What Does An Option Price Mean?

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- The standard answer is that option prices are the discounted risk-neutral mean of their payoff.
- To make this precise, let $P_t(K,T)$ and $C_t(K,T)$ denote the respective time $t \in [0,T]$ prices of a European put and call paying off $(K - S_T)^+$ and $(S_T - K)^+$ at T.
- Let $B_t(T)$ be the time $t \in [0,T]$ price of a zero coupon bond paying \$1 at T and let $\mathbb{Q}_T$ denote the associated forward measure.
- Then the forward prices of the put and call are just the means under this measure of their terminal payoff:

$$\hat{P}_T(K,T) \equiv \frac{P_t(K,T)}{B_t(T)} = E_t^{\mathbb{Q}_T}(K - S_T)^+$$

$$\hat{C}_t(K,T) \equiv \frac{C_t(K,T)}{B_t(T)} = E_t^{\mathbb{Q}_T}(S_T - K)^+.$$

Is That All They Mean?

- The conventional wisdom is that European option prices only convey information regarding the terminal risk-neutral distribution of the underlying asset price.
- However, we know that in classical models with independent stochastic volatility, the risk-neutral mean of the *volatility* realized over $[0, T]$ is approximately proportional to the forward price of an ATMF option:

$$E_t^{\mathbb{Q}_T} \sqrt{\frac{\int_0^T (dF_t/F_t)^2}{T}} \approx \sqrt{\frac{2\pi}{T}} \frac{\hat{C}_0(F_0, T)}{F_0}.$$

- Ignoring the typically tiny approximation error, this example illustrates that the (risk-neutral) mean of a path statistic (i.e. realized volatility) can be inferred from an option's price, provided that structure is imposed on the underlying price process (in this case, path continuity and independent volatility dynamics).

Does an Option Price Only Forecast Volatility?

- No! This presentation will show that by imposing weak structure on the underlying price process, option prices can convey the risk-neutral mean of several path statistics.
- All of our results arise from the ability to perfectly replicate the path-dependent payoff even though markets are incomplete. Time permitting, the replicating strategies will be explicated.
- We will also consider the information content of some simple option portfolios, eg. strangles, but we respect the reality that option strikes and maturities are discrete.

Def'n: Zero Risk-Neutral Drift

- Our first set of results assumes only that the underlying price process has zero risk-neutral drift.
- If the underlying is a spot price, then this implies quite strongly that the riskfree rate is the dividend yield.
- If the underlying is a forward price instead, then no restrictions are being placed on the dynamics.
- As a result, we henceforth will assume that the relevant path statistics are computed from forward prices denoted by $\{F_t, t \in [0, T]\}$. The forward contract has a maturity date T, but we don't indicate this argument for simplicity.

Def'n: (Forward) Time Value

- Time value of an option is classically defined as the difference between an option's price and its intrinsic value. It will be useful to introduce forward option prices and the corresponding notion of forward time value.
- Recall that $B_0(T)$ is the price of a zero coupon bond paying \$1 at T and that $\mathbb{Q}_T$ denotes the associated forward measure.
- Recall that $P_0(K,T) \equiv B_0(T)E^{\mathbb{Q}_T}(K - S_T)^+$ and $C_0(K,T) \equiv B_0(T)E^{\mathbb{Q}_T}(S_T - K)^+$ respectively denote the initial spot prices of a European put and call struck at K and maturing at T.
- Also recall that $\hat{P}_0(K,T) \equiv \frac{P_0(K,T)}{B_0(T)}$ and $\hat{C}_0(K,T) \equiv \frac{C_0(K,T)}{B_0(T)}$ respectively denote the initial forward prices of the put and call.
- Let $\theta(K,T) \equiv \min[\hat{P}_0(K,T), \hat{C}_0(K,T)]$ be the (forward) time value of an option. From put call parity, $\theta(K,T)$ is independent of the option's polarity:

$$\theta(K,T) = \hat{P}_0(K,T) - (K - F_0)^+ = \hat{C}_0(K,T) - (F_0 - K)^+.$$

Def'n: Overshoots

- Suppose that the valuation time is t_0 and consider a set of $n \geq 1$ future times $t_1, \ldots, t_n$, where $t_n \equiv T$ corresponds to the maturity of an option.
- Let $\{F_0, F_1, \ldots, F_n\}$ denote the corresponding forward prices and define the overshoots of the strike K over $[0, T]$ as the path-dependent payoff:

$$\mathcal{O}(K, T, n) \equiv \sum_{i=1}^{n} [1(F_{i-1} \leq K)(F_i - K)^+ + 1(F_{i-1} > K)(K - F_i)^+].$$

- In words, F overshoots K between t_{i-1} and t_i if and only if the forward price jumps over the strike K from above or below. In either case, the extent of the overshoot is the unsigned difference between the landing price F_i and the strike K.

