Filling invariants: homological vs. homotopical

Pallavi Dani

Louisiana State University

joint with A. Abrams, N. Brady, and R. Young

Isoperimetric inequality in the plane

Let γ be a simple loop in $\mathbb{R}^2$ of length L.

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If $A =$ area enclosed by γ then

$$A \leq \frac{L^2}{4\pi}.$$

Moreover $\frac{L^2}{4\pi}$ is the *smallest* function which work for all loops.

(Since if γ is a circle, $A = \frac{L^2}{4\pi}$)

$\frac{L^2}{4\pi}$ is the *isoperimetric function* of $\mathbb{R}^2$.

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- Riemannian manifold (area is defined as an integral)
- Cell complex (area is given by counting cells)

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A *filling* of γ is an extension $\tilde{\gamma} : D^2 \to X$. (Or $X^{(2)}$ in cell complex case)

The *filling area* of γ is

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The *isoperimetric function* (or *Dehn function*) of X is

$$\text{Dehn}_X(x) = \sup\{\text{FArea}(\gamma) \mid \text{ length}(\gamma) \leq x\}$$

Let G be a finitely presented group. If G acts *geometrically* (i. e. properly discontinuously and cocompactly) on spaces X_1 and X_2, then their Dehn functions have the same "growth type". More precisely,

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For example,
any polynomial of degree $d \simeq x^d$
If $\alpha, \beta \geq 1$, then $x^\alpha \simeq x^\beta \iff \alpha = \beta$
$e^x \simeq \lambda^x$ for all $\lambda > 1$
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Define $\mathrm{Dehn}_G(x) := \mathrm{Dehn}_{X_1}(x)$ (well defined up to $\simeq$)

Dehn functions are *quasi-isometry invariants*.
(Gromov, Bridson, Alonso)

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- For every x, there is a loop γ_x of length x such that $\mathrm{FArea}(\gamma_x) \succeq x^2$. (Take γ_x to be a circle with circumference x which lies in a plane). So

$$\mathrm{Dehn}_{\mathbb{Z}^n}(x) \succeq x^2$$

Thus $\mathrm{Dehn}_{\mathbb{Z}^n}(x) \simeq x^2$

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- Birget-Sapir-Rips (almost) characterize which computable functions occur as Dehn functions of finitely presented groups.

Let X be a cell complex with $H_1(X) = 0$.

If α is a 1-cycle, then a filling of α is an 2-chain β with $\partial\beta = \alpha$.

The *filling mass* of α is

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$\text{Dehn}_X(x)$ is called the *homotopical Dehn function of X*.

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Q. Are there fin. pres. groups for which $\text{Dehn}_G(x)$ and $\text{FA}_G(x)$ are different?

Homotopical Dehn function	Homological Dehn function
$X = k$-connected space	$X = k$-acyclic space
Boundaries: immersions $S^k \to X$ Fillings: extensions $D^{k+1} \to X$	Boundaries: k-cycles Fillings: $(k+1)$-chains

Higher dimensional filling functions

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$\mathrm{FVol}(\sigma)$=filling vol of a k-sphere σ	$\mathrm{FMass}(\alpha)$=filling mass of a k-cycle α
$\mathrm{Dehn}^k_X(x) = \sup_{\mathrm{Vol}(\sigma)\leq x}\{\mathrm{FVol}(\sigma)\}$	$\mathrm{FV}^k_X(x) = \sup_{\mathrm{Mass}(\alpha)\leq x}\{\mathrm{FMass}(\alpha)\}$

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Q. Are $\text{Dehn}^k_G(x)$ and $\text{FV}^k_G(x)$ ever different (for a fixed G and k)?

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However, spheres of dimension at least 2 can be filled just as easily by balls as by chains. More precisely:

Theorem (Gromov, Groft)

Let X be a k-connected space, with $k \geq 2$. Given $\sigma : S^k \to X^{(k)}$, if σ corresponds to a k-cycle α, and β is a $(k+1)$-chain with boundary α, then there exists an extension $\tilde{\sigma} : D^{k+1} \longrightarrow X$ such that

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So when $k \geq 2$, we have

$$\mathrm{Dehn}_X^k(x) \preceq \mathrm{FV}_X^k(x)$$

By a similar argument, cycles of dimension at least 3 are no harder to fill than spheres of the same dimension. More precisely,

Theorem (Groft)

Let X be a k-connected space, with $k \geq 3$. Then given any k-cycle α, there exists a sphere $\sigma : S^k \to X$ with $\mathrm{Vol}(\sigma) = \mathrm{Mass}(\alpha)$, such that

$$\mathrm{FMass}(\alpha) = \mathrm{FVol}(\sigma)$$

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Corollary

If $k \geq 3$, and G is a group which acts geometrically on a k-connected complex, then

$$\mathrm{Dehn}^k_G(x) \simeq \mathrm{FV}^k_G(x).$$

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Theorem (Young)

There exist groups G which act geometrically on 2-connected complexes such that $\mathrm{FV}_G^2(x)$ is strictly greater than $\mathrm{Dehn}_G^2(x)$.

