Complex Kleinian groups

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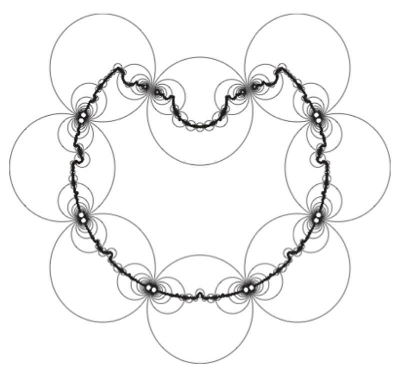

INTRODUCTION

These notes, written jointly by Angel Cano and José Seade, have been prepared for the lectures of Seade at the "GROUPS, GEOMETRY AND DYNAMICS" School and Discussion Meeting, to be held at the Center of Excellence in Mathematical Sciences of the Kumaun University, at Almora, Uttarakhand, India, during December 3-16.

Our aim is to provide an introduction to the geometric and dynamical study of discrete group actions on complex projective spaces, a subject that in dimension one goes back to Poincaré and others, but which is in its childhood when we look at higher dimensions. We refer to the bibliography in these notes, and to our recent monograph [10], for more on this subject.

The first section provides a glimpse of the classical theory of Kleinian groups of isometries of real hyperbolic spaces. That is our starting point and the paradigm of what comes next. This paves the ground for the following sections.

In Section 2 we enter into the topic we aim to study in these lectures: The theory of Complex Kleinian groups. These are by definition groups of holomorphic automorphisms of complex projective spaces $\mathbb{CP}^n$. For $n = 1$ this coincides with the setting discussed in Section 1, but in higher dimensions, new features appear. A significant difference between complex dimension 1 and higher dimensions, comes from the role played by "the limit set". While in dimension 1 that concept is classical, this is not so in higher dimensions. It turns out that the "limit set" introduced by R. Kulkarni in the late 1970s plays a key-role in this discussion.

In Section 3 we discuss the classification of the elements in $\mathrm{PSL}(n+1, \mathbb{C})$, following the classical classification of the elements in $\mathrm{PSL}(2, \mathbb{C})$ into elliptic, parabolic and loxodromic transformations. The material in this section is based on work by A. Cano, L. Loeza and J. P. Navarrete. We discuss here the geometry and dynamics of each type of transformations. We know from Navarrete's work that every elliptic or parabolic element in $\mathrm{PSL}(3, \mathbb{C})$ is conjugate to (respectively) an elliptic or parabolic element in $\mathrm{PU}(2, 1)$, but there are new types of loxodromic elements in $\mathrm{PSL}(3, \mathbb{C})$ that cannot exist in $PU(2, 1)$. It is interesting that when we come into higher dimensions, $n > 2$, the work of Cano and Loeza shows that one also has parabolic elements in $\mathrm{PSL}(n + 1, \mathbb{C})$ which do not exist in $\mathrm{PU}(n, 1)$: These are actually conjugate to parabolic elements in some projectivized group $\mathrm{PU}(k, l) \subset \mathrm{PSL}(n + 1, \mathbb{C})$.

Finally, Section 4 is a brief description of the structure of the Kulkarni limit set for complex Kleinian subgroups of $\mathrm{PSL}(3, \mathbb{C})$. This is based on work, published and also in progress, by A. Cano, W. Barrera, L. Loeza, J. P. Navarrete and J. Seade.

We are grateful to Krishnendu Gongopadhyay for inviting us to deliver these lectures, thus giving us the opportunity to disseminate our work on this beautiful subject, which is still full of interesting questions and research problems, waiting to be explored.

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1 KLEINIAN GROUPS IN REAL HYPERBOLIC SPACE

1.1 Inversions and the Möbius group

The material in this section is all well-known and there is a vast literature about it. Two general references for this section are Beardon's book [2] and the excellent notes of M. Kapovich [21].

Let us consider a classical type of transformations, which are analogous to reflections, the inversions. Given a circle $C = C(a,r)$ in the plane $\mathbb{R}^2$ with centre at a point $a \in \mathbb{R}^2$ and radius r, the inversion in C is the map $\iota = \iota(a,r)$ of the 2-sphere $\mathbb{S}^2 \cong \widehat{\mathbb{R}}^2 := \mathbb{R}^2 \cup \infty$ defined for each $z = (x,y) \neq a, \infty$ by:

$$\iota_{a,r}(x,y) \;=\; (a_1,a_2) \;+\; \frac{r^2}{|(x,y)-(a_1,a_2)|^2}\big(x-a_1, y-a_2\big)\,;$$

define $\iota(a) = \infty$ and $\iota(\infty) = a$. Notice that each $z = (x,y) \neq a, \infty$ is carried into the unique point $z' = (x',y')$ in the line determined by z and a which satisfies:

$$d(z,a)\cdot d(z',a) = r^2\,,$$

where $d(\ ,a)$ is the usual distance to a. We remark that for circles of maximal length (*i.e.*, radius 1 in the 2-sphere) this map is just a reflection in the corresponding line in $\mathbb{R}^2$.

Notice this formula is easily adapted to describing inversions in $(n-1)$-spheres in $\mathbb{S}^n \cong \mathbb{R}^n \cup \infty$.

It is an exercise to show that inversions are conformal maps, *i.e.*, they preserve angles. That is, if two curves in $\mathbb{S}^2$ meet with an angle θ, then their images under an inversion also meet with an angle θ. Moreover, one has that if C_1 , C_2 are circles in $\mathbb{S}^2$ and ι_1 is the inversion with respect to C_1, then $\iota_1(C_2) = C_2$ if and only if C_1 and C_2 meet orthogonally.

In fact the same statement holds in all dimensions:

Theorem 1.1 *Let C_1^{n-1} , C_2^{n-1} be spheres of dimension $n-1$ in $\mathbb{S}^n$ and ι_1 the inversion with respect to C_1. Then $\iota_1(C_2) = C_2$ if and only if C_1 and C_2 meet orthogonally.*

We now let $\text{Möb}(\mathbb{S}^n)$ be the group of diffeomorphisms of $\mathbb{S}^n \cong \widehat{\mathbb{R}} = \mathbb{R}^n \cup \{\infty\}$ generated by inversions on all $(n-1)$-spheres in $\mathbb{S}^n$, and let $\text{Möb}(\mathbb{B}^n)$ be the subgroup of $\text{Möb}(\mathbb{S}^n)$ consisting of maps that preserve the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$.

Notice that if the $(n-1)$-sphere $\mathcal{S}_1$ meets $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ orthogonally then $\mathcal{C} := \mathcal{S}_1 \cap \mathbb{S}^{n-1}$ is an $(n-2)$-sphere in $\mathbb{S}^{n-1}$ and the restriction to $\mathbb{S}^{n-1}$ of the inversion $\iota_{\mathcal{S}_1}$ coincides with the inversion on $\mathbb{S}^{n-1}$ defined by the $(n-2)$-sphere $\mathcal{C}$. In other words one has a canonical group homomorphism $\text{Möb}(\mathbb{B}^n) \to \text{Möb}(\mathbb{S}^{n-1})$.

Conversely, given an $(n-2)$-sphere $\mathcal{C}$ in $\mathbb{S}^{n-1}$ there is a unique $(n-1)$-sphere $\mathcal{S}$ in $\mathbb{S}^n$ that meets $\mathbb{S}^{n-1}$ orthogonally at $\mathcal{C}$. The inversion

$$\iota_{\mathcal{C}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$$

extends canonically to the inversion:

$$\iota_{\mathcal{S}} : \mathbb{B}^n \to \mathbb{B}^n ,$$

thus giving a canonical group homomorphism $\text{Möb}(\mathbb{S}^{n-1}) \to \text{Möb}(\mathbb{B}^n)$, which is obviously the inverse morphism of the previous one. Thence one has:

Lemma 1.2 *There is a canonical group isomorphism $\text{Möb}(\mathbb{B}^n) \cong \text{Möb}(\mathbb{S}^{n-1})$.*

Definition 1.3 We call $\text{Möb}(\mathbb{B}^n)$ (and also $\text{Möb}(\mathbb{S}^n)$) *the general Möbius group* of the ball (or of the sphere).

The subgroup $\text{Möb}_+(\mathbb{B}^n)$ of $\text{Möb}(\mathbb{B}^n)$ of words of even length consists of the elements in $\text{Möb}(\mathbb{B}^n)$ that preserve the orientation. This is an index two subgroup of $\text{Möb}(\mathbb{B}^n)$. Similar considerations apply to $\text{Möb}(\mathbb{S}^n)$. We call $\text{Möb}_+(\mathbb{B}^n)$ and $\text{Möb}_+(\mathbb{S}^n)$ *Möbius groups* (of the ball and of the sphere, respectively).

It is easy to see that $\text{Möb}(\mathbb{S}^n)$ includes:

- Euclidean translations: $t(x) = x+a$, where $a \in \mathbb{R}^n$. These are obtained by reflections on parallel hyperplanes.
- Rotations: $t(x) = Ox$, where $O \in \mathrm{SO}(n)$; obtained by reflections on hyperplanes through the origin.
- Homotecies, obtained by inversions on spheres with same centre and different radius.

