

Option Pricing

Mrinal K. Ghosh*

1 Introduction

We first introduce the basic terminology in option pricing.

Option: An option is the right, but not the obligation to buy (or sell) an asset under specified terms.

Holder of the Option: has the right without any obligation.

Writer of the Option: has no right, but is obliged to the holder to fulfill the terms of the option.

Call Option: gives the holder the right to buy something.

Put Option: gives the holder the right to sell something.

Asset: can be anything, but we consider only stocks which will be referred to as a primary security. An option is a derivative security.

Strike (or Exercise) Price: A prescribed amount at which the underlying asset may be bought or sold by the holder.

Expiration Date: a future time (date) after which the option becomes void.

European Option: can be exercised only on the expiry date.

American Option: can be exercised any time before and including the expiration date.

Short: selling an asset without actually possessing it.

Long: buy.

To summarize: A European call option is a contract with the following conditions: At a prescribed time in future (expiration date) the holder of the option may buy a prescribed asset for a prescribed price (strike price).

Note that the holder of the option has a right but no obligation, whereas the writer of the

*Department of Mathematics, Indian Institute of Science, Bangalore-12, India, email: mkg@math.iisc.ernet.in

option does have a potential obligation : he must sell the asset should the holder choose to exercise his call option. Since the option confers on its holder a right without any obligation it has some value, i.e., the holder must pay some amount (premium) at the time of opening the contract. At the same time, the writer of the option must have some compensation for the obligation he has assumed. Thus two central issues are:

- (a) How much would the holder of the option pay for this right, i.e., what is the value (price) of an option?
- (b) How can the writer of the option minimize the risk associated with his obligation ?

Nature of Option Price (value)

Let us examine the case of a European call option on a stock, whose price at time t is S_t . Let T = expiration date , K = the strike price. Now two cases arise:

- (i) $K > S_T$: the option will not be exercised.
- (ii) $K < S_T$: the holder makes a profit of $S_T - K$ by exercising the option.

Thus the price or value of the option at maturity is

$$C_T = (S_T - K)^+ = \max(S_T - K, 0).$$

Similarly for put option $P_T = (K - S_T)^+ = \max(K - S_T, 0)$. At the time of writing the option, S_T is unknown. Thus we have to deal with two problems :

- (i) How much the holder should pay to the writer at time $t = 0$ for an asset worth $(S_T - K)^+$ at time T ? This is the problem of pricing the option.
- (ii) How should the writer, who earns the premium initially, generate an amount $(S_T - K)^+$ at time T ? This is the problem of hedging the option.

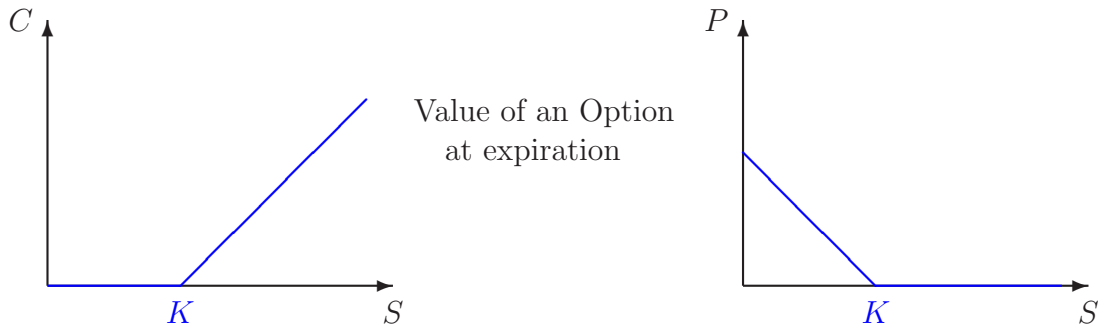


Figure 1: Call Option and Put Option

Time Value of Options

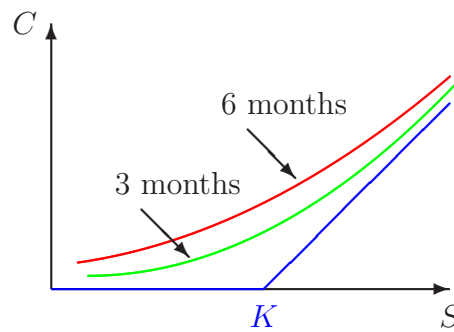


Figure 2: Time Value

We have thus far analyzed the value of the option C_T (or P_T) at the expiration date. At any time t between 0 to T , the option has a value C_t (or P_t). We say that a call option at time t is in the money, at the money, or out of the money depending on whether $S_t > K$, $S_t = K$, or $S_t < K$, respectively. Puts have reverse terminology.

It may be noted that even European options have a value at earlier times, since they provide potential for future exercise.

When there is a positive time to expiration, the value of the call option as a function of the stock price is a smooth curve rather than the decidedly kinked curve that applies at expiration. The value of option price for various expiration dates are shown in Figure 2. The problem is to determine the curve.

Factors Affecting the Option Price

- (1) The price of the stock S_t .
- (2) Time of expiration.

- (3) The strike price.
- (4) Volatility of the underlying stocks.
- (5) Prevailing interest rate.
- (6) Growth rate of the stock (in Black-Scholes model this does not appear explicitly in the option price).

Arbitrage and Put-Call Parity

Loosely speaking arbitrage means an opportunity to make risk free profit. Throughout we will assume the absence of arbitrage opportunities, i.e., there is no risk free profit available in the market. This will be made precise later. We now derive a formula relating European put and call prices, often referred to as put-call parity. Both the put and call which have maturity (expiration) T and exercise price K are contingent on the same underlying stock which is worth S_t at time t . We also assume that it is possible to borrow or invest money at a constant rate r . Let C_t and P_t denote respectively the prices of the call and the put at time t . Because of no arbitrage, the following equation called put-call parity holds for all $t < T$:

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

Indeed, assume that

$$C_t - P_t > S_t - Ke^{-r(T-t)}.$$

At time t , we buy a share of the stock and a put, and sell a call. The net value of the operation is

$$C_t - P_t - S_t.$$

If $C_t - P_t - S_t > 0$, we invest this amount at a rate r until T , whereas if it is negative we borrow it at the same rate. At time T , two outcomes are possible:

- $S_T > K$: the call is exercised, we deliver the stock, receive the amount K and clear the cash account to end up with a wealth $K + e^{r(T-t)}(C_t - P_t - S_t) > 0$.
- $S_T < K$: we exercise the put and clear the cash account to finish with a wealth

$$K + e^{r(T-t)}(C_t - P_t - S_t) > 0.$$

In both cases, we locked in a positive profit without making any initial endowment : this is an example of arbitrage opportunity.

Similarly, we can show that (*Exercise*)

$$C_t - P_t < S_t - Ke^{-r(T-t)}$$

will lead to an arbitrage opportunity.

Our exposition on Option Pricing is based on [2], [3], [4] and [5]. In particular the derivation and the explicit solution of Black-Scholes partial differential equation

(pde) are based on [2]; the Greeks are based on [4]. We also refer to [1] for an excellent non-technical exposition on derivatives.

We have thus far described option pricing in continuous time. We can also do the same in discrete time, i.e., when trading is done at discrete epochs $0, 1, 2, \dots, N$, where N is the expiration time.