Theorem (Abrams-Brady-D.-Young)

For every $d \in \mathbb{N}$, there is a finitely presented group H such that

$$\mathrm{FA}_H(x) \preceq x^5 \quad \text{and} \quad x^d \preceq \mathrm{Dehn}_H(x) \preceq x^{d+3}$$

There exists a finitely presented group H such that

$$\mathrm{FA}_H(x) \preceq x^5 \quad \text{and} \quad e^x \preceq \mathrm{Dehn}_H(x)$$

In each of the above cases, H is constructed as a subgroup of a CAT(0) group.

Let W be a simply connected 2-complex, and α is a loop in W.

To reduce $\mathrm{FMass}(\alpha)$ but not $\mathrm{FArea}(\alpha)$, attach a 2-complex Z along α, in which α bounds a 2-chain, but not a disk.

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Choose Z so that

- $\pi_1(Z) = \langle\langle\gamma\rangle\rangle$, where γ has infinite order.
- $\text{Mass}(Z)$ small compared to $\text{FArea}(\alpha)$

For example, $Z =$ presentation complex for an simple group containing an infinite-order element (such as Thompson's group T)

Form W' by attaching Z to W by identifying γ with α.

van Kampen's Theorem $\implies$ X' is simply connected.

In X',

$$\text{FMass}(\alpha) < \text{FArea}(\alpha)$$

Constructing a space with $\mathrm{FA}(x)$ less than $\mathrm{Dehn}(x)$

Let W and Z be as before.

To obtain a space X with

- $\mathrm{Dehn}_X(x) = \mathrm{Dehn}_W(x)$ but
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attach copies of Z along an infinite family of loops α_n such that

- length$(\alpha_n) \to \infty$
- An arbitrary loop can be decomposed into a sum of α_n's and therefore has small homological filling mass.

Group theoretic version

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- H acts geometrically on a space X as above

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Natural setting in which one sees many scaled copies of the same complex: certain spaces associated to kernels of homomorphisms to $\mathbb{Z}$

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Γ= defining graph

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- one generator for each vertex
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For example,

- Γ $=$ complete graph on n vertices $\implies A_\Gamma = \mathbb{Z}^n$
- $\Gamma = n$ vertices, no edges $\implies A_\Gamma =$ free group of rank n.
- $\Gamma =$ a square $\implies A_\Gamma = \mathbb{F}_2 \times \mathbb{F}_2$.

Salvetti complex $= K(A_\Gamma, 1)$.

- Start with a wedge of circles, one for each generator.
- Glue in a 2-torus for each edge of Γ

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- Glue in an n-torus for each complete subgraph with n-vertices.

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$K_\Gamma :=$ universal cover of this complex.
For example,

- If $A_\Gamma = \mathbb{Z}^n$ then $K_\Gamma = \mathbb{R}^n$ with a cube complex structure.
- If $A_\Gamma = \mathbb{F}_n$ then $K_\Gamma =$ a tree of valence $2n$.
- If $A_\Gamma = \mathbb{F}_2 \times \mathbb{F}_2$, then $K_\Gamma =$ a product of trees.

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The map $A_\Gamma \to \mathbb{Z}$ induces a height function $h : K_\Gamma \to \mathbb{R}$

- Defined on $K_\Gamma^{(1)}$ = Cayley graph of A_Γ by the homomorphism
- Extends to higher dimensional cells.

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Consider $H_\Gamma = \ker(A_\Gamma \to \mathbb{Z})$.

H_Γ acts geometrically and freely on level sets of h.

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1. H_Γ *is finitely presented if and only if Z is simply connected*

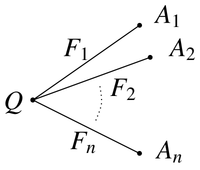
Theorem (Bestvina-Brady)

Let Γ be a graph, and Z be the flag complex on Γ. Let $H_\Gamma = \ker(h : A_\Gamma \to \mathbb{Z})$. Then

1. *H_Γ is finitely presented if and only if Z is simply connected*
2. *$h^{-1}(0)$ is homotopy equivalent to a wedge product of infinitely many copies of Z, indexed by the vertices in $K_\Gamma \setminus h^{-1}(0)$. In fact, $h^{-1}(0)$ is a union of infinitely many scaled copies of Z.*

H is the kernel of a homomorphism from a group G to $\mathbb{Z}$.

G is defined by the graph of groups on the right



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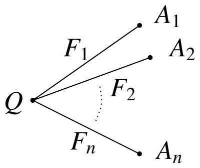

Here

- A_i is a RAAG defined by $Z^{(1)}$, where Z is a triangulated 2-complex with $\pi_1(Z)$ infinite simple.
- $F_i \cong F_2 \times F_2$
- Q is a group (defined in terms of labelled oriented graphs) such that
  - There is a map $h_Q : Q \to \mathbb{Z}$
  - $\mathrm{Dehn}_{\ker(h_Q)}(x) \simeq x^d$ or e^x ($\exists$ construction by Brady-Guralnik-Lee)

The maps h_{A_i}, h_Q extend to a map $h_G : G \to \mathbb{Z}$. Finally $H = \ker(h_G)$.

- H acts geometrically on a level set $h_G^{-1}(0)$ of a map $h_G : K_G \to \mathbb{R}$.
- $h_G^{-1}(0)$ has the structure of the space X described earlier.
- H is finitely presented (PL Morse theory arguments similar to Bestvina-Brady)