In fact one has:

Theorem 1.4 *The group Möb($\mathbb{S}^n$) of Möbius transformations is generated by the previous transformations: Translations, rotations and homotecies, together with the inversion: $t(x) = x/\|x\|^2$.*

In fact one has that $\text{Möb}_+(\mathbb{B}^n)$ contains the orthogonal group $\mathrm{SO}(n)$ as the stabilizer (or isotropy) subgroup at the origin 0 of its action on the open ball $\mathbb{B}^n$. The stabilizer of 0 under the action of the full group $\text{Möb}(\mathbb{B}^n)$ is $\mathrm{O}(n)$. This implies that $\text{Möb}_+(\mathbb{B}^n)$ acts transitively on the space of lines through the origin in $\mathbb{B}^n$. Moreover, $\text{Möb}_+(\mathbb{B}^n)$ clearly acts also transitively on the intersection with $\mathbb{B}^n$ of each ray through the origin. Thus it follows that $\text{Möb}_+(\mathbb{B}^n)$ acts transitively on $\mathbb{B}^n$. In other words we have:

Theorem 1.5 *The group $\text{Möb}_+(\mathbb{B}^n)$ acts transitively on the unit open ball $\mathbb{B}^n$ with isotropy* $\mathrm{Aff}(n)$*. Furthermore, this action extends to the boundary $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ and defines a canonical isomorphism between this group and the Möbius group $\text{Möb}_+(\mathbb{S}^{n-1})$.*

We remark that for $n > 2$, $\text{Möb}_+(\mathbb{S}^{n-1})$ is the group of (orientation preserving) conformal automorphisms of the sphere (see for instance Apanasov's book [1]). That is, we have:

Theorem 1.6 *For all $n > 2$ we have group isomorphisms*

$$\text{Möb}_+(\mathbb{B}^n) \cong \text{Möb}_+(\mathbb{S}^{n-1}) \cong \mathrm{Conf}_+(\mathbb{S}^{\mathrm{n}-1}) .$$

In fact the previous constructions show that every element in $\text{Möb}_+(\mathbb{B}^n)$ extends canonically to a conformal automorphism of the sphere at infinity $\mathbb{S}^{n-1}_\infty := \overline{\mathbb{H}}^n_{\mathbb{R}} \setminus \mathbb{H}^n_{\mathbb{R}}$ and conversely, every conformal automorphism of $\mathbb{S}^{n-1}_\infty$ extends to an element in $\text{Möb}_+(\mathbb{B}^n)$.

1.2 Hyperbolic space

We now use Theorem 1.5 to construct a model for hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$. We recall that a riemannian metric g on a smooth manifold M means a choice of a positive definite quadratic form on each tangent space T_xM, varying smoothly over the points in M. Such a metric determines lengths of curves as usual, and so defines a metric on M in the usual way, by declaring the distance between two points to be the infimum of the lengths of curves connecting them.

Now consider the open unit ball $\mathbb{B}^n$, its tangent space $T_0\mathbb{B}^n$ at the origin, and fix the usual riemannian metric on it, which is invariant under the action of $\mathrm{O}(n)$. Given a point $x \in \mathbb{B}^n$, consider an element $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{B}^n)$ with $\gamma(0) = x$. Let $D\gamma_0$ denote the derivative at 0 of the automorphism $\gamma : \mathbb{B}^n \to \mathbb{B}^n$. This defines an isomorphism of vector spaces $D\gamma_0 : T_0\mathbb{B}^n \to T_x\mathbb{B}^n$ and allows us to define a riemannian metric on $T_x\mathbb{B}^n$. In this way we get a riemannian metric at each tangent space of $\mathbb{B}^n$.

We claim that the above construction of a metric on the open ball is well defined, *i.e.*, that the metric one gets on $T_x\mathbb{B}^n$ does not depend on the choice of the element $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{B}^n)$ taking 0 into x. In fact, if $\eta \in \mathrm{M\ddot{o}b}(\mathbb{B}^n)$ is another element taking 0 into x, then $\eta^{-1} \circ \gamma$ leaves 0 invariant and is therefore an element in $\mathrm{O}(n)$. Since the orthogonal group $\mathrm{O}(n)$ preserves the metric at $T_0\mathbb{B}^n$, it follows that both maps, γ and η, induce the same metric on $T_x\mathbb{B}^n$. Hence this construction yields to a well-defined riemannian metric on $\mathbb{B}^n$.

It is easy to see that this metric is complete and homogeneous with respect to points, directions and 2-planes, so it has constant (negative) sectional curvature.

Definition 1.7 The open unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ equipped with the above metric serves as a model for the *hyperbolic n-space* $\mathbb{H}^n_{\mathbb{R}}$. The group $\mathrm{M\ddot{o}b}(\mathbb{B}^n)$ is its *group of isometries*, also denoted $\mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}})$, and its index two subgroup $\mathrm{M\ddot{o}b}_+(\mathbb{B}^n)$ is the *group of orientation preserving isometries* of $\mathbb{H}^n_{\mathbb{R}}$, $\mathrm{Iso}_+(\mathbb{H}^n_{\mathbb{R}})$.

In the sequel we denote the real hyperbolic space by $\mathbb{H}^n_{\mathbb{R}}$, to distinguish it from the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ (of real dimension $2n$) that we will consider later. Also, we denote by $\mathbb{S}^{n-1}_{\infty}$ the sphere at infinity, that is, the boundary of $\mathbb{H}^n_{\mathbb{R}}$ in $\mathbb{S}^n$. We set $\overline{\mathbb{H}}^n_{\mathbb{R}} := \mathbb{H}^n_{\mathbb{R}} \cup \mathbb{S}^{n-1}_{\infty}$.

Given that we have a metric in $\mathbb{H}^n_{\mathbb{R}}$, we can speak of length of curves, area, volume, and so on. We also have the concept of *geodesics*: curves that minimize (locally) the distance between points. These are the segments of curves in $\mathbb{H}^n_{\mathbb{R}}$ which are contained in circles that meet the boundary $\mathbb{S}^{n-1}_{\infty}$ orthogonally.

Notice that the constructions above show that every isometry of $\mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}})$ extends canonically to a conformal automorphism of the sphere at infinity $\mathbb{S}^{n-1}_{\infty}$ and conversely, every conformal automorphism of $\mathbb{S}^{n-1}_{\infty}$ extends to an isometry of $\mathbb{H}^n_{\mathbb{R}}$.

1.3 Kleinian groups

We now consider a subgroup $\Gamma \subset Iso(\mathbb{H}^n_{\mathbb{R}})$ and look at its action on the hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. We want to study how the orbits of points in $\mathbb{H}^n_{\mathbb{R}}$ (and in $\overline{\mathbb{H}}^n_{\mathbb{R}}$) behave under the action of Γ.

Definition 1.8 Let $\Gamma \subset \mathrm{Iso}(\mathbb{H}^{\mathrm{n}}_{\mathbb{R}})$ be a discrete subgroup. The *limit set* of Γ is the subset $\Lambda = \Lambda(\Gamma)$ of $\mathbb{S}^{n-1}_{\infty}$ of points which are accumulation points of orbits in $\mathbb{H}^{n}_{\mathbb{R}}$. That is,

$$\Lambda := \Big\{ y \in \mathbb{S}^{n-1}_{\infty} \,|\, y = \lim\{g_m(x)\} \ \text{ for some } x \in \mathbb{H}^{n}_{\mathbb{R}} \text{ and } \{g_m\} \ \text{ a sequence in } \mathrm{Iso}(\mathbb{H}^{\mathrm{n}}_{\mathbb{R}}) \Big\} .$$

By definition, this is a closed, invariant subset of $\mathbb{S}^{n-1}_{\infty}$ which is non-empty, unless Γ is finite. This is the set where the dynamics concentrate. It can happen that Λ is the whole sphere at infinity, as for instance in the previous example of the triangle subgroups of isometries of $\mathbb{H}^{2}_{\mathbb{R}}$.

Definition 1.9 A discrete subgroup of $\mathrm{Iso}(\mathbb{H}^{\mathrm{n}+1}) \cong \mathrm{Conf}(\mathbb{S}^{\mathrm{n}})$ is *Kleinian* if its limit set is not the whole sphere at infinity.

In the sequel we refer to these as *conformal Kleinian groups*, to distinguish them from the complex Kleinian groups that we shall study later.

We remark that nowadays the term "Kleinian group" is often used for an arbitrary discrete subgroup of hyperbolic motions, regardless of whether or not the region of discontinuity is empty.

Let us consider for a moment a more general setting. Let G be some Lie group acting on a smooth Riemannian manifold M by smooth maps.

Definition 1.10

i. The action of G is *discontinuous at* $x \in M$ if there is a neighbourhood U of x such that the set

$$\{g \in G \,|\, gU \cap U \neq \varnothing\}$$

is finite. The set of points in M at which G acts discontinuously is called the *region of discontinuity*. The action is *discontinuous on* M if it is discontinuous at every point in M.

ii. The action is *properly discontinuous* on an open invariant set $U \subset M$ if for each non empty compact set $K \subset M$ the set

$$\{g \in G \,|\, gK \cap K \neq \varnothing\} ,$$

is finite.

iii. The action is *equicontinuous* on an open invariant set $U \subset M$ if all the transformations have "equal variation". More precisely, the action is equicontinuous at a point $x_0 \in U$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(g(x_0), g(x)) < \varepsilon$ for all $g \in G$ and all x such that $d(x_0, x) < \delta$. The family is equicontinuous if it is equicontinuous at each point of U.

We remark that by Arzelà-Ascoli's theorem, equicontinuity is equivalent to demanding that the transformations defined by the group action form a normal family, *i.e.*, every sequence $\{g_n\} \subset G$ contains a subsequence which converges uniformly on compact sets in U.