Th'm: Time Value as Mean Overshoots

- A discrete form of the Tanaka Meyer formula is given by the following tautology:

$$(F_n - K)^+ = (F_0 - K)^+ + \sum_{i=1}^{n} 1(F_{i-1} > K)(F_i - F_{i-1}) + \mathcal{O}(K, T, n),$$

where recall that the overshoots by F of the strike K over $[0, T]$ is the path-dependent payoff:

$$\mathcal{O}(K, T, n) \equiv \sum_{i=1}^{n} [1(F_{i-1} \leq K)(F_i - K)^+ + 1(F_{i-1} > K)(K - F_i)^+].$$

- Taking the risk-neutral mean of both sides of the top equation, no arbitrage implies that *forward time value means overshoots:*

$$\theta(K, T) \equiv \hat{C}_0(K, T) - (F_0 - K)^+ = E_0^{\mathbb{Q}_T} \mathcal{O}(K, T, n).$$

Implications

- Recall that forward time value means overshoots:

$$\theta(K,T) \equiv \hat{C}_0(K,T) - (F_0 - K)^+ = E_0^{\mathbb{Q}_T} \mathcal{O}(K,T,n).$$

- Loosely speaking, the more often an underlying overshoots K over $[0,T]$, the greater is the time value of the option struck at K and maturing at T.
- Notice that the LHS is independent of n and hence so is the RHS.
- As n approaches infinity, the limiting value of $\mathcal{O}(K,T,n)$ can be zero or finite, depending on the nature of the underlying price process.
- If the underlying forward price process has continuous sample paths of bounded variation, then the limiting value of $\mathcal{O}(K,T,n)$ is zero, and options written on this process have zero time value.

Generalization

- Carr and Lee develop the notion of multiply exerciseable American options (called *hyper options*) which generalize the above result.
- In a hyper option, the n fixed times $t_1, \ldots, t_n$ are generalized to N stopping times, $\tau_1, \ldots, \tau_N$ which are chosen by the buyer of the hyper option. Each stopping time τ_i results in a payoff at T to the hyper option holder given by either $F_i - K$ if the hyper option is exercised as a call, or $K - F_i$ if the hyper option is exercised as a put.
- The initial polarity of the hyper option is determined by the hyper option seller, but this polarity switches sign upon each exercise by the buyer.
- Despite the fact that hyper options can be optimally exercised early an arbitrarily large number of times, the arbitrage-free value of a hyper option on the forward price is simply given by the price of the European option of the same polarity.

Def'n: Zero Risk-Neutral Drift with One Level First Kissed

- Our next set of results assumes that there is some distinguished spatial level K, for which the forward price F can only *first kiss*.
- If we let τ_K denote the first time that F crosses K, this means that $F_\tau = K$. The forward price can jump over K as often as it likes after this first kissing time.
- Note that the imposition of the existence of K is the weakest possible restriction in the direction of imposing sample path continuity.

Th'm: OTM Put as Down-and-In Call w. Strike = Barrier

- A *down-and-in call* with lower barrier $L \leq F_0$, strike $K \geq L$, and maturity T pays $1(\tau_L < T)(F_T - K)^+$ at T, where τ_L denotes the first time that F crosses L.
- Suppose that $K = L$ and that L is first kissed. Then no arbitrage $\Rightarrow$:

$$E_0^{\mathbb{Q}^T}[1(\tau_L < T)(F_T - L)^+] = \hat{P}_0(L, T).$$

  Hence, an out-of-the money put has the same value as a down-and-in call whose barrier is the put's strike.
- In other words, the price of an OTM put means a down-and-in call's payoff.

Implications

- Recall that if $K = L$ and if L is first kissed, then no arbitrage $\Rightarrow$:

$$E_0^{\mathbb{Q}_T}[1(\tau_L < T)(F_T - L)^+] = \hat{P}_0(L, T).$$

- Loosely speaking, the more likely it is that the underlying forward price drops below L and then rises above L before T, the more valuable is the out-of-the money put with strike L and maturity T.
- If the barrier L can be jumped over, then assuming that the ITM put with strike L has more value at τ_L than the OTM call with the same strike, no arbitrage implies:

$$E_0^{\mathbb{Q}_T}[1(\tau_L < T)(F_T - L)^+] \leq \hat{P}_0(L, T).$$

- It is interesting to note that if we take the initial put price as given, then increasing the number of ways that the down-and-in call can knock in has the effect of lowering its value.

