Quantitative geometry of hyperbolic manifolds, I

Peter Shalen

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(A set function A is *finitely additive* if $A(X \cup Y) = A(X) + A(Y)$ for any two disjoint subsets of S^2. It is *rotationally invariant* if $A(\rho(X)) = A(X)$ for every $\rho \in \mathrm{SO}(3)$.)

The Banach-Tarski paradox, cont'd

Let us call two subsets X and Y of $\mathbb{R}^3$ *equivalent* if for some integer N they have decompositions into disjoint subsets

$$X = X_1 \cup \cdots X_N$$

and

$$Y = Y_1 \cup \cdots Y_N$$

such that X_i and Y_i are isometric for $i = 1, \ldots, N$.

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Banach and Tarski used this to show that any two bounded sets with non-empty interior are equivalent ("the pea and the sun").

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(and similarly $x\overline{X} = F - X$, $y^{-1}Y = F - \overline{Y}$, $y\overline{Y} = F - Y$).

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gives rise to a decomposition

$$C' = \mathcal{X} \cup \overline{\mathcal{X}} \cup \mathcal{Y} \cup \overline{\mathcal{Y}} \cup \Omega,$$

where $\mathcal{X} = X \cdot \Omega$, $\overline{\mathcal{X}} = \overline{X} \cdot \Omega$, and so on.

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we obtain

$$A(C') = A(\mathcal{X}) + A(\overline{\mathcal{X}}) + A(\mathcal{Y}) + A(\overline{\mathcal{Y}}) + A(\Omega).$$

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Similarly,

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It's easy to find a rotation that carries the countable set C into its complement C', so $A(C) = 0$ and hence $A(S^2) = 0$.

Uniform restrictions on discrete groups

If Γ is a discrete, torsion-free subgroup of $\mathrm{Isom}_+(\mathbb{H}^n)$, then for any $p \in \mathbb{H}^n$, the set

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However, if one considers a larger set of elements $x_1, \ldots, x_k \in \Gamma$, under appropriate conditions one can sometimes give uniform conditions involving the distances $\mathrm{dist}(p, x_1 \cdot p), \ldots, \mathrm{dist}(p, x_k \cdot p)$ which imply that they cannot all be simultaneously small.

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Results of this kind turn out to be useful in studying geometric quantities associated to hyperbolic manifolds, such as volume, injectivity radius, diameter, etc.

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Theorem (Anderson-Canary-Culler-S. + Agol and Calegari-Gabai)

Let $k \geq 2$ be an integer and let F be a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$ which is freely generated by elements $x_1, \ldots, x_k$. Let p be any point of $\mathbb{H}^3$ and set $d_i = \mathrm{dist}(p, x_i \cdot p)$ for $i = 1, \ldots, k$. Then we have

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}.$$

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As a result, instead of getting a paradox, one gets an estimate.

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Let us assume for a moment that the series diverges for $s = D$. For every $s > D$ and every $p \in \mathbb{H}^3$ we define a Borel probability measure $\mu_{p,s}$ on the compact space $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup S_\infty$ by

$$\mu_{p,s} = \frac{1}{g(p,s)} \sum_{\gamma\in\Gamma} \exp(-s\,\mathrm{dist}(p,\gamma\cdot p))\delta_{\gamma\cdot p}$$

where $\delta_{\gamma\cdot p}$ denotes a Dirac mass concentrated at $\gamma \cdot p$.

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The support of μ_p is the limit set $\Lambda \subset S_\infty$ of Γ.

Under an element $\gamma \in \Gamma$, the measure transforms according to the law

$$d\mu_{\gamma^{-1}(p)} = \lambda_{\gamma,p}^{D}\, d\mu_p \tag{1}$$

for every $\gamma \in \Gamma$ and every $p \in \mathbb{H}^3$. Here D is the critical exponent, and $\lambda_{\gamma,p}$ is the "conformal expansion factor" with respect to the round metric on S_∞ centered at p; this means that for every $\zeta \in S_\infty$, the tangent map $d\gamma_\infty : T_\zeta(S_\infty) \to T_{\gamma_\infty(\zeta)}(S_\infty)$ changes lengths by a factor of $\lambda_{\gamma,p}(\zeta)$.