Notice also that, clearly, every properly discontinuous action is *a fortiori* discontinuous, but not conversely. For instance:

Example 1.11 (Kulkarni) Consider the map T in $\mathbb{R}^2$ given by $(x, y) \mapsto (\frac{1}{2}x, 2y)$ and all its iterates $\{T_n\}_{n\in\mathbb{Z}}$. This gives an action of $\mathbb{Z}$ in $\mathbb{R}^2$ which is discontinuous away from the origin 0. Notice that if we take a circle C around the origin, then its forward orbit accumulates on the whole $\{y\}$-axe, while the backwards orbit accumulates on the $\{x\}$-axe. So this action is not properly discontinuous on $\mathbb{R}^2 \setminus \{0\}$. Yet we notice that the action is properly discontinuous on $\mathbb{R}^2 \setminus \{x = 0\}$ and also on $\mathbb{R}^2 \setminus \{y = 0\}$.

This example shows that not every discontinuous action is properly discontinuous: Yet, for Kleinian groups one has (see the literature for a proof):

Theorem 1.12 *Let G be a discrete subgroup of* $\mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}})$. *Then G acts properly discontinuously on $\mathbb{H}^n_{\mathbb{R}}$ and its limit set is the complement of the region of discontinuity Ω of its action on $\mathbb{S}^{n-1}_\infty$. Furthermore, Ω is the largest region in $\mathbb{S}^{n-1}_\infty$ where the action is properly discontinuous, and it is also the largest region in the sphere where the action is equicontinuous.*

So, whenever we have a Kleinian group, the sphere $\mathbb{S}^{n-1}_\infty$ splits in two sets, which are invariant under the group action: the limit set Λ, where the dynamics concentrates, and the region of discontinuity Ω where the dynamics is "mild" and plays an important role in geometry, as we will see later.

1.4 The low dimensions: the group $\mathrm{PSL}(2, \mathbb{C})$

The two and three dimensional cases are classical and can be regarded simultaneously. Consider the open 3-ball $\mathbb{B}^3$ and its boundary $\partial\mathbb{B}^3$, which is the 2-sphere, that we regard as being the Riemann sphere $\mathbb{S}^2$, *i.e.*, the usual 2-sphere equipped with a complex structure, making it biholomorphic to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$, also called the *Cauchy* plane.

It is explained in many text books that in this dimension, an orientation preserving diffeomorphism of $\mathbb{S}^2$ is conformal if and only if it is holomorphic. This is essentially a consequence of the Cauchy-Riemann equations. Moreover, every holomorphic automorphism of the Riemann sphere is a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers such that $ad - bc = 1$.

Let us look now at the group $\mathrm{SL}(2, \mathbb{C})$ of 2×2 complex matrices with determinant 1. This group acts linearly on $\mathbb{C}^2$, so it acts on the complex projective line $\mathbb{CP}^1$ which is biholomorphic to the Riemann sphere $\mathbb{S}^2 \cong \mathbb{C}\cup\infty := \widehat{\mathbb{C}}$. The induced action of SL$(2,\mathbb{C})$ on $\mathbb{CP}^1$ is via the Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d}\ .$$

Thus one has a natural projection

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{Conf}_+(\mathbb{S}^2) \cong \mathrm{Iso}_+(\mathbb{H}^3_{\mathbb{R}}),$$

given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$. This is in fact a homomorphism of groups: the product of two matrices in SL$(2,\mathbb{C})$ maps to the composition of the corresponding Möbius transformations.

It is clear that the above projection is surjective. Furthermore, two matrices in SL$(2,\mathbb{C})$ define the same Möbius transformation if and only if they differ by multiplication by ± 1. Hence the group PSL$(2,\mathbb{C}) \cong$ SL$(2,\mathbb{C})/\{\pm I\}$ can be identified with the group of all Möbius transformations $\left\{\frac{az+b}{cz+d}\right\}$, which is isomorphic to the group of orientation preserving isometries of the hyperbolic 3-space. This coincides with the group of holomorphic automorphisms of the Riemann sphere; it also coincides with Conf$_+(\mathbb{S}^2)$, the group of orientation preserving conformal automorphisms on $\mathbb{S}^2$.

Summarizing:

Theorem 1.13 *One has the following isomorphisms of groups:*

$$\mathrm{Iso}_+(\mathbb{H}^3_{\mathbb{R}}) \ \cong\ M\ddot{o}b_+(\mathbb{B}^3) \ \cong\ \mathrm{Conf}_+(\mathbb{S}^2) \ \cong$$

$$\cong\ \left\{ \frac{az+b}{cz+d}\ ;\ a,b,c,d,\in\mathbb{C}\ ,\ ad-bc=1 \right\} \ \cong\ \mathrm{PSL}(2,\mathbb{C})\ .$$

Now recall that a Möbius transformation $\frac{az+b}{cz+d}$ with $ad-bc=1$ preserves the upper half plane $\mathcal{H}\subset\mathbb{C}$ if and only if a,b,c,d are real numbers. These correspond to compositions of inversions in $\widehat{\mathbb{C}} = \mathbb{C}\cup\infty$ on circles (or lines) orthogonal to the x-axis. Hence these are isometries of $\mathbb{H}^2_{\mathbb{R}}$ and one has:

Theorem 1.14

$$\mathrm{Iso}_+(\mathbb{H}^2_{\mathbb{R}}) \ \cong\ M\ddot{o}b_+(\mathbb{B}^2) \ \cong\ \left\{ \frac{\mathrm{az+b}}{\mathrm{cz+d}}\ ;\ \mathrm{a,b,c,d,}\in\mathbb{R}\ ,\ \mathrm{ad-bc=1} \right\} \ \cong\ \mathrm{PSL}(2,\mathbb{R})\,.$$

1.5 Geometric classification of the elements in $\mathrm{Iso}_+(\mathbb{H}^n_{\mathbb{R}})$

We now classify the elements of $\mathrm{Iso}_+(\mathbb{H}^n_{\mathbb{R}})$ in terms of their fixed points. We start with the case $n = 2$ which is classical and there is a vast literature about this topic (see for instance [23], [2]). By the theorem above, an isometry of the hyperbolic plane can be regarded as a Möbius transformation T given by $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d, \in \mathbb{R}$ and $ad - bc = 1$. The fixed points of T are the points where $T(z) = z$. These are the solutions of the equation:

$$z = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4bc}}{2c} .$$

Since the coefficients a, b, c, d are all real numbers we have the following three possibilities:

i) $(d - a)^2 + 4bc < 0$;

ii) $(d - a)^2 + 4bc = 0$;

iii) $(d - a)^2 + 4bc > 0$.

Assuming, as we do, $ad - bc = 1$, we have:

$$(d - a)^2 + 4bc = (a + d)^2 - 4 ,$$

and $a + d$ is the trace of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so we call $\mathrm{Tr}(T) := a + d$ *the trace* of T. Then the three cases above can be written as::

i) $0 \leq \mathrm{Tr}^2(T) < 4$. The map T is called *elliptic*;

ii) $\mathrm{Tr}^2(T) = 4$. The map T is called *parabolic*

iii) $\mathrm{Tr}^2(T) > 4$. The map T is called *hyperbolic*.

In the first case the map T has one fixed point in $\mathbb{H}^2_{\mathbb{R}}$, regarded as the upper half-plane $\mathbb{H} = \{Im\, z > 0\}$; the other fixed point is the complex conjugate of the previous one, so it is in the lower half-plane. In the second case T has only one fixed point (of multiplicity two) and this is contained in the x-axis (union ∞), which is the "boundary" of the hyperbolic plane, the sphere at infinity. In the third case T has two distinct fixed points, both contained in the sphere at infinity.

If T is elliptic then one can conjugate it by an automorphism of the Riemann sphere to make it have its fixed points at 0 and ∞, and T becomes a rotation around the origin, $T(z) = e^{i\theta} z$.

If T is parabolic then it is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to a map of the form $S(z) = z + k$, with $k \in \mathbb{R}$ constant. This map is a translation and has ∞ as its fixed point.

If T is hyperbolic then it is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to a map of the form $S(z) = \lambda^2 z$, with λ real and $\neq \pm 1$. This map has 0 and ∞ as fixed points and all other points move along straight lines through the origin. This description is good in some sense, but it is not satisfactory because the map S does not preserve $\mathbb{H}$, which is our model for $\mathbb{H}^2_{\mathbb{R}}$. To describe its dynamics in $\mathbb{H}$ it is better to consider its fixed points x_1, x_2, and assume for simplicity that both are finite and contained in the real-axis. These two points determine a unique geodesic in $\mathbb{H}^2_{\mathbb{R}}$, namely the unique half-circle in $\mathbb{H}$ with end-points x_1, x_2 and meeting orthogonally the x-axis. This geodesic is invariant under T. Moreover, given any other point $x \in \mathbb{H}$, there is a unique circle passing through x_1, x_2 and x. These circles fill out the whole space $\mathbb{C}$ and they are invariant under T, so they are unions of orbits. When the fixed points are taken to be 0 and ∞, these circles become the straight lines through the origin, or the meridians through the North and South poles if we think of T as acting on the Riemann sphere.

If we consider now an isometry T of $\mathbb{H}^3_{\mathbb{R}}$ and we think of it as a Möbius transformation with (possibly) complex coefficients, then we have again three possibilities:

i. The map T has two distinct fixed points which are both complex conjugate numbers. In this case T is said to be *elliptic*, as before. Again, T is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a rotation.

ii. The map T has only one fixed point which is real. In this case T is said to be *parabolic* and it is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a translation.

iii. The map T has two distinct fixed points which are both real numbers. In this case, as before, T is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a map of the form $z \mapsto \lambda^2 z$, but this time λ can be a complex number with $|\lambda| \neq 1$. In this case T is said to be *loxodromic*. Now T leaves invariant the geodesic in $\mathbb{H}^3$ that has end-points at the fixed points of T and the dynamics of all other points is a translation along that geodesic, together with a rotation around it. The number λ is called the *multiplier* of T. When this number is real the map is said to be *hyperbolic*, and in that case there is no rotation, only translation along the geodesic.