Def'n: Zero Risk-Neutral Drift with Two Levels Skipfree

- Our next set of results assumes that there are two distinguished spatial levels K and $H \geq K$, which the forward price F can never jump over.
- If we let τ_{KH} denote the first time that F exits the interval $[K, H]$, this means that either $F_{\tau_{KH}} = K$ or $F_{\tau_{KH}} = H$.

Def'n: Upcrosses

- Consider an interval (K, H) and a time set $[0, T]$.
- An upcross of the interval is completed at the first passage time of the underlying forward price from some level at or below K to some level at or above H. The underlying has to return to K or below to set up for another upcross.
- An upcross is partially completed at maturity if the underlying finishes in (K, H). The fraction of the upcross partially completed at maturity is then $\frac{F_T - K}{H - K}$.
- Consider as a path statistic the number of upcrosses, both completed and partial.

Th'm: Scaled Time Value as Mean Upcrosses

- Suppose we restrict the F dynamics so that K and H are skipfree over $[0, T]$.
- Consider a hyper call with $F_0 \leq K$ and suppose that we require that it be exercised as a call each time F first reaches H after touching K and furthermore, that it be exercised as a put when F subsequently reaches K.
- If we scale everything by $\frac{1}{H-K}$, then the total (forced) exercise proceeds matches the number of upcrosses, including any partial upcross at expiry.
- If $F_0 > K$, then we would consider a hyper put and force the first exercise at the first passage time to K rather than H. Afterwards, we would force exercise as above.
- Let $U(K, H, T)$ be the number of upcrosses by F of the interval $[K, H]$ in the time set $[0, T]$. No arbitrage implies:

$$E_0^{\mathbb{Q}_\mathbb{T}} U(K, H, T) = \frac{\hat{\Theta}_0(K, T)}{H - K},$$

where recall $\hat{\Theta}_0(K, T)$ is the forward time value of a European option with strike K and maturity T. Remarkably, $(H - K)E_0^{\mathbb{Q}_\mathbb{T}} U(K, H, T)$ is independent of H.

Pf: Scaled Time Value as Mean Upcrosses

- Assuming only zero risk-neutral drift and that K and H are skipfree, we now show that one can replicate a claim paying the number of upcrosses (both complete and partial) of a given interval (K, H).
- For example, if the interval is (K=100,H=110) and the underlying completes 3 upcrosses and finishes at 107, then the payoff on the claim paying the number of upcrosses is $3.70.
- To formally define the payoff of an upcrosser, let $\tau_0 \equiv 0$ and for $i = 1, 2, \ldots,$ recursively define the stopping times σ_n and τ_n by:

$$\sigma_i \equiv \inf\{i \geq \tau_{i-1} : F_t \leq K\} \qquad \tau_i \equiv \inf\{t \geq \sigma_i : F_t \geq H\},$$

where we adopt the usual convention that the infimum of the empty set is infinity. If we adopt the dual convention that the supremum of the empty set is zero, then the number of completed upcrosses by time t is:

$$n_t^u \equiv \max\{i : \tau_i \leq t\}, \qquad t \in [0, T].$$

Pf: Scaled Time Value as Mean Upcrosses (Con'd)

- At any time t, it will be useful to know whether or not F has been at or below K since the last upcross, if any. Accordingly, we also define:

$$n_t^d \equiv \max\{i : \sigma_i \leq t\}, \qquad t \in [0, T],$$

but stress that it is not necessarily the number of completed downcrosses.

- If $n_t^d = n_t^u$, then at time t, the first requirement for the next upcross has not been met, while if $n_t^d > n_t^u$, then at time t, it has.
- Let $V_t^u(K, H, T)$ denote the arbitrage-free value of an upcrosser at time $t \in [0, T]$.

$$V_T^u(K, H, T) = n_T^u + 1(n_T^d > n_T^u, F_T < H)\frac{(F_T - K)^+}{H - K}.$$

The last term gives credit for a partially completed upcross, if any.

Pf: Scaled Time Value as Mean Upcrosses (Con'd)

- We assume nothing about riskfree rates and dividends. When F can not skip over H or K, semi-static trading in European options replicates the payoff to an upcrosser perfectly. As a result, no arbitrage implies:

$$\begin{aligned} V_t^u(K,H,T) &= \frac{1}{H-K}\left[1(F_t \le K) + 1(n_t^d > n_t^u, F_t \in (K,H))\right] C_t(K,T) \\ &+ \frac{1}{H-K}\left[1(F_t \ge H) + 1(n_t^d = n_t^u, F_t \in (K,H))\right] P_t(K,T), \end{aligned}$$

- First suppose that $F_0 \le K$, so that the writer of the upcrosser uses the sale proceeds to buy $\frac{1}{H-K}$ calls of strike K. Since $F_0 \le K$, the calls are out-of-the-money (OTM). If the forward price never hits H before maturity and finishes below K, then the investor has no liability and the call finishes OTM.
- If the forward price never hits H before maturity but finishes between K and H, then the payoff from the calls covers the partially completed upcross.

