Quantitative geometry of hyperbolic manifolds, II

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A very natural example is to relate these geometrically defined invariants to classical topological invariants such as homology. Relating the geometry of a manifold to its homology is a very classical theme in differential geometry.

Volumes

Thurston showed, using results due to Jorgensen and Gromov, that the set of (finite) volumes of hyperbolic 3-manifolds, as a subset of $\mathbb{R}_+$, is well-ordered (by the usual order relation on $\mathbb{R}$). Its ordinal type is ω^ω.

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This means that the results below, which relate volume to the rank of homology, may be thought of either in terms of real numbers or in terms of ordinals. (The real number corresponding to a given ordinal is not known, but can typically being bounded above by some number R, explicitly producing a rich enough class of manifolds with volumes $< R$.)

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The bound on r_p is sharp when $p = 5$.

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$$p \notin \bigcup_C Z_\lambda(C),$$

where C ranges over the maximal cyclic subgroups of Γ, and

$$Z_\lambda(C) := \{z \in \mathbb{H}^3 : \mathrm{dist}(z, \gamma \cdot z) < \lambda \text{ for some } \gamma \in C - \{1\}\}.$$

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Hence M contains a ball of radius $\lambda/2$ if and only if the sets $Z_\lambda(C)$ fail to cover $\mathbb{H}^3$.

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Each $Z_\lambda(C)$ is a "cylinder" in the following sense: there exist a hyperbolic line A_C (the common "axis" of the non-trivial elements of C) and a number r depending on C and λ such that

$$Z_\lambda(C) = \{z \in \mathbb{H}^3 : \mathrm{dist}(z, A_C) < r\}.$$

Displacement Cylinders, cont'd

If we take $\lambda = \log(2k-1)$, the $\log(2k-1)$ theorem implies (formally) that if Γ is k-free and if $C_1, \ldots, C_k$ are maximal cyclic subgroups of Γ such that

$$Z_\lambda(C_1) \cap \cdots \cap Z_\lambda(C_k) \neq \emptyset$$

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(Recall the relevant part of the $\log(2k-1)$ Theorem: Let $k \geq 2$ be an integer and let F be a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$ which is freely generated by elements $x_1, \dots, x_k$.

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These ingredients interact via topology.

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Thus a discrete, purely loxodromic subgroup Γ of $\mathrm{Isom}_+(\mathbb{H}^3)$ and a number $\lambda > 0$ determine a Γ-labeled complex K and a labeling-compatible action of Γ on K.

Labeled complexes, cont'd

The consequence of the $\log(2k-1)$ theorem stated above implies:

(*) If Γ is k-free and we take $\lambda = \log(2k-1)$, then for every $(k-1)$-simplex $\Delta = (C_0, \ldots, C_{k-1})$, the free group $\Theta(\Delta)$ has rank less than k.

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Thus topological-combinatorial-group-theoretical results about contractible Γ-labeled complexes which admit labeling-compatible Γ-actions and satisfy (*) can imply geometric results, such as the existence of balls of certain radii in hyperbolic manifolds whose fundamental groups satisfy such conditions as k-freeness for suitable values of k.

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If $\pi_1(M)$ is 4-free then M contains a point P such that all loops of length $< \log 7$ based at P define elements of a single cyclic subgroup of $\pi_1(M, P)$.

If this cyclic subgroup were trivial, there would be a hyperbolic ball of radius $(\log 7)/2$ about P. The weaker conclusion is still geometrically meaningful, and gives volume estimates.

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Let g be an integer ≥ 2. Let M be a closed, simple ($\iff$ hyperbolic) 3-manifold such that $H_1(M; \mathbb{Z}_2)$ has rank at least $\max(3g-1, 6)$ *and $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group. Then M contains a closed surface F with* $1 < \mathrm{genus}(F) \leq g$ *which is incompressible in the sense that the inclusion homomorphism $\pi_1(F) \to \pi_1(M)$ is injective.*

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This is a fancier version of classical results due to Papkyriakopoulos about removing self-intersections of surfaces in 3-manifolds. It depends on the S.-Wagreich theorem stated above, a deep result due to Gabai, and Fisher's inequality from combinatorics.

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When M does contain a low-genus incompressible surface, a result due to Agol-Storm-Thurston often gives a good lower bound for the volume of M. This result depends on Perelman's work on the Ricci flow with surgeries.

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Now define $\mathrm{kish}(M, F)$ (the "kishkes" of X, sometimes called the "guts") to be the union of all components of $\overline{X - \Sigma}$ that have negative Euler characteristic.

Kishkes and volume

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I will illustrate how the above ingredients fit together to prove the theorems that I stated at the beginning.

Putting the ingredients together

Here is one of the theorems that I stated at the beginning.

Theorem (Culler-S.)

Let M be a closed, orientable hyperbolic 3-manifold. If $\operatorname{Vol} M \leq 3.08$ then the rank of $H_1(M; \mathbb{Z}_2)$ is at most 5.

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If 3-manifold, k is an integer, and $H_1(M; \mathbb{Z}_p)$ has rank at least $k+2$ for some prime p, then $\pi_1(M)$ either is k-free or has a subgroup isomorphic to a genus-g surface group for some g with $1 < g < k$.

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Applying this with $p = 2$ and $k = 3$, we deduce that $\pi_1(M)$ either is 3-free or contains a genus-2 surface group.

Putting the ingredients together, cont'd

If $\pi_1(M)$ is 3-free, we use another one of the theorems I stated earlier:

Theorem (Anderson-Canary-Culler-S.)

If $\pi_1(M)$ *is* 3*-free then* M *contains a hyperbolic ball of radius* $(\log 5)/2$.

Putting the ingredients together, cont'd

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Theorem (Anderson-Canary-Culler-S.)

If $\pi_1(M)$ is 3-free then M contains a hyperbolic ball of radius $(\log 5)/2$.

A theorem due to Böröczky and Florian about sphere-packing in hyperbolic space implies that if M contains a hyperbolic ball of radius $(\log 5)/2$ then $\mathrm{Vol}\, M > 3.08$.

Putting the ingredients together, cont'd

Now suppose that $\pi_1(M)$ contains a genus-2 surface group. In this case we use:

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The next step is to apply the Agol-Storm-Thurston theorem:

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The other results I mentioned at the beginning are proved by putting together the ingredients I have described in a similar way.