In order to give a similar classification in higher dimensions it is convenient to look at these transformations “from the inside” of the hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ (see [20] for a deeper and more complete description of this classification). Let T be an isometry of $\mathbb{H}^n_{\mathbb{R}}$ and pick up a point $p \in \mathbb{H}^n_{\mathbb{R}}$ such that the points p, $T(p)$ and $T^2(p)$ are not in an Euclidean straight line. Let $\mathcal{L}$ be the line that bisects the angle that they form, and look at the lines $T^{-1}(\mathcal{L})$, $\mathcal{L}$ and $T(\mathcal{L})$. There are three possibilities:

i. These three lines intersect in $\mathbb{H}^n_{\mathbb{R}}$.

ii. These three lines intersect at the $(n-1)$-sphere at infinity of $\mathbb{H}^n_{\mathbb{R}}$.

iii. These three lines do not intersect neither in $\mathbb{H}^n_{\mathbb{R}}$ nor at the sphere at infinity.

In the first case T has a fixed point at the meeting point of the three lines. The map T is said to be *elliptic*. These maps form an open set in $\mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}})$.

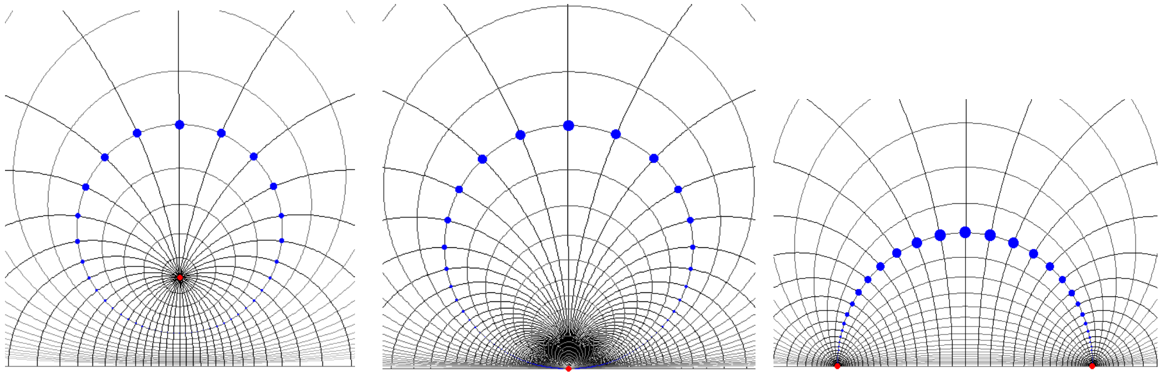

Figure 1: The three types of isometries

In the second situation the three lines are *parallel* in hyperbolic space and one has a fixed point at infinity. The map is a translation and it is said to be *parabolic*; this can be regarded as a limit case between the other two.

The last case is when the lines are *ultra-parallel*, *i.e.*, they do not meet in $\overline{\mathbb{H}}^n_{\mathbb{R}}$ (see Thurston's book for more on the topic). Now T leaves invariant the geodesic γ that minimises the length between the lines L and $T(L)$. In this case T is a translation along γ and a rotation around it. The end-points of γ are fixed points of T. These maps are called *loxodromic* (or just *hyperbolic*) and they also form an open set in $\mathrm{Iso}(\mathbb{H}^{\mathrm{n}}_{\mathbb{R}})$.

2 COMPLEX KLEINIAN GROUPS

In the previous section we studied discrete subgroups of isometries of real hyperbolic spaces $\mathbb{H}^n_{\mathbb{R}}$. When $n = 3$, the sphere at infinity is 2-dimensional and we can think of it as being the Riemann sphere $\mathbb{S}^2$, which is a complex 1-dimensional manifold, diffeomorphic to the projective line $\mathbb{CP}^1$. In this case one has that every (orientation preserving) element in the conformal group $\mathrm{Conf}_+(\mathrm{S}^2)$ is actually a Möbius transformation:

$$z \mapsto \frac{az+b}{cz+d}\,,$$

where a, b, c, d are complex numbers such that $ad - bz = 1$. The set of all such maps forms a group, which is isomorphic to the group $\mathrm{PSL}(2,\mathbb{C})$ of projective automorphisms of $\mathbb{CP}^1$:

$$\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\pm\mathrm{Id}\,,$$

where $\mathrm{SL}(2,\mathbb{C})$ is the group of 2×2 matrices with complex coefficients and determinant 1, and Id is the identity matrix. Hence, considering discrete subgroups of $\mathrm{Iso}_+(\mathbb{H}^3_{\mathbb{R}})$ is the same thing as considering discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

Let us focus now on studying discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, the group of automorphisms of the complex projective space $\mathbb{CP}^n$. We start by recalling some well-known facts about these spaces.

2.1 Complex projective space

We recall that the complex projective space $\mathbb{CP}^n$ is defined as:

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\sim\,,$$

where "$\sim$" denotes the equivalence relation given by $x \sim y$ if and only if $x = \alpha y$ for some nonzero complex scalar α. In short, $\mathbb{CP}^n$ is *the space of complex lines through the origin* in $\mathbb{C}^{n+1}$.

Consider for instance $\mathbb{CP}^1$. Every point here represents a complex line through the origin in $\mathbb{C}^2$. Recall that a complex line ℓ through the origin is always determined by a unit vector in it, say v, together with all its complex multiples. In other words, a unit vector v in $\mathbb{C}^2$ determines the complex line

$$\ell = \{\lambda \cdot v \mid \lambda \in \mathbb{C}\}\,.$$

Notice that the unit vectors in $\mathbb{C}^2$ form the 3-sphere $\mathbb{S}^3$, just as the unit vectors in $\mathbb{C}$ form the circle

$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid z = e^{i\theta}\ , \theta \in [0, 2\pi]\}\,.$$

Notice that the circle $\mathbb{S}^1$ acts on $\mathbb{C}^2$ in the obvious way: $e^{i\theta} \cdot (z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2))$. This action preserves distances in $\mathbb{C}^2$, so given a point $v \in \mathbb{S}^3 \subset \mathbb{C}^2$, its orbit under this $\mathbb{S}^1$-action is the set $\{(e^{i\theta} \cdot v\}$, which is a circle in $\mathbb{S}^3$ contained in the complex line determined by v. That is, *the intersection of* $\mathbb{S}^3$ *with every complex line through the origin in* $\mathbb{C}^2$ is a circle, and one has:

$$\mathbb{CP}^1 \cong \mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{S}^2\,.$$

The projection $\mathbb{S}^3 \to \mathbb{CP}^1 \cong \mathbb{S}^2$ is known as *the Hopf fibration*.

More generally, $\mathbb{CP}^n$ is a compact, connected, complex n-dimensional manifold, diffeomorphic to the orbit space $\mathbb{S}^{2n+1}/\mathrm{U}(1)$, where $\mathrm{U}(1) \cong \mathbb{S}^1$ is acting coordinate-wise on the unit sphere in $\mathbb{C}^{n+1}$. In fact, we usually represent the points in $\mathbb{CP}^n$ by *homogeneous coordinates* $(z_1 : z_2 : \cdots : z_{n+1})$. This means that we are thinking of a point in $\mathbb{CP}^n$ as being the equivalence class of the point $(z_1, z_2 : \cdots , z_{n+1})$ up to multiplication by non-zero complex numbers. Hence if, for instance, we look at points where the first coordinate z_1 is not zero, then the point $(z_1 : z_2 : \cdots : z_{n+1})$ is the same as $(1 : \frac{z_2}{z_1} : \cdots : \frac{z_{n+1}}{z_1})$. Notice this is just a copy of $\mathbb{C}^n$. That is, every point in $\mathbb{CP}^n$ that can be represented by a point $(z_1 : z_2 : \cdots : z_{n+1})$ with $z_1 \neq 0$, has a neighbourhood diffeomorphic to $\mathbb{C}^n$, consisting of all points with homogeneous coordinates $(1 : w_2 : \cdots : w_{n+1})$. Of course similar remarks apply for points where $z_2 \neq 0$ and so on. This provides the classical way for constructing an atlas for $\mathbb{CP}^n$ with $(n+1)$ coordinate charts.

Notice one has a projection $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$, a *Hopf fibration*, and the usual riemannian metric on $\mathbb{S}^{2n+1}$ is invariant under the action of $\mathrm{U}(1)$. Therefore this metric descends to a riemannian metric on $\mathbb{CP}^n$, which is known as the *Fubini-Study* metric.

It is clear that every linear automorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{CP}^n$, and it is well-known that every automorphism of $\mathbb{CP}^n$ arises in this way. Thus one has that the group of projective automorphisms is:

$$\mathrm{PSL}(n+1,\mathbb{C}) \;:=\; \mathrm{GL}(n+1,\mathbb{C})/(\mathbb{C}^*)^{n+1} \cong SL(n+1,\mathbb{C})/\mathbb{Z}_{n+1}\,,$$

where $(\mathbb{C}^*)^{n+1}$ is being regarded as the subgroup of diagonal matrices with a single nonzero eigenvalue, and we consider the action of $\mathbb{Z}_{n+1}$ (viewed as the roots of the unity) on $\mathrm{SL}(n+1,\mathbb{C})$ given by the usual scalar multiplication. Then $\mathrm{PSL}(n+1,\mathbb{C})$ is a Lie group whose elements are called projective transformations.