Pf: Scaled Time Value as Mean Upcrosses (Con'd)

- If the forward price does touch H before maturity, then at the first time before maturity that it does so, the investor sells the $\frac{1}{H-K}$ calls and buys $\frac{1}{H-K}$ puts. By put call parity:

$$C_t(K,T) - P_t(K,T) = B_t(F_t - K),$$

  this conversion results in enough money to buy $\frac{F_{\tau_1}-K}{H-K}$ bonds. Since $F_{\tau_1} = H$ under the skipfree assumption, one bond is purchased which just covers the increase in the intrinsic value of the investor's terminal liability due to the completed upcross.
- If the forward price never returns to K before maturity, then these $\frac{1}{H-K}$ puts expire worthless, but the liability of one dollar is covered.
- If after the first upcross, the forward price touches K before maturity, then at the first time that it does so, the investors sells the $\frac{1}{H-K}$ puts and buys $\frac{1}{H-K}$ calls. Since $F_{\sigma_1} = K$, PCP implies that this reversal is self financing. After this trade, the investor is left holding an OTM call with the forward price at K. As this was the investor's initial position, we are done when $F_0 \leq K$.

Pf: Scaled Time Value as Mean Upcrosses (Con'd)

- Now, consider the trading strategy when $F_0 > K$. The writer of the upcrosser can use the proceeds from the sale to buy $\frac{1}{H-K}$ OTM puts struck at K. If the forward price never hits or touches K, then no liability is due and the puts expire worthless.
- If the forward price does hit K, then at the first time prior to maturity that it does so, the investor sells the $\frac{1}{H-K}$ puts and buys $\frac{1}{H-K}$ calls. Afterwards, the investor is in the same position as when $F_0 \leq K$. Hence the investor can follow the strategy described above from then on.
- Thus, from then on the investor always holds $\frac{1}{H-K}$ calls while below K and $\frac{1}{H-K}$ puts while above H. What is held between K and H is exactly what was held when the corridor was last entered. Thus, if the corridor was last entered from below, the investor holds $\frac{1}{H-K}$ calls and if the corridor was last entered from above, the investor holds $\frac{1}{H-K}$ puts instead.
- We conclude that we have constructed a simple trading strategy in European options which replicates the payoff.

Def'n: Downcrosses

- Consider an interval (L, K) and a time set $[0, T]$.
- A downcross of the interval is completed at the first passage time of the underlying forward price from some level at or above K to some level at or below L. The underlying has to return to K or above to set up for another downcross.
- A downcross is partially completed at maturity if the underlying finishes in (L, K). The fraction of the downcross partially completed at maturity is then $\frac{K-F_T}{K-L}$.
- Consider as a path-statistic the number of downcrosses, both completed and partial.

Th'm: Scaled Time Value as Mean Downcrosses

- Now we restrict the forward price dynamics so that L and K are skipfree.
- Consider a hyper put with $F_0 \geq K$ and suppose that we require that it be exercised as a put each time F first reaches L after first touching K and furthermore, that it be exercised as a call when F subsequently reaches K.
- If we scale everything by $\frac{1}{K-L}$, then the total (forced) exercise proceeds matches the number of downcrosses.
- If $F_0 < K$, then we would consider a hyper call and force the first exercise at the first passage time to K rather than L. Afterwards, we would force exercise as above.
- Letting $D(L, K, T)$ be the number of downcrosses by F of the interval $[K, H]$ in the time set $[0, T]$. No arbitrage implies:

$$E_0^{\mathbb{Q}_{\mathbb{T}}} D(L, K, T) = \frac{\hat{\Theta}_0(K, T)}{K - L},$$

where recall that $\hat{\Theta}_0(K, T)$ is the forward time value of a European option with strike K and maturity T. Remarkably, $(K - L)E_0^{\mathbb{Q}_{\mathbb{T}}} D(L, K, T)$ is independent of L.

Def'n: Zero Risk-Neutral Drift with Three Levels Skipfree

- Our next set of results assumes that there are three distinguished spatial levels L, $K \geq L$ and $H \geq K$, which the forward price F can never jump over.
- By combining results on upcrosses and downcrosses, we can simultaneously generalize the previous results on each.