Exercise 1.1 (Bull spread) An investor who is bullish about a stock (believing that it will rise) may wish to construct a *bull spread* for that stock. One way to construct such a spread is to buy a call with a strike price K_1 and sell a call with the same expiration date but with a strike price $K_2 > K_1$. Draw the payoff curve for such a spread. Is the initial cost of the spread positive or negative?

Exercise 1.2 Let C_n and P_n be the call and the put prices respectively at time n on the same underlying with same strike price K and with the same expiration date N . If r denotes the risk free interest rate per period, show that the following put-call parity holds:

$$C_n - P_n = S_n - K(1 + r)^{-(N-n)}$$

where S_n is the price of the underlying at time n .

Exercise 1.3 You would like to speculate on a rise in the price of a certain stock. The current stock price is Rs. 290 and a 3-month call with a strike price of Rs. 300 costs Rs. 29. You have Rs. 58,000 to invest. Identify two alternative strategies, one involving an investment in the stock and the other involving in the option. What are the potential gains and losses from each?

2 Option Pricing in Discrete time

Let (Ω, \mathcal{F}, P) be a finite probability space, and $\{\mathcal{F}_n\}, n = 0, 1, \dots, N$, a filtration on it with $\mathcal{F}_N =$ the set of all subsets of Ω ; N is a positive integer which is the planning horizon. It will correspond to the maturity of the options. We will assume that $P(\{\omega\}) > 0$ for all $\omega \in \Omega$. We consider a market which consists of $(k + 1)$ financial assets, whose prices at time n ($0 \leq n \leq N$) are given by $S_n^0, S_n^1, \dots, S_n^k$ which are adapted (measurable) with respect to \mathcal{F}_n , i.e., these prices are based on the information available up to and including time n . Typically $\mathcal{F}_n = \sigma(S_m^0, S_m^1, \dots, S_m^k, m = 0, 1, \dots, n)$. Let $S_n = (S_n^0, S_n^1, \dots, S_n^k)$ denote the vector of prices at time n . The asset indexed by 0 is a risk free asset, and we assume that $S_0^0 = 1$. If the return on the risk free asset over one period of time is constant, and is equal to r , then

$$S_n^0 = (1 + r)^n.$$

Let $\beta_n = \frac{1}{S_n^0} = (1 + r)^{-n}$; β_n is interpreted as the discount factor, i.e, if an amount β_n is invested in the risk free asset at time 0, then its value at time n will be 1. In other words β_n is the present value (at time n) of 1. Since the rate of interest is constant, $\beta_n = \beta^n$, where $\beta = (1 + r)^{-1}$. The risk free asset is often referred to as the *numeraire*. The assets indexed by $i = 1, 2, \dots, k$ are all risky assets (for example stocks).

Definition 2.1 (Strategies) A trading strategy is defined as a (finite) sequence of random variables

$$\phi = \{(\phi_n^0, \phi_n^1, \dots, \phi_n^k), 0 \leq n \leq N\}$$

in \mathbb{R}^{k+1} , where ϕ_n^i denotes the number of assets i held at time n , and ϕ is predictable, i.e., for all $i = 0, 1, \dots, k$, ϕ_0^i is \mathcal{F}_0 -measurable, and for $n \geq 1$, ϕ_n^i is \mathcal{F}_{n-1} -measurable. This means that the position in the portfolio $(\phi_n^0, \phi_n^1, \dots, \phi_n^k)$ at time n is decided on the basis of the information available at time $(n-1)$, and kept until n when the new quotations are available. The value of the portfolio at time n is given by

$$V_n(\phi) = \langle \phi_n, S_n \rangle = \sum_{i=0}^k \phi_n^i S_n^i.$$

Let $\tilde{S}_n^i = \beta_n S_n^i = (1+r)^{-n} S_n^i$; \tilde{S}_n^i is called the discounted value of the asset i at time n . The discounted value of the portfolio at time n is given by

$$\tilde{V}_n(\phi) = \beta_n V_n(\phi) = \sum_{i=0}^k \phi_n^i \tilde{S}_n^i.$$

Definition 2.2 A strategy $\phi = \{(\phi_n^0, \phi_n^1, \dots, \phi_n^k), 0 \leq n \leq N\}$ is called *self-financing* if for all $n = 0, 1, \dots, N-1$,

$$\langle \phi_n, S_n \rangle = \langle \phi_{n+1}, S_n \rangle$$

i.e.,

$$\sum_{i=0}^k \phi_n^i S_n^i = \sum_{i=0}^k \phi_{n+1}^i S_n^i.$$

The interpretation of self-financing strategy is the following: at time n , once the new prices $S_n^0, S_n^1, \dots, S_n^k$ are quoted, the investor readjusts his position from ϕ_n to ϕ_{n+1} without bringing in or consuming any wealth.

Lemma 2.1 The following statements are equivalent:

- (a) The strategy ϕ is self-financing;
- (b) For any $n = 1, 2, \dots, N$

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^n \langle \phi_j, \Delta S_j \rangle$$

where $\Delta S_j = S_j - S_{j-1}$.

Proof. Let ϕ be self-financing. Then

$$\langle \phi_n, S_n \rangle = \langle \phi_{n+1}, S_n \rangle,$$

which implies

$$\langle \phi_{n+1}, S_{n+1} - S_n \rangle = \langle \phi_{n+1}, S_{n+1} \rangle - \langle \phi_{n+1}, S_n \rangle = V_{n+1}(\phi) - V_n(\phi).$$

Hence

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^n \langle \phi_j, \Delta S_j \rangle.$$

This proves that (a) implies (b). The converse follows using an analogous argument. ■

Remark 2.1 The above lemma shows that if an investor follows a self-financing strategy then the value of its portfolio is completely determined by the initial wealth $V_0(\phi)$ and $(\phi_n^1, \dots, \phi_n^k), 0 \leq n \leq N$. Similarly we can show that

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^n \langle \phi_j, \Delta \tilde{S}_j \rangle$$

where $\Delta \tilde{S}_j = \tilde{S}_j - \tilde{S}_{j-1}$. Thus for a self-financing strategy the discounted value of its portfolio is also completely determined by the initial wealth $V_0(\phi)$ and $(\phi_n^1, \dots, \phi_n^k), 0 \leq n \leq N$. In fact we can proceed a step further in this direction.

Lemma 2.2 For any predictable process $\{(\phi_n^1, \dots, \phi_n^k), 0 \leq n \leq N\}$ and for any initial wealth V_0 , there exists a unique predictable process $\{\phi_n^0, 0 \leq n \leq N\}$ such that the strategy $\phi = \{(\phi_n^0, \phi_n^1, \dots, \phi_n^k), 0 \leq n \leq N\}$ is self-financing and $V_0(\phi) = V_0$.

Proof. The proof is constructive. The self-financing condition implies

$$\tilde{V}_n(\phi) = \phi_n^0 + \phi_n^1 \tilde{S}_n^1 + \dots + \phi_n^k \tilde{S}_n^k = V_0 + \sum_{j=1}^n (\phi_j^1 \Delta \tilde{S}_j^1 + \dots + \phi_j^k \Delta \tilde{S}_j^k)$$

which defines ϕ_n^0 . The uniqueness of $\{\phi_n^0, 0 \leq n \leq N\}$ follows again from the self-financing property. ■

Remark 2.2 If $\phi_n^0 < 0$, we have borrowed an amount $|\phi_n^0|$ from the risk free asset. If $\phi_n^i < 0$ for $i \geq 1$, we say that we are short a number $|\phi_n^i|$ of asset i . Short-selling and borrowing are allowed, but the value of the portfolio must have a specified lower bound. To simplify matters we choose this number to be zero, i.e., the value of the portfolio must always be non-negative.

