Distribution of Values on Quadratic Surfaces

O. Sargent

Groups Geometry and Dynamics, 2012

Motivation

Consider the following:

- $P : \mathbb{R}^d \to \mathbb{R}^s$ a polynomial map.
- $X \subseteq \mathbb{R}^d$ defined over $\mathbb{Q}$ such that $\left|X \cap \mathbb{Z}^d\right| = \infty$.

Question

What can one say about $P\left(X \cap \mathbb{Z}^d\right) \subseteq \mathbb{R}^s$?

- When is it dense? (conditions on P and X?)
- If it is dense, then exactly how dense?

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Special Cases

$X = \mathbb{R}^d$, $P = (L_1, \ldots, L_s)$ a linear map.

- If $s < d$ and $\alpha_1 L_1 + \cdots + \alpha_s L_s \notin \mathbb{Q}$ unless $\alpha_i = 0$ for all i, then $\overline{P(\mathbb{Z}^d)} = \mathbb{R}^s$.
- Classical.

$X = \mathbb{R}^d$, P a quadratic form.

- If $d > 2$ and P is indefinite, non degenerate, not a multiple of a rational form, then $\overline{P(\mathbb{Z}^d)} = \mathbb{R}$.
- Oppenheim Conjecture, proved by G. Margulis in 1989.

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Today: Another Special Case (Density).

- $Q : \mathbb{R}^d \to \mathbb{R}$, indefinite quadratic form, non degenerate, rational coefficients.
- $M = (L_1, \ldots, L_s) : \mathbb{R}^d \to \mathbb{R}^s$, linear map.
- $X_a(\mathbb{K}) = \{x \in \mathbb{K}^d : Q(x) = a\}$. ($\mathbb{K} = \mathbb{Z}$ or $\mathbb{R}$)
- Take $a \in \mathbb{Q}$ such that $|X_a(\mathbb{Z})| = \infty$.

Theorem (1. (O.S 2012))

If

1. $d > 2s$ *and rank* $\left(Q|_{Ker(M)}\right) > 2$,
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Remarks about the conditions of Theorem 1.

- $d > 2s$ probably not necessary. $(d > s+2$ should work??)
- $\operatorname{rank}\left(Q|_{\operatorname{Ker}(M)}\right) > 2$ is probably necessary although so far no counterexamples?
- $Q|_{\operatorname{Ker}(M)}$ is indefinite is possibly too strong but it implies that $X_a(\mathbb{R}) \cap \left\{x \in \mathbb{R}^d : M(x) = b\right\}$ is non compact which is necessary.
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Today: Another Special Case (Quantification).

Theorem (2. (O.S 2012))

If

1. $d > 2s$ *and rank* $\left(Q|_{Ker(M)}\right) = d - s$,
2. $Q|_{Ker(M)}$ *has signature* (r_1, r_2) *where* $r_1 \geq 1$ *and* $r_2 \geq 3$,
3. $\alpha_1 L_1 + \cdots + \alpha_s L_s \notin \mathbb{Q}$ *unless* $\alpha_i = 0$ *for all* i.

Then there exists $C_0 > 0$ *such that for all* $\theta > 0$ *there exists* $T_0 > 0$ *such that for all* $T > T_0$ *and* $R \subseteq \mathbb{R}^s$*- compact with smooth boundary*

$$(1-\theta)\, C_0 Vol(R)\, T^{d-s-2} \leq \left|\left\{v \in \mathbb{Z}^d : Q(v) = a, M(v) \in R, \|v\| \leq T\right\}\right| \leq (1+\theta)\, C_0 Vol(R)\, T^{d-s-2}.$$

Remarks about the conditions of Theorem 2.

- C_0 is such that

$$C_0 \mathrm{Vol}(R)\, T^{d-s-2} \sim \mathrm{Vol}\left(X_a(\mathbb{R}) \cap \left\{v \in \mathbb{R}^d : M(v) \in R, \|v\| \leq T\right\}\right)$$

  evaluated w.r.t the Haar measure on $X_a(\mathbb{R})$.

- The condition that $\mathrm{rank}\left(Q|_{\mathrm{Ker}(M)}\right) = d - s$ should be able to be relaxed to $\mathrm{rank}\left(Q|_{\mathrm{Ker}(M)}\right) > 2$.

- The cases where $Q|_{\mathrm{Ker}(M)}$ has signature $(1,2)$ or $(2,2)$ are 'exceptional and there are more integer points than expected by a factor of $\log T$.

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Ideas about the proof of Theorem 1: Strategy.

- Use Ratner's Theorem.

Theorem (Ratner's Orbit closure Theorem, 1990)

G - connected Lie group. $U \leq G$, generated by 1-parameter unipotent subgroups. $\Gamma \leq G$ a lattice. Then for all $x \in G/\Gamma$, $\overline{Ux} = Fx$ for F a closed connected subgroup $U \leq F \leq G$.

- For our purpose, set:
  - $G_Q = SO(Q)^o = \{g \in SL_d(\mathbb{R}) : Q(gx) = Q(x)\}^o$- connected Lie group.
  - $\Gamma_Q = G_Q \cap SL_d(\mathbb{Z})$ - lattice because Q is rational.
  - $H_{Q,M} = \{g \in G_Q : M(gx) = M(x)\}$.
- Look at $H_{Q,M} \curvearrowright G_Q/\Gamma_Q$, if $H_{Q,M}$ is generated by 1-p unip. s.g then use Ratner!

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- Look at $H_{Q,M} \curvearrowright G_Q/\Gamma_Q$, if $H_{Q,M}$ is generated by 1-p unip. s.g then use Ratner!

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- An equivalence relation:
  - $(Q, M) \sim (Q', M') \iff \exists g_d \in GL_d(\mathbb{R}), g_s \in GL_s(\mathbb{R})$ such that $\forall x \in \mathbb{R}^d \, (Q(g_d x), g_s M(g_d x)) = (Q'(x), M'(x))$.
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1. *rank* $\left(Q|_{Ker(M)}\right) = d - s$ *and* $(Q, M) \sim \left(Q'(x_1, \dots, x_s) + \sum_{i=s+1}^{p} x_i^2 - \sum_{i=p+1}^{d} x_i^2, (x_1, \dots, x_s)\right) = (Q_1, M_1)$
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What is $H_{Q,M}$?

- Notation: I_n - $n \times n$ identity matrix, $I_{n_1,n_2} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{pmatrix}$.
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Intermediate Subgroups.

- The task now is to determine closed connected subgroups, F such that $H_{Q,M} \leq F \leq G_Q$.
- In case 1, if s is relatively large there are quite a few possibilities for F.
- In case 2. Let $A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & I_{d-2} & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}$.
  - Then $F = H_{Q,M}$, $F = gAH_{Q_2,M_2}g^{-1}$ or $F = G_Q$.
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F invariant subspaces.

- Show that, if V is an F invariant subspace of dimension smaller than $d-s$, then V is defined over $\mathbb{Q}$.
  - This works, by using the fact that $H_{Q,M} \leq F$ and so any F invariant subspace must be $H_{Q,M}$ invariant.
  - Leads to two types of F invariant subspaces:
    - ⋆ those contained in $\langle x_1, \ldots, x_s \rangle$
    - ⋆ those that contain $\langle x_{s+1}, \ldots, x_d \rangle$
  - The assumption that $d > 2s$ means the latter type have dimension larger than $d-s$.
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Thank you for listening!