Def'n: Exits and Local Return Variance

- Consider an interval (L, H) containing an option strike $K > 0$ and suppose that the forward price cannot jump over L, K, or H.
- Consider the payoff at T of $d \cdot D(L, K, T) + u \cdot U(K, H, T)$, where u and d are arbitrary real constants.
- As $D(L, K, T)$ and $U(K, H, T)$ can both be replicated using a T maturity option struck at K and its underlying asset, it follows that any linear combination of them is also replicable with the same assets.
- If $u = d = 1$, then the replicable payoff is the number of exits of the interval (L, H) which emanate from K.
- If $u = \frac{H-K}{K} > 0$ and $d = \frac{L-K}{K} < 0$, then the replicable payoff is the local return at K realized upon exiting (L, H).
- Suppose $u = f\left(\frac{H-K}{K}\right)$ and $d = f\left(\frac{L-K}{K}\right)$ for some function $f(R)$ which is bounded for $R \in \left(\frac{L-K}{K}, \frac{H-K}{K}\right)$. Then the replicable payoff is any function f of the local return at K realized upon exiting (L, H).
- In particular, by setting $f(R) = R^2$, one can replicate the local return variance at the strike K.

The General Replication Recipe

- We now show how to replicate the general payoff $d \cdot D(L, K, T) + u \cdot U(K, H, T)$, where u and d are arbitrary real constants.
- In general, the replicating strategy combines a static position in an initially OTM option of strike K, along with semi-dynamic trading in the underlying futures. The option position is independent of the polarity, while the futures trading strategy depends on it. The futures trading strategy also depends on a pure jump process F^-, which is the futures price at the last trade.
- If $F_0 \leq K$, then one holds $\frac{d}{K-L} + \frac{u}{H-K}$ initially OTM calls and one is also short $\frac{d1(F_t^- = K)}{K-L} + \frac{u1(F_t^- > K)}{H-K}$ futures contracts at each time $t \in [0, T]$.
- If $F_0 > K$, then one holds $\frac{d}{K-L} + \frac{u}{H-K}$ initially OTM puts, but one is now long $\frac{u1(F_t^- = K)}{H-K} + \frac{d1(F_t^- < K)}{K-L}$ futures contracts at each time $t \in [0, T]$.
- The arbitrage-free price paid at time 0 is the initial replication cost $\left(\frac{d}{K-L} + \frac{u}{H-K}\right) TV_0^e(K)$.

Interpreting the General Replication Recipe

- It can be shown that when calls are held long, then forwards are held short, while when puts are held long, then forwards are also held long.
- The forwards trading strategy can be interpreted as a poor man's delta-hedge of the initially OTM option.
- The terminology arises because the forward trading strategy is independent of both any estimate of volatility, either historical or implied. Also the number of forwards held takes at most three values and is only changed infrequently. As a result, it will not replicate the initially OTM option payoff.
- However, the option replication error does replicate the desired payoff, hopefully making the poor man rich.

Th'm: Scaled Time Value as Local Return Variance

- Applying the general replicating recipe when $d = \left(\frac{K-L}{K}\right)^2$ and $u = \left(\frac{H-K}{K}\right)^2$, for $L \leq K \leq H$, we have that the arbitrage-free price to pay at time 0 for the local return variance paid at T is $\left(\frac{K-L}{K^2} + \frac{H-K}{K^2}\right) TV_0^e(K)$.
- This price reflects just the static component, which is $\left(\frac{K-L}{K^2} + \frac{H-K}{K^2}\right)$ OTM options.
- If $F_0 \leq K$, then one is also dynamically short $\frac{(K-L)1(F_t^- = K)}{K^2} + \frac{(H-K)1(F_t^- > K)}{K^2}$ forward contracts at each time $t \in [0, T]$.
- If $F_0 > K$, then one is instead dynamically long $\frac{(H-K)1(F_t^- = K)}{K^2} + \frac{(K-L)1(F_t^- < K)}{K^2}$ forward contracts at each time $t \in [0, T]$.
- One can alternatively set $d = \left[\ln\left(\frac{L}{K}\right)\right]^2$ and $u = \left[\ln\left(\frac{H}{K}\right)\right]^2$, provided that $L > 0$. The details of the replicating strategy are left to you.

Def'n: Running Maximum Never Jumps Up

- Our next set of results imposes a pair of restrictions on the risk-neutral process governing the forward price F of the underlying asset.
- First, we explore an alternative way to impose a mild version of sample path continuity.
- Instead of assuming that some spatial levels are skipfree, we will instead forbid up jumps in the path of the running maximum of F.
- Hence, at each $t \in [0, T]$, we allow the possibility of down jumps in F_t and we allow the possibility of up jumps of limited size in F_t, but we give zero probability to up jumps in F_t which are sufficiently large so that $M_t \equiv \max_{s \in [0,t]} F_s$ could increase by a jump.