Definition 2.3 A strategy $\phi = \{(\phi_n^0, \dots, \phi_n^k), 0 \leq n \leq N\}$ is said to be *admissible* if it is self-financing, and $V_n(\phi) \geq 0$ for any $n = 0, 1, \dots, N$.

We now introduce the concept of **ARBITRAGE** which means the possibility of risk-less profit. A precise definition is given below.

Definition 2.4 An arbitrage strategy is an admissible strategy with zero initial value and positive final value with a positive probability.

Definition 2.5 The market is said to be *viable* if there is no arbitrage opportunity.

Lemma 2.3 If the discounted prices of assets $\{\tilde{S}_n^i\}, i = 1, 2, \dots, N$, are martingales (with respect to $\{\mathcal{F}_n\}$) then for any self-financing strategy ϕ , the discounted value of the portfolio $\{\tilde{V}_n(\phi)\}$ is also a martingale.

Proof. Exercise (Hint: use Remark 2.1). ■

Definition 2.6 A probability measure $P^* \equiv P$ is said to be an *equivalent martingale measure* (EMM) if the discounted asset prices $\{\tilde{S}_n^i\}$ are martingales with respect to P^* . Such a probability measure is also referred to as a *risk neutral measure*.

Theorem 2.4 The market is viable (i.e., arbitrage free) if and only if there exists an EMM P^* .

Proof. Assume that there exists a probability measure $P^* \equiv P$ such that $\{\tilde{S}_n^i\}$ are martingale with respect to P^* . Then for any self-financing strategy ϕ , $\{\tilde{V}_n(\phi)\}$ is also a martingale by Lemma 2.3. Thus

$$E^*[\tilde{V}_N(\phi)] = E^*[\tilde{V}_0(\phi)].$$

If ϕ is admissible and $V_0(\phi) = 0$, then

$$E^*[\tilde{V}_N(\phi)] = 0$$

which implies that

$$\tilde{V}_N(\phi) = 0.$$

Thus the market is viable. The converse is more involved. It is proved using a “Separation Theorem”. We omit it. ■

Definition 2.7 (Contingent Claims) An \mathcal{F}_N -measurable function $H \geq 0$ is called a contingent claim (of maturity N). For example for a European call option on the underlying S^1 with strike price K

$$H = (S_N^1 - K)^+ = \max(S_N^1 - K, 0).$$

For a European put on the same asset with the same strike price K ,

$$H = (K - S_N^1)^+ = \max(K - S_N^1, 0).$$

Definition 2.8 A contingent claim defined by H is *attainable* if there exists an admissible strategy ϕ worth H at time N , i.e., $V_N(\phi) = H$.

Definition 2.9 (Complete Market) The market is said to be *complete* if every contingent claim is attainable, i.e., if H is a contingent claim, then there exists an admissible strategy ϕ such that $V_N(\phi) = H$. The strategy ϕ is often referred to as a strategy *replicating* the contingent claim H .

Theorem 2.5 A viable market is complete if and only if there exists a *unique* probability measure $P^* \equiv P$ under which the discounted prices $\{\tilde{S}_n^i\}$ are martingales.

Proof. First assume that the market is viable and complete. Let H be a contingent claim. Let ϕ be an admissible strategy such that $V_N(\phi) = H$. Since ϕ is self-financing

$$\beta_N H = \tilde{V}_N(\phi) = V_0(\phi) + \sum_{j=1}^N \langle \phi_j, \Delta \tilde{S}_j \rangle.$$

If P_1 and P_2 are two equivalent martingale measures then by Lemma 2.3, $\{\tilde{V}_n(\phi)\}$ is a martingale under both P_1 and P_2 . Thus

$$E^{P_1}[\tilde{V}_N(\phi)] = E^{P_2}[\tilde{V}_N(\phi)].$$

This implies

$$E^{P_1}[H] = E^{P_2}[H].$$

Since H is arbitrary, it follows that $P_1 = P_2$.

Conversely assume that the market is viable and incomplete. Then there exists a contingent claim H which is not attainable. Let \tilde{L} be the set of all random variables of the form

$$V_0 + \sum_{n=1}^N \langle \phi_n, \Delta \tilde{S}_n \rangle$$

where V_0 is a \mathcal{F}_0 -measurable and $\{(\phi_n^1, \dots, \phi_n^k)\}$ is predictable. Then $\beta_N H$ does not belong to \tilde{L} . Thus \tilde{L} is a proper subset of $L^2(\Omega, \mathcal{F}, P^*)$ where P^* is an EMM. Thus there exists a non-zero random variable X which is orthogonal to \tilde{L} . Set

$$P^{**}(\omega) = \left(1 + \frac{X(\omega)}{2 \max_{\omega} |X(\omega)|}\right) P^*.$$

Then P^{**} is a probability measure and $P^{**} \equiv P^* \equiv P$. Moreover

$$E^{**} \left[\sum_{n=1}^N \langle \phi_n, \Delta \tilde{S}_n \rangle \right] = 0$$

for any predictable ϕ . Hence $\{\tilde{S}_n^i\}$ is a P^{**} martingale. Thus there are two equivalent martingale measures. ■

Pricing and Hedging of European Options

Assume that the market is viable and complete. Let P^* denote the unique EMM. Let H be a contingent claim and ϕ the corresponding replicating strategy, i.e., $V_N(\phi) = H$. Since $\{\tilde{V}_n(\phi)\}$ is a P^* -martingale, we have for $n = 0, 1, \dots, N - 1$

$$\tilde{V}_n(\phi) = E^*[\tilde{V}_N(\phi)|\mathcal{F}_n].$$

This implies

$$(1 + r)^{N-n}V_n(\phi) = E^*[H|\mathcal{F}_n].$$

Thus at any time the value of an admissible strategy ϕ replicating H is completely determined by H itself. Hence

$$V_n(\phi) = (1 + r)^{-(N-n)}E^*[H|\mathcal{F}_n]$$

is the wealth needed at time n to replicate H at time N . Thus this quantity is the price of the option at time n . If at time 0, an agent sells an option for $(1 + r)^{-N}E^*[H]$, he can follow a replicating strategy ϕ to generate an amount H at time N . In other words the investor is *perfectly hedged*.