There is a classical way of decomposing the projective space that paves the way for studying complex hyperbolic geometry. For this we think of $\mathbb{C}^{n+1}$ as being a union $N_- \cup N_0 \cup N_+$, where each of these sets consists of the points $(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1}$ satisfying that $|z_{n+1}|^2$ is, respectively, larger, equal or smaller than $|z_1|^2 + \cdots + |z_n|^2$. It is clear that each of these sets is a complex cone, that is, union of complex lines through the origin in $\mathbb{C}^{n+1}$, with (deleted) vertex at 0.

Obviously

$$S := \{(z_1, \cdots, z_{n+1}) \in N_0 \,|\, z_{n+1} = 1\,\}\,,$$

is a sphere of dimension $(2n-1)$, and N_0 is the union of all complex lines in $\mathbb{C}^{n+1}$ joining the origin $0 \in \mathbb{C}^{n+1}$ with a point in S; each such line meets S in a single point. Hence the projectivisation $[S] = (N_0 \setminus \{0\})/\mathbb{C}^*$ of N_0 is a $(2n-1)$-sphere in $\mathbb{CP}^n$ that splits this space in two sets, which are the projectivisations of N_- and N_+. The set N_0 is often called *the cone of light*.

Similarly, notice that the projectivisation of N_- is an open $(2n)$-ball $\mathbb{B}$ in $\mathbb{CP}^n$, bounded by the sphere $[S]$. This ball serves as model for complex hyperbolic geometry, as we will see in the following section, where we describe its full group of holomorphic isometries, which is naturally a subgroup of projective transformations. This gives a natural source of discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, those coming from complex hyperbolic geometry.

2.2 Complex Kleinian groups

Recall from the previous section that the action of a subgroup $G \subset \mathrm{PSL}(n+1,\mathbb{C})$ is *properly discontinuous* on an invariant open set $U \subset \mathbb{CP}^n$ if for every compact set $K \subset U$ one has that the set

$$\{g \in G \,|\, gK \cap K \neq \varnothing\}$$

is finite.

Definition 2.1 A discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{C})$ is *complex Kleinian* if there exists a non-empty open invariant set in $\mathbb{CP}^n$ where the action is properly discontinuous.

As we know already, for $n = 1$, $\mathbb{CP}^1$ is the Riemann sphere, $\mathrm{PSL}(2,\mathbb{C})$ can be regarded as being the group of (orientation preserving) isometries of the hyperbolic space $\mathbb{H}^3_{\mathbb{R}}$ and we are in the situation envisaged previously, of classical Kleinian groups.

Notice that in this classical case, there is a particularly interesting class of Kleinian subgroups of $\mathrm{PSL}(2,\mathbb{C})$: Those which are conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. This latter group can be regarded as the group of Möbius transformations with real coefficients:

$$z \mapsto \frac{az+b}{cz+d} \quad , \quad ad - bc = 1 \ , \ a, b, c, d \in \mathbb{R} \, .$$

These are the Möbius transformations that preserve the upper half plane in $\mathbb{C}$. And if we identify the Riemann sphere with the extended plane $\mathbb{C} \cup \infty$, via stereographic projection, these are the conformal automorphisms of the sphere that preserve the Southern hemisphere, *i.e.*, they leave invariant a 2-ball in $\mathbb{S}^2$. Equivalently, these are subgroups of $\mathrm{Iso}\,\mathbb{H}^3_{\mathbb{R}}$ which actually are groups of isometries of the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$. These are called *Fuchsian groups*. In higher dimensions, this role is played by the so-called complex hyperbolic groups. These are, by definition, subgroups of $\mathrm{PSL}(n+1,\mathbb{R})$ which act on $\mathbb{CP}^n$ leaving invariant a certain open ball of complex dimension n, which serves as model for complex hyperbolic geometry. In the subsection below we speak a few words about this interesting subject. We will come back to it later.

2.3 Complex hyperbolic and complex affine groups

Let us describe first two specially important subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$: The group $\mathrm{PU}(n,1)$ of holomorphic isometries of the complex hyperbolic space, and the complex affine group. We start with $\mathrm{PU}(n,1)$.

Let us look at the subset $[N_-]$ of $\mathbb{CP}^n$ consisting of points whose homogeneous coordinates satisfy:

$$|z_1|^2 + \cdots |z_n|^2 < |z_{n+1}|^2 \, . \tag{2.2}$$

As noticed above, this set is an open ball $\mathbb{B}$ of real dimension $2n$ and its boundary,

$$[N_0] := \{(z_1 : \cdots : z_{n+1}) \in \mathbb{CP}^n \,\big|\, |z_1|^2 + \cdots |z_n|^2 = |z_{n+1}|^2\} \, ,$$

is a sphere of real dimension $2n-1$. This set $[N_-]$ is the usual starting point for complex hyperbolic geometry; for this one needs to introduce a metric, which is known as the Bergman metric. We shall do that in a way similar to the one we used for real hyperbolic space.

Let $\mathrm{U}(n)$ be the unitary group. By definition, its elements are the $(n) \times (n)$ matrices which satisfy

$$\langle Uz, Uw \rangle = \langle z, w \rangle \, ,$$

for all complex vectors $z = (z_1..., z_n)$ and $w = (w_1, ..., w_n)$, where $\langle \cdot, \cdot \rangle$ is the usual hermitian product on $\mathbb{C}^n$: $\langle z, w \rangle = \sum_{i=1}^{n} z_i \cdot \bar{w}_i$. This is equivalent to saying that the columns of U form an orthonormal basis of $\mathbb{C}^n$ with respect to the hermitian product.

We now let $\mathrm{U}(n,1)$ be the subgroup of $\mathrm{GL}(n+1,\mathbb{C})$ of transformations that preserve the quadratic form

$$Q(z_1, \cdots, z_{n+1}) = |z_1|^2 + \cdots |z_n|^2 - |z_{n+1}|^2 . \tag{2.3}$$

In other words, an element $U \in \mathrm{GL}(n+1,\mathbb{C})$ is in $\mathrm{U}(n,1)$ if and only if $Q(z) = Q(Uz)$ for all points in $\mathbb{C}^{n+1}$. Let $\mathrm{PU}(n,1)$ be its projectivization. Then the action of $\mathrm{PU}(n,1)$ on $\mathbb{CP}^n$ leaves invariant the set $[N_-]$. To see this, recall that a point in $\mathbb{CP}^n$ is in $[N_-]$ if and only if its homogeneous coordinates satisfy equation (2.3). If $(z_1 : \cdots : z_{n+1})$ is in $[N_-]$ and γ is in $\mathrm{PU}(n,1)$, then the point $\gamma(z_1 : \cdots : z_{n+1})$ is again in $[N_-]$. Therefore the group $\mathrm{PU}(n,1)$ *acts on* the ball $[N_-] \cong \mathbb{B}^{2n}$.

Recall that to construct the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ we considered the unit open ball $\mathbb{B}^n$ in $\mathbb{R}^{n+1}$, and we looked at the action of the Möbius group $\mathrm{Möb}_+(\mathbb{B}^n)$ on this ball. This action was transitive with isotropy $\mathrm{O}(n,\mathbb{R})$. So we can consider the usual metric at the space $T_0(\mathbb{B}^n)$, tangent to the ball at the origin, and spread it around using that the action is transitive; we get a well-defined metric on the ball using the fact that the isotropy $\mathrm{O}(n,\mathbb{R})$ preserves the usual metric.

Let us now do the analogous construction for the ball $[N_-]$ using the action of $\mathrm{PU}(n,1)$: It is an exercise to show that this action is transitive, with isotropy $\mathrm{PU}(n)$. Let P be the center of this ball, $P := (0 : 0 : \cdots : 0 : 1)$. We equip the tangent space $T_P([N_-]) \cong \mathbb{C}^n$ with the usual hermitian metric, and spread this metric around $[N_-]$ using the action of $\mathrm{PU}(n,1)$. Since the isotropy $\mathrm{PU}(n)$ preserves the metric in $T_P([N_-])$ we get a well-defined metric on the ball $[N_-] \cong \mathbb{B}^{2n}$. This is the Bergman metric on the ball $[N_-]$, which thus becomes a model for the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, with $\mathrm{PU}(n,1)$ as its group of holomorphic isometries. Its boundary $[N_0]$ is the sphere at infinity $\mathbb{S}^{2n-1}_\infty$.

Since the action of $\mathrm{PU}(n,1)$ on $\mathbb{H}^n_{\mathbb{C}}$ is by isometries, then one has (by general results of groups of transformations) that every discrete subgroup of $\mathrm{PU}(n,1)$ acts discontinuously on $\mathbb{H}^n_{\mathbb{C}}$. Hence, regarded as a subgroup of $\mathrm{PU}(n+1)$, such a group acts on $\mathbb{CP}^n$ with non-empty region of discontinuity.

The subgroups of $\mathrm{PU}(n,1)$ are usually known as *complex hyperbolic groups*, and from the previous discussion we deduce:

Every complex hyperbolic discrete group is a complex Kleinian group,

a statement that generalises to higher dimensions the well-known fact that every Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is Kleinian when regarded as a subgroup of $\mathrm{PSL}(2,\mathbb{C})$.