Def'n: Arithmetic Put Call Symmetry

- We will need a second restriction on the risk-neutral process governing the forward price of the underlying asset.
- We say that Arithmetic Put Call Symmetry (APCS) holds at a particular time $t \geq 0$ for a particular maturity $T \geq t$ if a put maturing at T has the same market price as the co-terminal call struck the same distance away from F_t:

  $$P_t(K_p, T) = C_t(K_c, T),$$

  for all strikes K_p, K_c satisfying: $K_p = F_t + \triangle K, K_c = F_t - \triangle K$, where $\triangle K \in \mathbb{R}$.
- APCS implies that options of the same moneyness have the same value.

Def'n: Call on Terminal Drawdown

- Terminal drawdown is defined as the difference between the continuously-monitored maximum of some asset price over some period $[0, T]$ and the asset's final price at T. It measures the ex post regret from selling the asset for S_T at T, rather than for its historical maximum $M_T \equiv \max_{t \in [0,T]} S_t$ over $[0, T]$.
- A European call written on the terminal drawdown $DD_T \equiv M_T - S_T$ is a contingent claim that pays $(DD_T - K)^+$ at T. A long position in this call provides insurance against large realizations of terminal drawdown.

Ass'n: Maximum Never Jumps and APCS

- We now formally assume that the running maximum M of the forward price never increases by a jump, and that Arithmetic Put Call Symmetry (APCS) holds at all times t for which the running maximum increases.
- All of our requirements are met by the class of Ocone martingales:

$$dF_t = a_t dW_t, \qquad t \in [0, T]$$

where the absolute volatility process a evolves independently of W.

- Setting $a_t = a$ leads to the Bachelier model for the forward price.

Th'm: Strangle as Mean Payoff from a Drawdown Call

- Suppose that F is a $\mathbb{Q}_T$ martingale whose running maximum is continuous and for which APCS holds at all times τ when $dM_\tau > 0$. Then under frictionless markets, no arbitrage implies that for $K_d \geq 0, t \in [0, T]$:

$$E_t^{\mathbb{Q}_T}(DD_T - K_d)^+ = \hat{P}_t(M_t - K_d, T) + \hat{C}_t(M_t + K_d, T), \qquad t \in [0, T], K_d \geq 0,$$

where recall that $DD_T \equiv_{t \in [0,T]}^{\max} S_t - S_T$ is the terminal drawdown.

- In words, a drawdown call is replicated by always holding a strangle centered at the running maximum M_t, and whose width is the strike K_d of the drawdown call.

- The strategy is self-financing because the cash outflow required to move the put strike up when the running maximum increases infinitessimally is financed by the cash inflow received from moving the call strike up (given that APCS is in fact holding at such times).

- Setting $K_d = 0$ yields the result that a straddle struck at the running maximum is just the mean of the terminal drawdown.

Def'n: Calls on Trading Gains eg. Passport Options

- Consider a dynamic trading strategy in a single risky asset for which shareholdings oscillate randomly between ± 1.
- Now consider European call options written on the gains from such a binary trading strategy. An example is given by passport options, which are over-the-counter options written on the gains from a dynamic strategy for which shareholdings can vary in the interval $[-1, 1]$.
- Since the payoff is convex, the optimal trading strategy can be shown to be binary, and hence passport options can be correctly valued as a call option written on the gains from a binary trading strategy .

Passport Option Term Sheet

Merrill Lynch

GLOBAL FOREIGN EXCHANGE

30th April, 2004

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EURUSD Passport Option

Receive the net cumulative profit from spot positions of up to EUR10mio in EURUSD, with losses underwritten

Counterparty A: Counterparty B:	Merrill Lynch Capital Services
Notional:	EUR10mio
Trade Date: Settlement Date:	30th April, 2004 4th May, 2004
Expiry Date: Maturity Date:	2nd August, 2004 4th August, 2004
Counterparty B pays:	2.95% * Notional on Settlement Date
Counterparty A pays	max (0, Net USD value of permitted spot trades) on Maturity Date
Permitted Spot Trades	Counterparty B may have long or short position of no greater than Notional in EURUSD, from the Trade date to the Valuation Date, subject to the trading conditions.
Trading Conditions	Counterparty B may alter this position up to 3 times a day. Counterparty B must execute all position amendments through Counterparty A All amendments must be carried out between 8am and 5pm London time, or on an order basis
Calculation Agent:	Merrill Lynch Capital Services

Def'n: Running P&L from a Binary Trading Strategy

- A dynamic trading strategy is *binary* when the shareholdings can only be ± 1.
- The running P&L π_t from a binary trading strategy is defined by $\pi_t \equiv \int_0^t c_s dF_s, \in [0, T]$, where $c_s = \pm 1$ and F has zero risk-neutral drift.
- A call written on the terminal P&L from a binary trading strategy has a terminal payoff at T of $(\pi_T - k)^+$, where $k \in \mathbb{R}$ is the strike price.