For a European call on an underlying with price S_n at time n , $H = (S_N - K)^+$, $K =$ the strike price. Hence the price of this option at time n is given by

$$(1 + r)^{-(N-n)}E^*[(S_N - K)^+|\mathcal{F}_n].$$

For a European put on the same underlying with the same strike price, the option price at time n is given by

$$(1 + r)^{-(N-n)}E^*[(K - S_N)^+|\mathcal{F}_n].$$

Cox-Ross-Rubinstein (CRR) Model

In this model we consider a risky asset whose price is S_n at time n , $0 \leq n \leq N$, and a risk free asset whose return is r over one period of time. Thus the price of the risk free asset at time n is given by

$$S_n^0 = (1 + r)^n, S_0^0 = 1.$$

Thus the discount factor is given by

$$\beta_n = \beta^n = (1 + r)^{-n}.$$

The risky asset is modelled as follows: between two consecutive periods, the relative price change is either a or b , where $-1 < a < b$, with strictly positive probability for each event, i.e.,

$$S_{n+1} := \begin{cases} S_n(1 + a) & \text{with probability } p_0 > 0 \\ S_n(1 + b) & \text{with probability } (1 - p_0) > 0. \end{cases}$$

The initial stock price S_0 is prescribed. The set of possible states is then

$$\Omega := \{1 + a, 1 + b\}^N.$$

Each N -tuple represents the successive values of $\frac{S_{n+1}}{S_n}$, $n = 0, 1, \dots, N - 1$. Let $\mathcal{F}_n = \sigma(S_0, S_1, \dots, S_n)$. Let

$$T_n = \frac{S_n}{S_{n-1}}, \quad n = 1, 2, \dots, N.$$

If $(x_1, x_2, \dots, x_N) \in \Omega$, set

$$P(\{x_1, \dots, x_N\}) = P(T_1 = x_1, \dots, T_N = x_N).$$

Note that P is completely determined by T_1, \dots, T_N . We first show that the discounted price $\{\tilde{S}_n\}$ is a martingale if and only if

$$E[T_{n+1}|\mathcal{F}_n] = 1 + r, \quad n = 0, 1, \dots, N - 1.$$

Indeed

$$E[\tilde{S}_{n+1}|\mathcal{F}_n] = \tilde{S}_n$$

if and only if

$$E\left[\frac{\tilde{S}_{n+1}}{\tilde{S}_n}|\mathcal{F}_n\right] = 1$$

if and only if

$$E[T_{n+1}|\mathcal{F}_n] = 1 + r.$$

Next we claim that for the market to be viable (i.e., arbitrage free) $r \in (a, b)$, i.e., $a < r < b$. Indeed, if the market is viable then there exists probability measure P^* such that $\{\tilde{S}_n\}$ is a P^* -martingale. Thus

$$E^*[T_{n+1}|\mathcal{F}_n] = 1 + r$$

which implies

$$E^*T_{n+1} = 1 + r.$$

Since T_{n+1} is either equal to $1 + a$ or $1 + b$ with positive probability, it follows that

$$1 + a < 1 + r < 1 + b.$$

We thus have

$$a < r < b.$$

Let

$$p := \frac{b - r}{b - a} > 0.$$

We now show that $\{\tilde{S}_n\}$ is a P^* -martingale if and only if T_1, T_2, \dots, T_N are independent and identically distributed (iid) with

$$P^*(T_i = 1 + a) = p = 1 - P^*(T_i = 1 + b), \quad i = 1, 2, \dots, N.$$

Suppose that $\{T_i\}$ are iid with the above distribution. Then

$$E^*[T_{n+1}|\mathcal{F}_n] = E^*[T_{n+1}] = p(1+a) + (1-p)(1+b) = 1+r.$$

Thus $\{\tilde{S}_n\}$ is a P^* -martingale.

Conversely assume that $\{\tilde{S}_n\}$ is a P^* -martingale. Then

$$E^*[T_{n+1}|\mathcal{F}_n] = 1+r, \quad n = 0, 1, \dots, N-1$$

which implies that

$$E^*[T_{n+1}] = 1+r.$$

Thus

$$(1+a)P^*(T_{n+1} = 1+a) + (1+b)(1 - P^*(T_{n+1} = 1+a)) = 1+r$$

which implies

$$P^*(T_{n+1} = 1+a) = \frac{b-r}{b-a} = p.$$

Therefore T_i 's are identically distributed. Again

$$E^*[T_{n+1}|\mathcal{F}_n] = 1+r$$

which implies

$$(1+a)E^*[I\{T_{n+1} = 1+a\}|\mathcal{F}_n] + (1+b)E^*[I\{T_{n+1} = 1+b\}|\mathcal{F}_n] = 1+r.$$

But

$$E^*[I\{T_{n+1} = 1+a\}|\mathcal{F}_n] + E^*[I\{T_{n+1} = 1+b\}|\mathcal{F}_n] = 1.$$

Thus

$$P^*(T_{n+1} = 1+a|\mathcal{F}_n) = p = 1 - P^*(T_{n+1} = 1+b|\mathcal{F}_n).$$

Proceeding inductively it can be shown that (*Exercise*)

$$P^*(T_1 = x_1, T_2 = x_2, \dots, T_n = x_n) = p_1 p_2 \dots p_n$$

where $p_i = p$ if $x_i = 1+a$, $p_i = 1-p$ if $x_i = 1+b$. Hence T_i 's are independent. Also note that the joint distribution of T_1, T_2, \dots, T_n is uniquely determined under P^* . Hence P^* is unique. Thus the *market* is *viable* and *complete*. Using this we first derive the *put-call parity*:

$$C_n - P_n = S_n - K(1+r)^{-(N-n)}.$$

Under P^* , we have

$$\begin{aligned} C_n - P_n &= (1+r)^{-(N-n)} E^*[(S_N - K)^+ - (K - S_N)^+|\mathcal{F}_n] \\ &= (1+r)^{-(N-n)} E^*[(S_N - K)|\mathcal{F}_n] \\ &= (1+r)^n E^*[\tilde{S}_N|\mathcal{F}_n] - K(1+r)^{-(N-n)} \\ &= (1+r)^n \tilde{S}_n - K(1+r)^{-(N-n)} \\ &= S_n - K(1+r)^{-(N-n)}. \end{aligned}$$

We next derive a formula for C_n in terms of a, b, r . We have

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} E^*[(S_N - K)^+ | \mathcal{F}_n] \\ &= (1+r)^{-(N-n)} E^*[(S_n \prod_{i=n}^N T_i - K)^+ | \mathcal{F}_n]. \end{aligned}$$

Let $C(n, x)$ stand for C_n when $S_n = x$. Then

$$C(n, x) = (1+r)^{-(N-n)} \sum_{j=0}^{N-n} \frac{(N-n)!}{(N-n-j)!j!} p^j (1-p)^{N-n-j} [x(1+a)^j (1+b)^{N-n-j} - K]^+.$$

We now find the replicating strategy for a call. Let ϕ_n be the number of risky asset at time n , and ϕ_n^0 the number of risk free asset at time n . Then

$$\phi_n^0 (1+r)^n + \phi_n S_n = C(n, S_n).$$

This implies

$$\begin{aligned} \phi_n^0 (1+r)^n + \phi_n S_{n-1} (1+a) &= C(n, S_{n-1} (1+a)) \\ \phi_n^0 (1+r)^n + \phi_n S_{n-1} (1+b) &= C(n, S_{n-1} (1+b)). \end{aligned}$$

Putting $S_{n-1} = x$, we have

$$\phi_n = \phi(n, x) = \frac{C(n, x(1+b)) - C(n, x(1+a))}{x(b-a)}.$$

Finally we derive the asymptotic formula as $N \rightarrow \infty$. We wish to compute the price of a call or put with maturity $T > 0$ on a single stock with strike price K . Put $r = \frac{RT}{N}$, $R =$ instantaneous interest rate (also referred to as the *spot interest rate*). Note that

$$e^{RT} = \lim_{N \rightarrow \infty} (1 + \frac{RT}{N})^N.$$

Also put

$$\ln \left(\frac{1+a}{1+r} \right) = -\frac{\sigma}{\sqrt{N}}.$$

Then using *Central Limit Theorem* we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} P_0^{(N)} &= \lim_{N \rightarrow \infty} \left(1 + \frac{RT}{N}\right)^{-N} E^*[K - S_0 \prod_{n=1}^N T_n] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (K e^{-RT} - S_0 e^{-\frac{\sigma^2}{2+\sigma y}})^+ e^{-\frac{y^2}{2}} dy \\ &= K E^{-RT} \Phi(-d_2) - S_0 \Phi(-d_1) \end{aligned}$$

where

$$d_1 = \frac{(\ln \frac{S_0}{K} + RT + \frac{\sigma^2}{2})}{\sigma}, \quad d_2 = d_1 - \sigma.$$

Exercise 2.1 Consider the CRR model described above. Suppose that $r \leq a$. Show that this leads to an arbitrage opportunity. Deduce the same conclusion if $r \geq b$.