Now we look at complex affine groups. For this we recall that there is another classical way of constructing the projective space, and this also plays a significant role for producing discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$. This is by thinking of $\mathbb{CP}^n$ as being the union of $\mathbb{C}^n$

and the "hyperplane at infinity":

$$\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}.$$

A way for doing so is by writing

$$\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C} = \{(Z, z_n) \mid Z = (z_1, ..., z_n) \in \mathbb{C}^n \text{ and } z_n \in \mathbb{C}\}.$$

Then every point in the hyperplane $\{(Z, 1)\}$ determines a unique line through the origin in $\mathbb{C}^{n+1}$, *i.e.*, a point in $\mathbb{CP}^n$; and every point in $\mathbb{CP}^n$ is obtained in this way except for those corresponding to lines (or "directions") in the hyperplane $\{(Z, 0)\}$, which form the "hyperplane at infinity" $\mathbb{CP}^{n-1}$. It is clear that every affine map of $\mathbb{C}^{n+1}$ leaves invariant the hyperplane at infinity $\mathbb{CP}^{n-1}$. Furthermore, every such map carries lines in $\mathbb{C}^{n+1}$ into lines in $\mathbb{C}^{n+1}$, so the map naturally extends to the hyperplane at infinity. This gives a natural inclusion of the affine group

$$\mathrm{Aff}(\mathbb{C}^n) \cong \mathrm{GL}(n, \mathbb{C}) \ltimes \mathbb{C}^n,$$

in the projective group $\mathrm{PSL}(n + 1, \mathbb{C})$.

In particular, if Γ is a discrete subgroup of $\mathrm{Aff}(\mathbb{C}^n)$ which are isometries in $\mathbb{C}^n$ with respect to the usual Hermitian metric, then they have a non-empty region of discontinuity in $\mathbb{CP}^n$. We get:

Every discrete group of Euclidean isometries in $\mathbb{C}^n$ is complex Kleinian.

Yet, the problem of deciding whether or not an arbitrary discrete subgroup of $\mathrm{Aff}(\mathbb{C}^n)$ is discrete in $\mathrm{PSL}(n + 1, \mathbb{C})$ can be rather subtle.

2.4 The Kulkarni limit set and examples

In the first section of these notes we defined the limit set of a Kleinian group in the classical way, as the set of accumulation points of the orbits. This is indeed a good definition in that setting in all possible ways: its complement Ω is the largest region of discontinuity for the action of the group on the sphere, and Ω is also the region of equicontinuity of the group, *i.e.*, the set of points where the group forms a normal family.

It would be nice to have such a "universal" concept in the setting of complex Kleinian groups. Alas this is not possible in general and there is not a "correct" concept of limit set. As we will see, there can be several definitions of "limit set", each having its own interest, its own characteristics and leading to interesting results. Yet, one has that for complex dimension 2, "generically" the various natural possible definitions of a limit set coincide (see [9]). We do not know whether or not there is a similar statement in higher dimensions.

Indeed the question of giving "the definition" of limit set can be rather subtle, as pointed out by R. Kulkarni in the general setting of discrete group actions [22], and in [28] for the particular setting we envisage here. This is illustrated by the following example, taken from [24]. Let $\gamma \in \mathrm{PSL}(3, \mathbb{C})$ be the projectivisation of the linear map $\tilde{\gamma}$ given by:

$$\tilde{\gamma} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

where $\alpha_1\alpha_2\alpha_3 = 1$ and $|\alpha_1| < |\alpha_2| < |\alpha_3$. We denote by Γ the cyclic subgroup of $\mathrm{PSL}(3, \mathbb{C})$ generated by γ. Each α_i corresponds to a 1-dimensional eigenspace in $\mathbb{C}^3$, hence to a fixed point of γ in $\mathbb{CP}^2$, that we denote by e_i. The point $\{e_1\}$ is a repelling point while $\{e_3\}$ is an attractor. The projective lines $\overleftrightarrow{e_1, e_2}$ and $\overleftrightarrow{e_2, e_3}$ are both invariant lines. The orbits of points in the line $\overleftrightarrow{e_1, e_2}$ accumulate in e_1 going backwards, and they accumulate in e_2 going forwards. Similar considerations apply to the line $\overleftrightarrow{e_2, e_3}$. Thus e_2 is a saddle point

The orbit of each point in $\mathbb{CP}^2 \setminus (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3})$ accumulates at the points $\{e_1, e_3\}$, and it is not hard to see that Γ forms a normal family at all points in $(\mathbb{H}^2_{\mathbb{C}} \cup \mathbb{S}^3_\infty) \setminus \{e_1, e_3\}$. It is not hard to show that: one has (see [24] or [10, Chapter 3] for the proof):

i. Γ acts discontinuously on $\Omega_0 = \mathbb{CP}^2 - (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_3, e_2})$, and also on $\Omega_1 = \mathbb{CP}^2 - (\overleftrightarrow{e_1, e_2} \cup \{e_3\})$ and $\Omega_2 = \mathbb{CP}^2 - (\overleftrightarrow{e_3, e_2} \cup \{e_1\})$.

ii. Ω_1 and Ω_2 are the maximal open sets where Γ acts properly discontinuously; and Ω_1/Γ and Ω_2/Γ are compact complex manifolds. (In fact they are Hopf manifolds).

iii. Ω_0 is the largest open set where Γ forms a normal family.

It follows that even if the set of accumulation points of the orbits consists of the points $\{e_1, e_2, e_3\}$, in order to actually get a properly discontinuous action we must remove a larger set. Furthermore, in this example we see that there is not a largest region where the action is properly discontinuous, since neither Ω_1 nor Ω_2 is contained in the other.

So one has several candidates to be called as "limit set":

- The points $\{e_1, e_2, e_3\}$ where all orbits accumulate. But the action is not properly discontinuous on all of its complement. Yet, this definition is good if we make this group conjugate to one in $\mathrm{PU}(1, 2)$ and we restrict the discussion to the "hyperbolic disc" $\mathbb{H}^2_{\mathbb{C}}$ contained in $\mathbb{CP}^2$. This corresponds to taking the Chen-Greenberg limit set of Γ, that we shall define below.

- The two lines $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_3, e_2}$, which are attractive sets for the iterations of γ (in one case) or γ^{-1} (in the other case). This corresponds to Kulkarni's limit set of Γ, that we define below, and it has the nice property that the action on its complement is properly discontinuous and also, in this case, equicontinuous. And yet, the proposition above says that away from either one of these two lines the action of Γ is discontinuous. So this region is not "maximal".

- Then we may be tempted to taking as limit set the complement of the "maximal region of discontinuity", but there is no such region: there are two of them, the complements of each of the two invariant lines, so which one we choose?

- Similarly we may want to define the limit set as the complement of "the equicontinuity region". In this particular example, that definition may seem appropriate. The problem is that this would rule out important cases, as for instance the Hopf manifolds, which can not be written in the form U/G where G is a discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$ acting equicontinuously on an open set U of $\mathbb{CP}^2$. Moreover, there are examples where Γ is the fundamental group of certain compact complex surfaces (Inoue surfaces) and the action of Γ on $\mathbb{CP}^2$ has no points of equicontinuity.

Thus one may have different definitions of the limit set, each having interesting properties. Yet, the following notion of limit set, introduced by Ravi Kulkarni in [22], does play a major role in the theory of complex Kleinian groups, and there is evidence that in complex dimension 2, this is the good concept to look at. This definition of a limit set applies in a very general setting of a discrete group G acting on a smooth manifold X, and it has the important property of assuring that complement of the limit set, is an open invariant set where the group acts properly discontinuously.

For this, recall that given a family $\{A_\beta\}$ of subsets of X, where β runs over some infinite indexing set B, a point $x \in X$ is a *cluster (or accumulation) point* of $\{A_\beta\}$ if every neighbourhood of x intersects A_β for infinitely many $\beta \in B$.

Given a manifold X and a group G of discrete diffeomorphisms of X, let $L_0(G)$ be the closure of the set of points in X with infinite isotropy group. Let $L_1(G)$ be the closure of the set of cluster points of orbits of points in $X - L_0(G)$, *i.e.*, the cluster points of the family $\{\gamma(x)\}_{\gamma\in G}$, where x runs over $X - L_0(G)$.

Finally, let $L_2(G)$ be the closure of the set of cluster points of $\{\gamma(K)\}_{\gamma\in G}$, where K runs over all the compact subsets of $X - \{L_0(G) \cup L_1(G)\}$. We have:

Definition 2.4 i. Let X be as above and let G be a group of homeomorphisms of X. The *Kulkarni limit set* of G in X is the set

$$\Lambda_{\mathrm{Kul}}(G) := L_0(G) \cup L_1(G) \cup L_2(G).$$

ii. The *Kulkarni region of discontinuity* of G is

$$\Omega_{\mathrm{Kul}}(G) \subset X := X - \Lambda_{\mathrm{Kul}}(G).$$

It is easy to see that the set $\Lambda_{\mathrm{Kul}}(G)$ is closed in X and it is G-invariant (it can be empty). The set $\Omega_{\mathrm{Kul}}(G)$ (which also can be empty) is open, G-invariant, and G acts properly discontinuously on it.

When G is a Möbius (or conformal) group, the classical definitions of the *limit set* and the *discontinuity set* coincide with the above definitions.