Th'm: Put or Call as Passport Call

- Assume that interest rates and dividends are zero and that the price process F defining the reference binary strategy is a $\mathbb{Q}_T$ martingale for which APCS holds at all times τ when parity changes.
- Under frictionless markets, no arbitrage $\Rightarrow$ for $k \in \mathbb{R}, t \in [0, T]$:

$$E_t^{\mathbb{Q}_\mathbb{T}}(\pi_T - k)^+ = 1(c_t = 1)C_t(F_t^- - \pi_t^- + k, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - k, T),$$

where recall $\pi_T \equiv \int_0^T c_s dF_s$, and where F^- and π^- indicate the forward price and P&L at the time of the last trade at or before t.

Interpretation

- Under the assumptions of the last slide:

$$E_t^{\mathbb{Q}_\mathbb{T}}(\pi_T - k)^+ = 1(c_t = 1)C_t(F_t^- - \pi_t^- + k, T) + 1(c_t = -1)P_t(F_t^- + \pi_t^- - k, T),$$

  where recall $\pi_T \equiv \int_0^T c_s dF_s$, and where F^- and π^- indicate the forward price and P&L at the time of the last trade at or before t.

- Define the *parity* at t of the gains process π as the call indicator c_t. Then the parity of the standard option held at t (i.e. call or put) matches the parity at t of the underlying gains process.

- Define the moneyness of the passport call as $\pi_t - k$, the moneyness of the standard call as $F_t - K$, and the moneyness of the standard put as $K - F_t$. Then the result states that the passport call always has the same value as the standard option with the same parity and with the same moneyness as the passport at the last switch time.

- Notice that jumps of arbitrary size and sign are allowed.

Passport Put Options

- Now consider a *put* option on the gains from a binary trading strategy.
- At its expiry T, the payoff is $(k - \pi_T)^+$, where $k \in \mathbb{R}$ is the put strike.
- One can show that the passport put always has the same value as a standard option with the same moneyness at the last switch time. The parity of the standard option held at t is now the *opposite* of the parity at t of the gains process.
- Hence, under the same assumptions as for a passport call, no arbitrage $\Rightarrow$ for $k \in \mathbb{R}, t \in [0, T]$:

$$E_t^{\mathbb{Q}_T}(k-\pi_T)^+ = 1(c_t = -1)C_t(F_t^- + \pi_t^- - k, T) + 1(c_t = 1)P_t(F_t^- - \pi_t^- + k, T),$$

where F^- and π^- indicate the forward price and P&L at the time of the last trade at or before t.

Ass'n: Barrier First Kissed and APCS

- Recall that we could represent an OTM put as a down-and-in call whose barrier was equal to its strike K provided that the forward price F could only *first kiss* K.
- We can also represent an OTM put as a down-and-in call whose barrier L is less than its strike K, but we have to also assume that Arithmetic Put Call Symmetry (APCS) holds if and when the barrier L is first kissed.

Th'm: OTM Put as a Down-and-In Call

- Recall that a down-and-in call with lower barrier $L \leq F_0$, strike $K \geq L$, and maturity T pays $1(\tau_L < T)(F_T - K)^+$ at T, where τ_L denotes the first time that F crosses L.
- Suppose that L is first kissed and that APCS holds whenever L is first kissed. Then no arb. $\Rightarrow$:

$$E_0^{\mathbb{Q}_T}[1(\tau_L < T)(F_T - K)^+] = \hat{P}_0(2L - K, T).$$

- Now suppose L can be jumped over and APCS holds at this jump time. Whenever lowering S_{τ_L} below L raises the put value and/or lowers the call value, then no arb. implies:

$$E_0^{\mathbb{Q}_T}[1(\tau_L < T)(F_T - K)^+] \leq \hat{P}_0(2L - K, T).$$

- It is interesting to again note that if we take the initial put price as given, then increasing the number of ways that the down-and-in call can knock in has the effect of lowering its value.

Def'n: Options on Upcrosses

- Recall that for a given spatial interval (K, H), the scaled time value $\frac{1}{H-K}\theta(K, T)$ could be interpreted as the mean number of upcrosses of (K, H), so long as K and H are skipfree.
- We now show that if we further assume APCS, then a standard OTM option can be interpreted as the mean of a call on the number of upcrosses of (K, H). The terminal payoff of the latter is:

$$\left[n^u_{T-} + 1(n^d_{T-} > n^u_{T-}, F_T < H)\frac{(F_T - K)^+}{H - K} - k\right]^+,$$

where for simplicity, we assume that the call strike is a positive integer k.

