1', and set $F_1 = F_0 \cup \partial N_1'$. If $\overline{\chi}(\mathrm{kish}(M, F_1))$ is not very small in comparison with n, take $F = F_0$. If $\overline{\chi}(\mathrm{kish}(M, F_1))$ is very small in comparison with n, continue.

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem.

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem. While many classical results in 3-manifold theory, beginning with Dehn's Lemma, treat the problem of desingularizing surfaces, the particular result needed here is new:

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem. While many classical results in 3-manifold theory, beginning with Dehn's Lemma, treat the problem of desingularizing surfaces, the particular result needed here is new:

Proposition D

Let M be compact, orientable, irreducible, atoroidal 3-manifold.

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem. While many classical results in 3-manifold theory, beginning with Dehn's Lemma, treat the problem of desingularizing surfaces, the particular result needed here is new:

Proposition D

Let M be compact, orientable, irreducible, atoroidal 3-manifold. Let G be a finitely generated subgroup of $\pi_1(M)$.

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem. While many classical results in 3-manifold theory, beginning with Dehn's Lemma, treat the problem of desingularizing surfaces, the particular result needed here is new:

Proposition D

Let M be compact, orientable, irreducible, atoroidal 3-manifold. Let G be a finitely generated subgroup of $\pi_1(M)$. Let T denote the image of G under the natural homomorphism $\pi_1(M) \to H_1(M; \mathbb{Z}_2)$.

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem. While many classical results in 3-manifold theory, beginning with Dehn's Lemma, treat the problem of desingularizing surfaces, the particular result needed here is new:

Proposition D

Let M be compact, orientable, irreducible, atoroidal 3-manifold. Let G be a finitely generated subgroup of $\pi_1(M)$. Let T denote the image of G under the natural homomorphism $\pi_1(M) \to H_1(M;\mathbb{Z}_2)$. Assume that $\dim T \leq (\dim H_1(M;\mathbb{Z}_2)) - 2$.

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem. While many classical results in 3-manifold theory, beginning with Dehn's Lemma, treat the problem of desingularizing surfaces, the particular result needed here is new:

Proposition D

Let M be compact, orientable, irreducible, atoroidal 3-manifold. Let G be a finitely generated subgroup of $\pi_1(M)$. Let T denote the image of G under the natural homomorphism $\pi_1(M) \to H_1(M;\mathbb{Z}_2)$. Assume that $\dim T \leq (\dim H_1(M;\mathbb{Z}_2)) - 2$. Then there is a (possibly disconnected) compact, 3-dimensional submanifold $\mathcal{B}$ of M, having incompressible boundary,

Some ingredients in proof of top. theorem, cont'd

This sketch shows that "desingularizing" an immersed surface in M, i.e. replacing it with an embedded surface having similar properties, is another major step in the proof of the Topological Theorem. While many classical results in 3-manifold theory, beginning with Dehn's Lemma, treat the problem of desingularizing surfaces, the particular result needed here is new:

Proposition D

Let M be compact, orientable, irreducible, atoroidal 3-manifold. Let G be a finitely generated subgroup of $\pi_1(M)$. Let T denote the image of G under the natural homomorphism $\pi_1(M) \to H_1(M; \mathbb{Z}_2)$. Assume that $\dim T \leq (\dim H_1(M; \mathbb{Z}_2)) - 2$. Then there is a (possibly disconnected) compact, 3-dimensional submanifold $\mathcal{B}$ of M, having incompressible boundary, such that

$$\dim T + \dim \check{T} - \dim(T \cap \check{T}) \leq \overline{\chi}(G) - \overline{\chi}(\mathcal{B}),$$

where $\check{T}$ denotes the image of the inclusion homomorphism $H_1(\mathcal{B}; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$.

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold.

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold. Let G be a finitely generated, freely indecomposable subgroup of $\pi_1(M)$, and set $p = \overline{\chi}(G)$.

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold. Let G be a finitely generated, freely indecomposable subgroup of $\pi_1(M)$, and set $p = \overline{\chi}(G)$. Then M has a compact, irreducible submanifold M_0 such that

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold. Let G be a finitely generated, freely indecomposable subgroup of $\pi_1(M)$, and set $p = \overline{\chi}(G)$. Then M has a compact, irreducible submanifold M_0 such that

1. *$i : \partial M_0$ is incompressible;*

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold. Let G be a finitely generated, freely indecomposable subgroup of $\pi_1(M)$, and set $p = \overline{\chi}(G)$. Then M has a compact, irreducible submanifold M_0 such that

1. *$i : \partial M_0$ is incompressible;*
2. *the image of i contains a conjugate of G;*

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold. Let G be a finitely generated, freely indecomposable subgroup of $\pi_1(M)$, and set $p = \overline{\chi}(G)$. Then M has a compact, irreducible submanifold M_0 such that

1. *$i : \partial M_0$ is incompressible;*
2. *the image of i contains a conjugate of G; and*
3. $\overline{\chi}(M_0) \leq p$.

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold. Let G be a finitely generated, freely indecomposable subgroup of $\pi_1(M)$, and set $p = \overline{\chi}(G)$. Then M has a compact, irreducible submanifold M_0 such that

1. *$i : \partial M_0$ is incompressible;*
2. *the image of i contains a conjugate of G; and*
3. *$\overline{\chi}(M_0) \leq p$.*

Remarkably, the proof of this requires the celebrated theorem (proof recently completed by Agol) that π_1 of a hyperbolic 3-manifold is LERF (or subgroup separable in the language of Mahan Mj's talk).

Some ingredients in proof of top. theorem, cont'd

Proposition D is proved by combining the Papakyriakpoulos-Shapiro-Whitehead method of double coverings with the following result:

Theorem

Let M be a compact, orientable, irreducible 3-manifold. Let G be a finitely generated, freely indecomposable subgroup of $\pi_1(M)$, and set $p = \overline{\chi}(G)$. Then M has a compact, irreducible submanifold M_0 such that

1. *$i : \partial M_0$ is incompressible;*
2. *the image of i contains a conjugate of G; and*
3. $\overline{\chi}(M_0) \leq p$.

Remarkably, the proof of this requires the celebrated theorem (proof recently completed by Agol) that π_1 of a hyperbolic 3-manifold is LERF (or subgroup separable in the language of Mahan Mj's talk). This means that every finitely generated subgroup is an intersection of finite-index subgroups.