3 Black-Scholes Theory

Let (Ω, \mathcal{F}, P) be a complete probability space and $\{\mathcal{F}_t\}$ a filtration which is right continuous, and complete w.r.t P . Let \mathcal{Y} be the class of stochastic processes $\phi = \{\phi_t, t \geq 0\}$ of the form

$$\phi_t(\omega) = \psi_0(\omega)I_{\{0\}}(t) + \sum_{j=1}^{n-1} \psi_j(\omega)I_{(t_j, t_{j+1}]}(t)$$

where $0 = t_0 < t_1 < \dots < t_n$, ψ_j is a bounded T_{t_j} - adapted random variable for $0 \leq j < n$, $n \geq 1$. Let $\bar{\Omega} = [0, \infty] \times \Omega$. The predictable σ -field \mathcal{P} is defined to be the smallest σ -field on $\bar{\Omega}$ with respect to which every element of \mathcal{Y} is measurable.

We consider a market consisting of one stock and one bond. The stock price $S = \{S_t, t \geq 0\}$ is assumed to follow a geometric Brownian motion (gBM) given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0, \quad (3.1)$$

i.e.,

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} \quad (3.2)$$

where $\{W_t\}$ is a standard Brownian motion. Without any loss of generality we assume that $\{\mathcal{F}_t\} = \{\mathcal{F}_t^W\}$, i.e., the basic filtration is generated by the Brownian motion.

Since $ES_t = S_0 e^{\mu t}$, μ may be treated as the mean growth rate of return from the stock; $\sigma > 0$ is the volatility. It measures the *standard deviation* of S_t in the logarithmic scale. Indeed, the standard deviation of $\log S_t$ is $\sigma\sqrt{t}$. The price of the bond at time t is given by

$$B_t = e^{-rt} \quad (3.3)$$

where r is the rate of interest which is assumed to be constant. One can show that this market is *complete*, i.e., any *contingent claim* can be replicated by an admissible strategy. We assume that the market is *viable*, i.e., there does not exist any arbitrage opportunity. We derive the price of an option on the stock S_t .

A trading strategy is a pair $\phi = (\phi^0, \phi^1)$ of predictable processes, where ϕ_t^0 is the number of bonds and ϕ_t^1 is the number of stocks the investor holds at time t . The value of the portfolio corresponding to the strategy $\phi = (\phi_t^0, \phi_t^1)$ at time t is given by

$$V_t(\phi) := \phi_t^0 B_t + \phi_t^1 S_t. \quad (3.4)$$

The strategy $\phi = (\phi^0, \phi^1)$ is self-financing if no fresh investment is made at any time $t > 0$ and there is no consumption. We work with self-financing strategies only. The gain accrued to the investor via the strategy ϕ up to time t is given by

$$G_t(\phi) := \int_0^t \phi_u^0 dB_u + \int_0^t \phi_u^1 dS_u. \quad (3.5)$$

For a self-financing strategy ϕ :

$$V_t(\phi) := V_0(\phi) + G_t(\phi). \quad (3.6)$$

Suppose that a European call option with terminal time T and strike price K itself is traded in the market. Let C_t denote the price of this call option at time t . Then

$$C_t = C(t, S_t) \tag{3.7}$$

for some function $C : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$

The augmented market with B_t, S_t, C_t is assumed to be viable. The following technical result plays an important role in deriving Black-Scholes partial differential equation. For a proof we refer to the book [2].

Lemma 3.1 Suppose $\phi = (\phi^0, \phi^1)$ is a self-financing strategy such that

$$V_T(\phi) = C(T, S_T) = (S_T - K)^+.$$

Then

$$V_t(\phi) = C(t, S_t) \quad \forall t \in [0, T].$$

Black-Scholes Partial Differential Equations

We look for a strategy $\phi = (\phi^0, \phi^1)$ such that

$$C_T = V_T(\phi).$$

In view of the previous lemma $C_t = V_t(\phi)$ for all t . Further we assume that $C(t, x)$ is a smooth function of t and x . Now

$$\begin{aligned} V_t = V_t(\phi) &= \phi_t^0 B_t + \phi_t^1 S_t \\ &= V_0(\phi) + \int_0^t \phi_u^0 dB_u + \int_0^t \phi_u^1 dS_u. \end{aligned}$$

Therefore

$$dV_t = \phi_t^0 dB_t + \phi_t^1 dS_t.$$

Using (3.1) and (3.3) it follows that

$$dV_t = r\phi_t^0 B_t dt + \phi_t^1 (\mu S_t dt + \sigma S_t dW_t)$$

i.e.,

$$dV_t = (r\phi_t^0 B_t + \mu\phi_t^1 S_t) dt + \sigma\phi_t^1 S_t dW_t. \tag{3.8}$$

Since $C(t, x)$ is a smooth function, by Ito's formula

$$\begin{aligned} dC_t &= dC(t, S_t) \\ &= \left\{ \frac{\partial}{\partial t} C(t, S_t) + \mu \frac{\partial}{\partial x} C(t, S_t) S_t + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2}{\partial x^2} C(t, S_t) \right\} dt \\ &+ \sigma \frac{\partial}{\partial x} C(t, S_t) S_t dW_t. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), and the above lemma, we get

$$\phi_t^1 = \frac{\partial C(t, S_t)}{\partial x} \quad (3.10)$$

and

$$r\phi_t^0 B_t + \mu\phi_t^1 S_t = \frac{\partial C(t, S_t)}{\partial t} + \mu \frac{\partial C(t, S_t)}{\partial x} S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t, S_t)}{\partial x^2}. \quad (3.11)$$

From (3.10) and (3.11) we get

$$r\phi_t^0 B_t = \frac{\partial C(t, S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t, S_t)}{\partial x^2}. \quad (3.12)$$

Again

$$\phi_t^1 S_t + \phi_t^0 B_t = V_t(\phi) = C(t, S_t).$$

Therefore

$$\phi_t^0 = \frac{1}{B_t} \left[C(t, S_t) - \frac{\partial C(t, S_t)}{\partial x} S_t \right]. \quad (3.13)$$

Therefore

$$\frac{\partial C(t, S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t, S_t)}{\partial x^2} + rS_t \frac{\partial C(t, S_t)}{\partial x} - rC(t, S_t) = 0, \quad 0 \leq t < T$$

with $C(T, S_T) = (S_T - K)^+$.

Thus the option price process $C_t = C(t, S_t)$ satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + rx \frac{\partial C}{\partial x} - rC = 0 \quad (3.14)$$

with the boundary condition :

$$C(T, x) = (x - K)^+. \quad (3.15)$$

The equation (3.14) is referred to as the Black-Scholes pde.