For instance, in the example above, where G is generated by:

$$\tilde{\gamma} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

with $\alpha_1\alpha_2\alpha_3 = 1$ and $|\alpha_1| < |\alpha_2| < |\alpha_3$, one has that the sets $L_0(G)$ and $L_1(G)$ are equal, and they consist of the three points $\{e_1, e_2, e_3\}$, while $L_2(G)$ consists of the lines $\overleftrightarrow{e_1, e_2}$ and $\overleftrightarrow{e_2, e_3}$, passing through the saddle point. Hence $\Lambda_{\mathrm{Kul}}(G)$ consists of two projective lines.

$$\Lambda_{\mathrm{Kul}}(G) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3} \,.$$

Let us give few other examples:

Example 2.5 *Now let G be the cyclic group generated by the projectivization of the map:*

$$\tilde{\gamma} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \,, \ \textit{with}\, |\alpha| \neq 1 \,.$$

Then $L_0 = L_1 = L_2$ is the union of the line $\overleftrightarrow{e_1, e_2}$ and the point e_3. Hence $\Lambda_{\mathrm{Kul}}(G)$ is now:

$$\Lambda_{\mathrm{Kul}}(G) = \overleftrightarrow{e_1, e_2} \cup \{e_3\} \,.$$

Example 2.6 *Consider now the cyclic group generated by the projectivization of the map:*

$$\tilde{\gamma} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,.$$

Then 1 is the only eigenvalue. Now we have $L_0 = L_1 = \{e_1\}$ and $L_2 = \overleftrightarrow{e_1, e_2}$. Hence $\Lambda_{\mathrm{Kul}}(G)$ now consists of a single line:

$$\Lambda_{\mathrm{Kul}}(G) = \overleftrightarrow{e_1, e_2} \,.$$

Example 2.7 *Now consider the group generated by the matrix 2.5 together with a new generator:*

$$\widetilde{G} = \left\langle \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} , \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \ |\alpha| \neq 1 \right\rangle$$

It is easy to see that the second matrix permutes the invariant lines $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_2, e_3}$ and $\overleftrightarrow{e_3, e_1}$. Hence one gets thta $\Lambda_{\mathrm{Kul}}(G)$ is:

$$\Lambda_{\mathrm{Kul}}(G) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3} \cup \overleftrightarrow{e_3, e_1} \,.$$

One can further check (see [12]) that in this example, the quotient $\Omega_{\mathrm{Kul}}(G)/G$ is a compact orbifold which implies that this group is quasi-cocompact.

To finish this section, we recall that we know from [22] that if G is a discrete group in $\mathrm{PSL}(n+1,\mathbb{C})$, then its action on the Kulkarni set $\Omega_{\mathrm{Kul}}(G)$ is properly discontinuous. Hence, if $\Omega_{\mathrm{Kul}}(G) \neq \emptyset$, then G is complex Kleinian. Similarly, we know from [11] that the action of $G \subset \mathrm{PSL}(n+1,\mathbb{C})$ on the equicontinuity set $\mathrm{Eq}(G)$ is properly discontinuous. Hence, if $\mathrm{Eq}(G) \neq \emptyset$, then G is complex Kleinian. We state these claims as a theorem:

Theorem 2.8 *Let G be a discrete subgroup of $G \subset \mathrm{PSL}(n+1,\mathbb{C})$ which satisfies either one (or both) of the following conditions (which are not always equivalent):*

- *Its Kulkarni region of discontinuity is non-empty, $\Omega_{\mathrm{Kul}}(G) \neq \emptyset$;*
- *Its equicontinuity region is non-empty, $\mathrm{Eq}(G) \neq \emptyset$.*

Then there is a non-empty open invariant set in $\mathbb{CP}^n$ where G acts properly discontinuously, and therefore G is complex Kleinian.

As noted above, this includes all discrete subgroups of complex hyperbolic isometries, as well as all discrete groups of isometries of $\mathbb{C}^n$ with respect to the usual Hermitian metric.

3 ON THE CLASSIFICATION OF PROJECTIVE AUTOMORPHISMS

In a previous section we described the classification of the elements in $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Iso}_+\mathbb{H}^3_{\mathbb{R}}$, and also for groups of isometries in higher dimensional real hyperbolic spaces; we refer to M. Kapovich's excellent notes for more on that subject. This classification has been, and continuous to be, extended by various authors to the groups of isometries in different settings, as for instance by W. Goldman [16], J. Parker, K. Gongopadhyay and others (see [14], [13], [17], [19], [7]).

Here we briefly describe the classification of the elements in the projective group $\mathrm{PSL}(n+1,\mathbb{C})$. We start by recalling in more detail, the classical case, presenting it in a way that paves the way for the generalizations that we describe in higher dimensions, which are essentially taken from the work of J. P. Navarrete [25], for the case $n = 2$, and by A. Cano and L. Loeza [8] in higher dimensions.

3.1 The classical case

Recall that $\mathrm{PSL}(2,\mathbb{R})$ can be regarded as being the group of isometries of the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ and its elements are classified into three types: elliptic, parabolic and loxodromic depending on the number and position of the fixed points: An element $\gamma \in PSL(2,\mathbb{R})$ is elliptic if it has two fixed point in the extended complex plane $\widehat{\mathbb{C}}$ and these two points are conjugate (so γ has one fixed point in the interior of the hyperbolic plane); γ is parabolic if it has only one fixed point, and it is hyperbolic (or loxodromic) if it has two fixed point contained in the extended real line $\widehat{\mathbb{R}}$.

Recall that this classification can be given in terms of the trace of γ, and this extends to the elements in $\mathrm{PSL}(2,\mathbb{C})$ as follows: Let $\gamma \in \mathrm{PSL}(2,\mathbb{C})$ be represented by a matrix:

$$\tilde{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \ a,b,c,d \in \mathbb{C}\,,$$

with determinant $ad - bc = 1$. Define the trace of γ by $\mathrm{Tr}(\gamma) = \mathrm{a} + \mathrm{d}$. Then:

Characterization 1 *The transformation γ is:*

i. Elliptic if its trace is a real number and $\mathrm{Tr}(\gamma)^2 < 4$;

ii. Parabolic if $\mathrm{Tr}(\gamma)^2 = 4$;

iii. Loxodromic if $\mathrm{Tr}(\gamma)^2 \notin [0,4]$. If γ is loxodromic and $\mathrm{Tr}(\gamma)$ is real, then γ is said to be hyperbolic.

We remark that there are several other equivalent ways of describing this classification, as for instance:

i) By their normal forms and the eigenvalues;

ii) By their fixed points and their local dynamics at the fixed points;

iii) By their limit set.

Let us recall these classifications. For this notice that every matrix $\tilde{\gamma}$ in $\mathrm{SL}(2,\mathbb{C})$ is congugate to a matrix of one of the following types:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \lambda \in \mathbb{C}\,,$$

depending on whether or not it is diagonalizable.

In the first case γ is parabolic, and in the second case it is either loxodromic or elliptic depending on the number λ, which is called the multiplier of γ. The map is elliptic if and only if its multiplier has norm 1, *i.e.* $|\lambda| = 1$; otherwise the map is loxodromic.

In short,

Characterization 2 *The transformation γ is:*

i. Parabolic if every lifting to $\mathrm{SL}(2,\mathbb{C})$ is non-diagonalizable, and in that case its eigenvalues have both norm 1.

ii. Elliptic if it has a lifting which is diagonalizable with eigenvalues of norm 1.

iii. Loxodromic if it has a lifting which is diagonalizable with at least one eigenvalue of norm $\neq 1$.

It follows that γ is parabolic if and only if it has exactly one fixed point in the projective line $\mathbb{CP}^1$. In this case, up to conjugation, we can assume that the fixed points is ∞ in the extended complex plane $\widehat{\mathbb{C}} \cong \mathbb{CP}^1$, or the North pole in the Riemann sphere. Then the map is just a translation.

All non-parabolic transformations have two fixed points, which up to conjugation can be assumed to be the points $0, \infty$ in the extended complex plane $\widehat{\mathbb{C}} \cong \mathbb{CP}^1$, or else the South and North poles in the Riemann sphere. If the eigenvalues have both norm 1, then the map is a rotation, either by a rational angle, if γ has finite order, or by an irrational angle.

If γ is loxodromic then one eigenvalue, say λ^+, has norm greater than 1, and the other, say λ^+, has norm smaller than 1, because their product is 1. Then λ^+ determines an attractive fixed point x_+ in $\mathbb{CP}^1$ and λ^- determines a repelling fixed point x_-. In fact, for all $x \in \mathbb{CP}^1 \setminus x_-$ one has that the sequence of iterates $\{\gamma_n(x)\}$ converges to x_+, while the sequence $\{\gamma_n^{-1}(x)\}$ converges to x_- for all $x \neq x_+$.

In short:

Characterization 3 *The transformation γ is:*

i. Parabolic if it has only one fixed point in $\mathbb{CP}^1$;

ii. Elliptic if it has two fixed points in $\mathbb{CP}^1$ and around each fixed point γ is conjugate to a rotation.

iii. Loxodromic if it has two fixed points in $\mathbb{CP}^1$, one of these being an attractor and the other a repelling point.

We observe that if γ is parabolic, say given by $z \mapsto z+1$, then γ leaves invariant the point ∞ as well as all the lines in $\mathbb{C}$ parallel to the real axis. In $\mathbb{CP}^1$ these lines become circles passing through ∞. if γ is elliptic, say a rotation around the origin in $\mathbb{C}$, then it leaves invariant all the circles centered at 0. And if γ is loxodromic, say with an attractive point at 0, then for every open disc D in around the origin we have $\gamma(\overline{D}) \subset D$, where $\overline{D}$ is its closure. That is, Γ is a contraction in D. We have:

Characterization 4 *The transformation γ is:*

i. Parabolic if it leaves invariant a family of circles in $\mathbb{CP}^1$ which pass through a given point p and determine a foliation of $\mathbb{CP}^1 \setminus p$;

ii. Elliptic if it leaves invariant a family of circles that define a foliation of $\mathbb{CP}^1$ minus two points;

iii. Loxodromic if there exists an open set $U \subset \mathbb{CP}^1$ such that $\gamma(\overline{U}) \subset U$).