Th'm: OTM Option as Call on Upcrosses

- Assume that the price process F is a $\mathbb{Q}_T$ martingale which never jumps over K or H and for which APCS holds at all exit times of (K, H).
- Then under frictionless markets no arbitrage implies that for any positive integer k, $t \in [0, T]$:

$$E_t^{\mathbb{Q}_\mathbb{T}}\left[n^u_{T-} + 1(n^d_{T-} > n^u_{T-}, F_T < H)\frac{(F_T - K)^+}{H - K} - k\right]^+$$

$$= (n^u_t - k)^+ \quad + \frac{1(n^d_t > n^u_t)}{H - K}\hat{C}_t(K + 2(H - K)(k - n^u_t)^+, T)$$

$$+ \frac{1(n^d_t = n^u_t)}{H - K}\hat{P}_t(K - 2(H - K)(k - n^u_t)^+, T).$$

Proof: OTM Option as Call on Upcrosses

- Suppose that an investor sells a call on upcrosses at $t = 0$. We will show that the nature of the hedging strategy depends on whether or not the call on upcrosses is in-the-money.
- At initiation, the call on upcrosses is out-of-the-money, and so long as this remains true, the hedger flips back and forth between OTM calls and puts of different strikes.
- If the call on upcrosses goes into the money before maturity, then the hedger can flip back and forth between calls and puts both struck at K. Alternatively, the hedger can just hold the call static and synthesize puts whenever needed by shorting forward contracts with delivery price K.

Proof: OTM Option as Call on Upcrosses

- At $t = 0$, $n^u_t = 0$ and suppose that the investor purchases $\frac{1}{H-K}$ units of the OTM option struck $2(H-K)k$ dollars away from K, i.e.:

$$\begin{aligned} &\frac{1}{H-K}\hat{C}_0(K + 2(H-K)k, T) \quad \text{if} \quad F_0 \leq K \\ &\frac{1}{H-K}\hat{P}_0(K - 2(H-K)k, T) \quad \text{if} \quad F_0 > K. \end{aligned}$$

- At each time that n^u or n^d increases while $n^u \leq k$, the hedger switches the polarity and strike of the option held. The new strike is chosen so that the trade is self-financing.
- Hence, at each time τ that n^u increases to catch up to n^d, we have $n^u_{\tau-} = n^u_\tau - 1$.
- The hedger sells his holding in $\frac{1}{H-K}$ calls struck at $K + 2(H-K)(k - n^u_\tau + 1)$ and buys a position in $\frac{1}{H-K}$ puts struck at $K - 2(H-K)(k - n^u_\tau)$.
- Notice that the average of the two strikes is H. At each time τ that n^u increases to catch up to n^d, we have $F_\tau = H$, so APCS implies that this trade is self-financing.

Proof: OTM Option as Call on Upcrosses

- While $n^u \leq k$, consider each time σ that n^d increases above n^u.
- We have $n^u_{\sigma-} = n^u_\sigma$, so the hedger sells his holding in $\frac{1}{H-K}$ puts struck at $K - 2(H-K)(k - n^u_\sigma)$ dollars and buys a position in $\frac{1}{H-K}$ calls struck at $K + 2(H-K)(k - n^u_\sigma)$ dollars.
- Notice that the average of the two strikes is K. At each time σ that n^d increases above n^u, we have $F_\sigma = K$, so APCS implies that this trade is also self-financing.

Proof: OTM Option as Call on Upcrosses

- Define a round trip as an upcross followed by a subsequent first return to K. At the end of each such round trip, the forward price is at K and the strikes being traded are each $2(H - K)$ dollars closer to K than they were at the beginning of the round trip.
- When a round trip ends with the number of upcrosses equal to k, the hedger is holding $\frac{1}{H-K}$ puts struck at K as the forward price returns to K. This ends the first regime.
- The second regime begins with the sale of these puts which from put call parity generates exactly enough cash to buy $\frac{1}{H-K}$ calls struck at K.
- After this point in time, the hedger can create the number of upcrosses beyond k, by simply following the strategy for creating upcrosses, which was explicated earlier **Q.E.D.**

Summary and Extensions

- Under weak assumptions such as zero drift, first kiss, skip-freedom, and/or price symmetry, forward option prices convey the risk-neutral mean of many financially relevant path statistics.
- Almost of the results that rely on APCS can be extended to a driving uncertainty whose normal volatility is affine or even quadratic.
- When option prices are transparent, the risk-neutral mean of the path-statistics also become transparent. Hence, we conclude that there is surprisingly deep wisdom in listing options.