Explicit Solution of Black-Scholes PDE

Introduce the new variables τ, ς by

$$\tau = \gamma(T - t), \quad \varsigma = \alpha \left\{ \log \frac{x}{K} + \beta(T - t) \right\}, \quad (3.16)$$

where α, β, γ are constants to be chosen later. Define the function $y(\tau, \varsigma)$ by

$$C(t, x) = e^{-r(T-t)} y(\tau, \varsigma). \quad (3.17)$$

Then

$$\begin{aligned} \frac{\partial C}{\partial t} &= r e^{-r(T-t)} y + e^{-r(T-t)} \frac{\partial y}{\partial t} \\ \frac{\partial y}{\partial t} &= \frac{\partial y}{\partial \varsigma} \frac{\partial \varsigma}{\partial t} - \frac{\partial y}{\partial \tau} \gamma = -\alpha\beta \frac{\partial y}{\partial \varsigma} - \gamma \frac{\partial y}{\partial \tau}. \end{aligned}$$

Hence

$$\frac{\partial C}{\partial t} = re^{-r(T-t)}y - \alpha\beta e^{-r(T-t)}\frac{\partial y}{\partial \varsigma} - \gamma e^{-r(T-t)}\frac{\partial y}{\partial \tau}. \quad (3.18)$$

Also

$$\begin{aligned} \frac{\partial C}{\partial x} &= e^{-r(T-t)}\frac{\partial y}{\partial x} = e^{-r(T-t)}\frac{\partial y}{\partial \varsigma}\frac{\partial \varsigma}{\partial x} \\ &= e^{-r(T-t)}\frac{\partial y}{\partial \varsigma}\frac{\alpha}{x}. \end{aligned}$$

Therefore

$$x\frac{\partial C}{\partial x} = \alpha e^{-r(T-t)}\frac{\partial y}{\partial \varsigma}. \quad (3.19)$$

Also

$$\begin{aligned} \frac{\partial^2 C}{\partial x^2} &= e^{-r(T-t)}\frac{\partial}{\partial x}\left[\frac{\partial y}{\partial \varsigma}\frac{\partial \varsigma}{\partial x}\right] \\ &= e^{-r(T-t)}\left[\frac{\partial^2 y}{\partial \varsigma^2}\left(\frac{\partial \varsigma}{\partial x}\right)^2 + \frac{\partial y}{\partial \varsigma}\frac{\partial^2 \varsigma}{\partial x^2}\right]. \end{aligned}$$

Substituting

$$\frac{\partial \varsigma}{\partial x} = \frac{\alpha}{x}, \quad \frac{\partial^2 \varsigma}{\partial x^2} = -\frac{\alpha}{x^2}$$

in the above relation, we obtain

$$\frac{\partial^2 C}{\partial x^2} = e^{-r(T-t)}\left[\frac{\alpha^2}{x^2}\frac{\partial^2 y}{\partial \varsigma^2} - \frac{\alpha}{x^2}\frac{\partial y}{\partial \varsigma}\right].$$

Therefore

$$x^2\frac{\partial^2 C}{\partial x^2} = e^{-r(T-t)}\left[\alpha^2\frac{\partial^2 y}{\partial \varsigma^2} - \alpha\frac{\partial y}{\partial \varsigma}\right]. \quad (3.20)$$

Then we have

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} - rC \\ = e^{-r(T-t)}\left[ry - \alpha\beta\frac{\partial y}{\partial \varsigma} - \gamma\frac{\partial y}{\partial \tau} + \frac{\sigma^2}{2}\left(\alpha^2\frac{\partial^2 y}{\partial \varsigma^2} - \alpha\frac{\partial y}{\partial \varsigma}\right) + r\alpha\frac{\partial y}{\partial \varsigma} - ry\right]. \end{aligned}$$

Simplifying and using (3.14), we obtain

$$-\alpha\beta\frac{\partial y}{\partial \varsigma} - \gamma\frac{\partial y}{\partial \tau} + \frac{\sigma^2}{2}\left[\alpha^2\frac{\partial^2 y}{\partial \varsigma^2} - \alpha\frac{\partial y}{\partial \varsigma}\right] + r\alpha\frac{\partial y}{\partial \varsigma} = 0 \quad (3.21)$$

The coefficient of $\frac{\partial y}{\partial \varsigma}$ in (3.21) is equal to

$$-\alpha\beta - \frac{\alpha\sigma^2}{2} + r\alpha = \alpha\left[r - \beta - \frac{\sigma^2}{2}\right].$$

Choose $\beta = r - \frac{\sigma^2}{2}$. Then (3.21) becomes

$$-\gamma \frac{\partial y}{\partial \tau} + \frac{1}{2} \sigma^2 \alpha^2 \frac{\partial^2 y}{\partial \zeta^2} = 0.$$

Now choose $\gamma = \sigma^2 \alpha^2$ (where $\alpha \neq 0$ is arbitrary). Then Black-Scholes pde (3.14) becomes

$$\frac{\partial y}{\partial \tau} = \frac{1}{2} \frac{\partial^2 y}{\partial \zeta^2}. \quad (3.22)$$

Choosing $\alpha = 1$, the boundary condition becomes:

$$C(T, x) = y \left(0, \log \frac{x}{K} \right).$$

Thus

$$y(0, u) = K (e^u - 1)^+. \quad (3.23)$$

The unique solution of (3.22) with the boundary condition (3.23) (in the class of function not growing faster than e^{ax^2}) is given by

$$\begin{aligned} y(\tau, \varsigma) &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-(u-\varsigma)^2/2\tau} K (e^u - 1)^+ du \\ &= \frac{K}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-(u-\varsigma)^2/2\tau} (e^u - 1)^+ du. \end{aligned}$$

Thus

$$y(\tau, \varsigma) = K e^{\varsigma + \frac{1}{2}\tau} \Phi \left(\frac{\varsigma}{\sqrt{\tau}} + \sqrt{\tau} \right) - K \Phi \left(\frac{\varsigma}{\sqrt{\tau}} \right). \quad (3.24)$$

Now in terms of t, x , note that

$$\alpha = 1, \quad \beta = r - \frac{\sigma^2}{2}, \quad \gamma = \sigma^2.$$

Thus

$$\begin{aligned} e^{\varsigma + \frac{1}{2}\tau} &= \frac{x}{K} e^{r(T-t)} \\ \frac{\varsigma}{\sqrt{\tau}} + \sqrt{\tau} &= \frac{\log \frac{x}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &:= d_1(x, T-t) \end{aligned}$$

and

$$\frac{\varsigma}{\sqrt{\tau}} = \frac{\log \frac{x}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} := d_2(x, T-t). \quad (3.25)$$

Then

$$C(t, x) = x \Phi(d_1(x, T-t)) - K e^{-r(T-t)} \Phi(d_2(x, T-t)). \quad (3.26)$$

Remark 3.1 Note that in deriving Black-Scholes pde, we have chosen

$$\phi_t^0 = \frac{1}{B_t} \left\{ C(t, S_t) - \frac{\partial C}{\partial x}(t, S_t) \right\} \quad (3.27)$$

$$\phi_t^1 = \frac{\partial C(t, S_t)}{\partial x}. \quad (3.28)$$

The value function is then given by

$$V_t(\phi) = \phi_t^0 B_t + \phi_t^1 S_t = C(t, S_t).$$

The gain $G_t(\phi)$ is given by

$$\begin{aligned} G_t(\phi) &= \int_0^t \phi_u^0 dB_u + \int_0^t \phi_u^1 dB_u \\ &= r \int_0^t \left\{ C(u, S_u) - \frac{\partial C(u, S_u)}{\partial x} S_u \right\} du + \int_0^t \frac{\partial C(u, S_u)}{\partial x} dS_u \\ &= \int_0^t \frac{\partial C}{\partial x} dS_u + \int_0^t \left\{ \frac{\partial C}{\partial u} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial x^2} \right\} du \\ &= C(t, S_t) - C(0, S_0) \\ &= V_t(\phi) - V_0(\phi). \end{aligned}$$

Thus $V_t(\phi) = V_0(\phi) + G_t(\phi)$

Therefore $\phi = (\phi^0, \phi^1)$ is self-financing. Since $V_T(\phi) = C(T, S_T)$, it follows that ϕ is a hedging strategy.