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Note that if A_p denotes the area measure centered at p, normalized to have total mass 1, the ordinary change of variable formula gives

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For simplicity, suppose that the Poincaré series

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diverges at the critical exponent $s = D$.

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and we define $\nu_{p,s;\overline{X}}$, $\nu_{p,s;Y}$ and $\nu_{p,s;\overline{Y}}$ similarly.

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After refining the sequence of values of $s > D$ that defined μ, we may arrange that $\nu_{p,s;X}$, $\nu_{p,s;\overline{X}}$, $\nu_{p,s;Y}$ and $\nu_{p,s;\overline{Y}}$ converge weakly to measures $\nu_{p;X}, \ldots, \nu_{p;\overline{Y}}$.

The "paradoxical" decomposition of Patterson-Sullivan measure, cont'd

For simplicity, suppose that the Poincaré series

$$g(s) = \sum_{\gamma \in F} \exp(-s\,\mathrm{dist}(p, \gamma \cdot p))$$

diverges at the critical exponent $s = D$. For each $s > D$, we set

$$\nu_{p,s;X} = \frac{1}{g(s)} \sum_{\gamma \in X} \exp(-s\,\mathrm{dist}(p, \gamma \cdot p))\delta_{\gamma \cdot p}$$

and we define $\nu_{p,s;\overline{X}}$, $\nu_{p,s;Y}$ and $\nu_{p,s;\overline{Y}}$ similarly.

After refining the sequence of values of $s > D$ that defined μ, we may arrange that $\nu_{p,s;X}$, $\nu_{p,s;\overline{X}}$, $\nu_{p,s;Y}$ and $\nu_{p,s;\overline{Y}}$ converge weakly to measures $\nu_{p;X}, \ldots, \nu_{p;\overline{Y}}$. We then have

$$\mu = \nu_{p;X} + \nu_{p;\overline{X}} + \nu_{p;Y} + \nu_{p;\overline{Y}}.$$

The "paradoxical" decomposition of Patterson-Sullivan measure, cont'd

The measures $\nu_{p;X}, \ldots, \nu_{p;\overline{Y}}$ satisfy the analogue of (1), e.g.

$$d\nu_{\gamma^{-1}(p);X} = \lambda^{D}_{\gamma,p}\, d\nu_{p;X},$$

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Combining these, taking $\gamma = x^{-1}$ (say), and integrating over S_∞, we get

$$\int \lambda^{D}_{x^{-1};p}\, d\nu_{p;X} = 1 - \int d\nu_{p;\overline{X}}.$$

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Theorem (Anderson-Canary-Culler-S. + Agol and Calegari-Gabai)

Let F be a discrete (purely loxodromic) subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$ which is freely generated by elements x and y.

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$$\frac{1}{1+\exp(\mathrm{dist}(p, x \cdot p))} + \frac{1}{1+\exp(\mathrm{dist}(p, y \cdot p))} \leq \frac{1}{2}.$$

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For the proof, work with the Patterson-Sullivan measure $\mu = \mu_p$ centered at p, and its decomposition

$$\mu = \nu_X + \nu_{\overline{X}} + \nu_Y + \nu_{\overline{Y}}$$

(with p now suppressed from notation).

A sketch of the proof of the $\log(2k-1)$ theorem. cont'd

Pretend for the moment that F is cocompact, so $D = 2$, and $\mu =$ normalized area measure $A = A_p$ centered at p.

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Pretend for the moment that F is cocompact, so $D = 2$, and $\mu =$ normalized area measure $A = A_p$ centered at p.Then $\nu_X, \ldots, \nu_{\overline{Y}}$ are bounded above by A.

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Set $\lambda = \lambda_{x^{-1}}$. We've seen

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$$\int_{C_{\alpha_X}} \lambda^2 \, dA \geq 1 - \beta_X. \qquad (2)$$

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Fix any point $p \in \mathbb{H}^3$, and consider the real-valued function defined on Δ by

$$(x, y) \mapsto \frac{1}{1 + \exp(\mathrm{dist}(p, x \cdot p))} + \frac{1}{1 + \exp(\mathrm{dist}(p, y \cdot p))}.$$

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