4 The Greeks

The derivatives of the function $C(t, x)$ in (3.26) with respect to t, x are referred to as the Greeks.

Three important Greeks are :

Delta Δ

$$\Delta = \Delta(t) := \frac{\partial C(t, x)}{\partial x} = \Phi(d_1(x, T-t)), \quad (4.29)$$

Theta Θ

$$\begin{aligned} \Theta = \Theta(t) &:= \frac{\partial C(t, x)}{\partial t} \\ &= -rK e^{-r(T-t)} \Phi(d_2(x, T-t)) \\ &\quad - \frac{\sigma x}{2\sqrt{T-t}} \Phi'(d_1(x, T-t)), \end{aligned} \quad (4.30)$$

where d_1, d_2 are as in (3.25).

Gamma Γ

$$\begin{aligned}
 \Gamma &= \Gamma(t) = \frac{\partial^2 C(t, x)}{\partial x^2} \\
 &= \Phi'(d_1(x, T-t)) \frac{\partial}{\partial x} d_1(x, T-t) \\
 &= \frac{1}{\sigma x \sqrt{T-t}} \Phi'(d_1(x, T-t)). \tag{4.31}
 \end{aligned}$$

Since Φ and Φ' are always positive Δ and Γ are always positive and Θ is always negative. We now explain the significance of these Greeks. If at time t the stock price is x , then the short option hedge calls for holding $\Delta = \frac{\partial C}{\partial x}(t, x)$ shares of stock, a position whose value is

$$x \frac{\partial C}{\partial x}(t, x) = x \Phi(d_1(x, T-t)).$$

The hedging portfolio value from (3.26) is given by

$$C(t, x) = x \Phi(d_1(x, T-t)) - K e^{-r(T-t)} \Phi(d_2(x, T-t)).$$

Therefore the amount invested in the bond (or money market) must be

$$C(t, x) - x \frac{\partial C}{\partial x}(t, x) = -K e^{-r(T-t)} \Phi(d_2(x, T-t))$$

a negative number. Thus to hedge a short position in a call option one must borrow money. To hedge a long position in a call option one does the opposite, i.e., to hedge a long call position one should hold $-\frac{\partial C}{\partial x}(t, x)$ shares of stock (i.e. have a short position in the stock) and invest $K e^{-r(T-t)} \Phi(d_2(x, T-t))$ in the bond (or money market).

Since $\Delta > 0, \Gamma > 0$, for a fixed t , $C(t, x)$ is increasing and convex in the variable x .

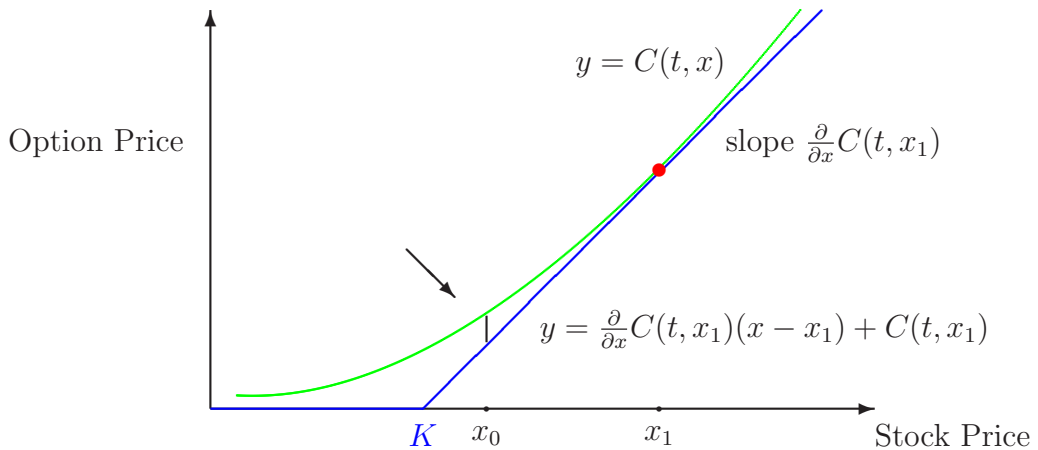


Figure 3: Delta- neutral Position

Suppose at time t the stock price is x_1 , and we wish to take a long position in the option and hedge it. We do this by purchasing the option for $C(t, x_1)$, shorting $\frac{\partial C}{\partial x}(t, x_1)$ shares of stock, which generates an income $x_1 \frac{\partial C}{\partial x}(t, x_1)$, and investing the difference.

$$M = x_1 \frac{\partial C}{\partial x_1}(t, x) - C(t, x_1)$$

in the bond (or money market). We wish to consider the sensitivity to stock price changes of the portfolio that has three components: long option, short stock, and long bond. The initial portfolio value

$$C(t, x_1) - x_1 \frac{\partial C}{\partial x}(t, x) + M$$

is zero at the moment t when we set up these positions. If the stock price were to instantaneously fall to x_0 and we do not change our positions in the stock or bond, then the value of the option we hold would fall to $C(t, x_0)$ and the liability due to our short position in the stock would decrease to $x_0 \frac{\partial C}{\partial x}(t, x_1)$. Our total portfolio value would be

$$C(t, x_0) - x_0 \frac{\partial C}{\partial x} C(t, x_1) + M = C(t, x_0) - \frac{\partial C}{\partial x} C(t, x_1)(x_0 - x_1) - C(t, x_1).$$

This is the difference at x_0 between the curve $y = C(t, x)$ and the line $y = \frac{\partial C}{\partial x} C(t, x_1)(x - x_1) + C(t, x_1)$ (as shown by a vertical segment above x_0 in the figure). Since this difference is positive, our portfolio benefits from an instantaneous drop in the stock price.

On the other hand, if the stock price were to rise instantaneously to x_2 and we do not change our position in the stock or bond, the value of the option would rise to $C(t, x_2)$, and the liability due to our short position in stock would increase to $x_2 \frac{\partial C(t, x_1)}{\partial x}$. Our total portfolio value would be

$$\begin{aligned} C(t, x_2) - x_2 \frac{\partial C(t, x_1)}{\partial x} + M \\ = C(t, x_2) - \frac{\partial C(t, x_1)}{\partial x}(x_2 - x_1) - C(t, x_1). \end{aligned}$$

This is the difference at x_2 between the curve $y = C(t, x)$ and the line $y = \frac{\partial C(t, x_1)}{\partial x}(x_2 - x_1) + C(t, x_1)$ in the above figure. Since the difference is positive, so our portfolio benefits from an instantaneous rise in stock price.

The portfolio we have set up is called *delta-neutral* and *long gamma*. The portfolio is long gamma because it benefits from the convexity of $C(t, x)$ as described above. If there is an instantaneous rise or an instantaneous fall in the stock price, the value of the portfolio increases. A long gamma portfolio is profitable in terms of high stock volatility.

“Delta-neutral” refers to the fact that the line in the above figure is tangent to the curve $y = C(t, x)$. Therefore, when the stock price makes a small move, the change of portfolio value due to the corresponding change in option price is nearly offset by the change in value of our short position in the stock. The straight line is a good approximation to the option price for small stock price moves. If the straight line were steeper than the option price at

the starting point x_1 , then we would be *short delta*; an upward move in the stock price would hurt the portfolio because the liability from the short position in stock would rise faster than the value of the option. On the other hand a downward move would increase the portfolio because the option price would fall more slowly than the rate of decrease in the liability from the short stock position. Unless an investor has a view on the market, he tries to set up portfolios that are delta-neutral. If he expects high volatility, he would at the same time try to choose the portfolio to be long gamma.

The portfolio described above may at first appear to offer an arbitrage opportunity. When we let time move forward, not only does the long gamma position offers an opportunity for profit, but the positive investment in the bond enhances this opportunity. The drawback is that the theta Θ (as in (4.30)) , the derivative of $C(t, x)$ w.r.t time is negative. As we move forward in time, the curve $y = C(t, x)$ is shifting downwards. In principle, the portfolio can lose money because the curve $y = C(t, x)$ is shifting downward more rapidly than the bond investment and the long gamma position generate income. The essence of hedging argument is that if the stock is really a GBM with a known σ , then so long as we continuously re-balance our portfolio, all these effects exactly cancel.

There are two other Greeks vega and rho; $\frac{\partial C}{\partial \sigma}$ is called the vega and $\frac{\partial C}{\partial r}$ is called the rho. With a suitable balance of the underlying stock and other derivatives, hedgers can eliminate the short term dependence of the portfolio movement in time, stock price, volatility or interest rate.

5 Risk Neutral Valuation by a Change of Measure

In this section we revisit the Black-Scholes theory using a martingale method. Let the market be given as in the previous section. Consider the stochastic process $\tilde{W}_t = W_t + qt$ where q is a constant. Under the probability measure P , $\{W_t\}$ is a standard Brownian motion, but \tilde{W}_t is not a standard Brownian motion (for $q \neq 0$). However, if we change the underlying probability measure for an appropriate probability measure \tilde{P} , \tilde{W}_t can be shown to be a standard Brownian motion under \tilde{P} . Using Girsanov's theorem the following holds.

Theorem 5.1 The following statement holds:

- The stochastic process

$$M_t = \exp\{-qW_t - \frac{1}{2}q^2t\}, t \in [0, T]$$

is a martingale with respect to the filtration $\{\mathcal{F}_t\}$ under the probability measure P .

- The relation

$$\tilde{P}(A) = \int_A M_T(\omega) dP(\omega)$$

defines a probability measure \tilde{P} which is equivalent to P . We often write this relationship as $\frac{d\tilde{P}}{dP} = M_T$.

- Under the new probability measure \tilde{P} the process \tilde{W}_t is a standard Brownian motion.

We now get back to the Black-Scholes model. Note that under the market probability measure P

- B_t grows at the rate r
- Expected growth rate of S_t is μ .

We change P to P^* (equivalent to P) under which both B_t and S_t grow at the same rate. Set

$$\frac{dP^*}{dP} = \exp\left(-\int_0^T \frac{\mu - r}{\sigma} dW_t - \frac{1}{2} \int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 dt\right). \quad (5.32)$$

Set

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t.$$

Then $P \equiv P^*$. Under P^* , W_t^* is a standard Brownian motion. Also under P^* , S_t satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

Thus under P^* the growth rates of B_t and S_t are the same.

Let

$$S_t^* := e^{-rt} S_t$$

. Then

$$dS_t^* = \sigma S_t^* dW_t^*.$$

Thus $\{S_t^*\}$ is a martingale under P^* . Hence the market is *arbitrage free*. The probability measure P^* is the risk neutral measure for this model.

The arbitrage free price of call option at time t is given by

$$\varphi_t = E^*[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t].$$

The uniqueness of risk neutral measure P^* gives the completeness of the market, i.e., *every contingent claim is attainable by a self financing strategy*. By Markov property

$$\varphi_t = E^*[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t] = E^*[e^{-r(T-t)}(S_T - K)^+ | S_t] = C(t, S_t).$$

One can then show that the function $C(t, x)$ satisfies the Black-Scholes pde described earlier. Alternatively, since S_t is a gBM with parameters r, σ , one can evaluate the above expectation directly to get the Black-Scholes formula (*Exercise*).

Remark 5.1 In the exercises given below, unless otherwise mentioned, the unit of time should be taken as one year. The starting time of a call is taken to be 0.

Exercise 5.1 Let $C(x, T, \sigma, r, K)$ denote the price of a European call (at time 0). Show that the function C is

- (a) decreasing and convex in K ;
- (b) increasing and convex in x ;
- (c) increasing, but neither convex nor concave, in r, σ and T .

Exercise 5.2 The prices of a certain stock follows a geometric Brownian motion (gBM) with parameters $\mu = 0.4, \sigma = 0.24$. If the present price of the stock is 40, what is the probability that a European call option, having four months until its expiration time and with a strike price of $K = 42$, will be exercised? (A stock whose price at the time of expiration of a call option is above the strike price is said to be finish *in the money*).

Exercise 5.3 To determine the probability that a European call option finishes in the money, is it enough to specify the five parameters K, S_0, r, T , and σ ? Explain your answer; if it is “no”, what else is required?

Exercise 5.4 The price of a certain stock follows a gBM with parameters $\mu = 0.1, \sigma = 0.3$. The present price of the stock is 95.

- (a) If $r = 0.04$, find the no-arbitrage price of a European call option that expires in three months and has exercise price $K = 100$.
- (b) What is the probability that the call option in part (a) is worthless at the time of expiry?

Exercise 5.5 What should the price of a European call option if the strike price is equal to zero?

Exercise 5.6 What should the price of a European call option become as the exercise time becomes larger and larger? Explain your answer with both intuitive reasoning and detailed analysis.

Exercise 5.7 What should be price of a European (K, T) call option become as the volatility parameter becomes smaller and smaller?

Exercise 5.8 Consider a stock whose price is given by

$$dS_t = r(t)S_t dt + \sigma(t)dW^*(t)$$

where $r(t)$ and $\sigma(t)$ are deterministic functions of t and W^* is a Brownian motion under the risk-neutral measure P^* .

- (a) Show that S_T is of the form $S_0 E^X$, where X is a normal random variable. Determine the mean and variance of X .
- (b) Find the risk-neutral valuation of a European call option on S_T .
- (c4) Let $C(T, x, K, r, \sigma)$ denote value of a European call option at time zero with parameters T, x, K, r, σ in Black-Scholes model. Show that $C(T, S_0, K, \frac{1}{T} \int_0^T r(t) dt, \sqrt{\frac{1}{T} \int_0^T \sigma^2(t) dt})$ is a risk-neutral valuation of the option price in the above model.

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