Finally we can give also a dynamical classification of these transformations. Notice that if γ is an elliptic element of finite order, then its limit set $\Lambda(\langle\gamma\rangle)$ is empty. And if γ is elliptic element with infinite order, then every point in $\mathbb{CP}^1_{\mathbb{C}}$ is an accumulation point of some orbit, so the limit set is everything: $\Lambda(\langle\gamma\rangle) = \mathbb{CP}^1$.

If γ is parabolic, then its limit set consists of its fixed point, and if γ is loxodromic, then it consists of its two fixed points. Hence we have:

Characterization 5 *The transformation γ is:*

i. Parabolic if its limit set in $\mathbb{CP}^1$ consists of one point;

ii. Loxodromic if its limit set consists of two points;

iii. Elliptic if its limit set is either empty or the whole $\mathbb{CP}^1$.

3.2 Classification of the elements in $\mathrm{PSL}(3,\mathbb{C})$

It is natural to expect that for the elements in $\mathrm{PSL}(3,\mathbb{C})$ one should have classifications of its elements in the vein of those given by the characterizations 1 to 4 above for the elements in $\mathrm{PSL}(2,\mathbb{C})$. This is indeed so, and that is the work started by Juan Pablo Navarrete in [25] (see also [10, Chapter 5]), and refined in [8]. Of course that the starting point is Goldman's classification of the elements in $\mathrm{PU}(2,1) \subset \mathrm{PSL}(3,\mathbb{C})$. These are the elements in $\mathrm{PSL}(3,\mathbb{C})$ that preserve the ball $\mathbb{B}^4$ in $\mathbb{CP}^2$ of points whose homogeneous coordinates $(z_1 : z_2 : z_3)$ satisfy:

$$|z_1|^2 + |z_2|^2 < |z_3|^2 .$$

The boundary $\partial\mathbb{B}^4$ is a 3-sphere. Then, an element $g \in \mathrm{PU}(2,1)$ is said to be *elliptic* if it has a fixed point inside $\mathbb{B}^4$, *parabolic* if it has exactly one fixed point in $\partial\mathbb{B}^4$, and *loxodromic* if it has two fixed points in $\partial\mathbb{B}^4$. I fact this same classification extends to higher dimensions, for the elements in $\mathrm{PU}(n,1)$. The only difference is that for $n = 2$ Goldman also gives a classification in terms of the trace.

This classification has been extended to real, complex and quaternionic hyperbolic spaces of "low dimensions" by various authors, as for instance Parker, Gongopadhyay and several others (see for instance [14], Cao-Go, Gongo, Gongo-Parsad , Gongo-Par-Par).

In fact the trichotomy of classifying the elements as elliptic, parabolic or loxodromic, extends to the group of isometries of all Gromov-hyperbolic spaces of negative curvature, though the classification is in a different vein (see for instance [7]). Notice however that the projective space $\mathbb{CP}^n$ is not Gromov-hyperbolic, neither it has a metric which is invariant under the action of $\mathrm{PSL}(n+1,\mathbb{C})$. And yet, we will see that the partition into elliptic, parabolic and loxodromic elements extends naturally to $\mathrm{PSL}(n+1,\mathbb{C})$, with one significant difference between the cases $n = 2$ and $n > 2$: In the first case one also has the classification in terms of the trace, and we do not know yet how to extend this to higher dimensions.

We consider first the case of $\mathrm{PSL}(3,\mathbb{C})$, following [25] and [8] (see also [10]). Let g be an element in $\mathrm{PSL}(3,\mathbb{C})$ and consider all its iterates $g^n := g \circ g^{n-1}$, for all $n \in \mathbb{Z}$ (with $g_1 := g$, $g_0 := Id$ and $g^{-n} := (g^{-1})^n$). In other words, we are considering the cyclic group generated by g, that we denote by $\langle g \rangle$. The element g is represented by a matrix $\tilde{g}$ in $\mathrm{GL}(3,\mathbb{C})$, unique up to multiplication by non-zero complex numbers.

Such a matrix $\tilde{g}$ has three eigenvalues, say $\lambda_1, \lambda_2, \lambda_3$, which may or may not be equal, and if they are distinct, they may or may not have equal norms: These facts make big differences in their geometry and dynamics, as we will see in the sequel. These, together with the corresponding Jordan canonical form of $\tilde{g}$, yield to the geometric and dynamical characterisations of the elements in $\mathrm{PSL}(3,\mathbb{C})$ that we give in this section. Similar considerations can be used also in higher dimensions, as we explain in the following section. Yet, in the case of $\mathrm{PSL}(3,\mathbb{C})$ there is also an algebraic classification in terms of the trace.

Coming back to our considerations in $\mathrm{PSL}(3,\mathbb{C})$, notice also that what really matters are the ratios amongst the λ_i, since multiplication of a matrix by a scalar, multiplies all its eigenvalues by that same scalar. Recall also that each eigenvalue determines a one dimensional space of eigenvectors in $\mathbb{C}^3$, so its projectivisation fixes the corresponding point in $\mathbb{CP}^2$. Distinct eigenvalues give rise to distinct fixed points in $\mathbb{CP}^2$. Also, every two points in $\mathbb{CP}^2$ determine a unique projective line; if the two points are fixed by g, then the corresponding line is g-invariant.

Let us use this information to have a closer look of the dynamics of g by considering a lifting $\tilde{g} \in \mathrm{SL}(3,\mathbb{C})$ and looking at its Jordan canonical form. One can check that this must be of one of the following three types:

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \text{where } \lambda_3 = (\lambda_1\lambda_2)^{-1}\,.$$

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \text{where } \lambda_3 = (\lambda_1)^{-2}\,.$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 3.1 Consider an element $g \in \mathrm{PSL}(3,\mathbb{C})$. Then:

i. g is *elliptic* if has a lifting to $\mathrm{SL}(3,\mathbb{C})$ which is diagonalizable with unitary eigenvalues.

ii. g is *parabolic* if has a lifting to $\mathrm{SL}(3,\mathbb{C})$ which is non-diagonalizable with unitary eigenvalues.

iii. g is *loxodromic* if has a lifting to $\mathrm{SL}(3,\mathbb{C})$ which has at least one eigenvalue which is not unitary.

Let us see what happens in each case. In the first case, when the lifting is diagonalizable, the images of e_1, e_2, e_3 are fixed by the corresponding map in $\mathbb{CP}^2$; for simplicity we denote the corresponding images by the same letters (to avoid having too many brackets $[e_i]$). One also has at least three invariant projective lines in $\mathbb{CP}^2$: $\mathcal{L}_1 := \overleftrightarrow{e_1, e_2}$, $\mathcal{L}_2 : \overleftrightarrow{e_2, e_3}$ and $\mathcal{L}_3 := \overleftrightarrow{e_1, e_3}$.

Up to re-numbering the eigenvalues, there are three essentially different possibilities (though a closer look at them shows that there are actually certain subcases):

i) $|\lambda_1| < |\lambda_2| < |\lambda_3|$.

ii) $|\lambda_1| = |\lambda_2| < |\lambda_3|$ (could be $|\lambda_1| < |\lambda_2| = |\lambda_3|$, but this is similar).

iii) $|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$

In case the point e_1 is repelling, e_2 is a saddle and e_3 is an attractor. Notice that the restriction of g to each of the three lines $\mathcal{L}_i$ is a loxodromic transformation in the group of automorphisms of this line, that we can identify with $\mathrm{PSL}(2,\mathbb{C})$: it has two fixed points in the line, one is repelling and the other is attracting.

Each point in $\mathcal{L}_1$ determines a unique projective line passing through that point and e_3, and the union of all these lines fills up the whole space $\mathbb{CP}^2$. In other words, the points in $\mathcal{L}_1$ parameterise the pencil $\{\mathcal{L}_y\}_{e_3}$of projective lines in $\mathbb{CP}^2$ passing through e_3. Since the line $\mathcal{L}_1$ is g-invariant, and e_3 is a fixed point of g, it follows that each element in this pencil is carried by g into another element of the pencil. Furthermore, we can say that this is happening in a "loxodromic" way in the following sense: if we start with a point x in one of these lines, then the g-orbit of x will travel from line to line, converging towards e_3 under the iterates of g, and getting closer and closer to the line $\mathcal{L}_1$ under the iterates of g^{-1}, thus converging to a fixed point in this line. This kind of transformations will correspond to the so called subclass of **strongly-loxodromic** elements in [25, 10].

Notice that in this case, since e_3 is an attracting fix point, we can choose a small enough "round ball" U containing e_3 such that $g(\overline{U}) \subset U$. This is relevant because that is a property which characterises loxodromic elements (see **??**).

Now consider the case

$$|\lambda_1| = |\lambda_2| < |\lambda_3|,$$

with the matrix still being diagonal. We can assume $|\lambda_1| = 1$. As before, the three points e_i are fixed points, the three lines are g-invariant and g carries elements in the pencil $\{\mathcal{L}_y\}_{e_3}$ into elements of this same pencil in a "loxodromic" way, as in the previous case, since the eigenvalue λ_3 has larger norm. The difference with the previous case is that the restriction of g to the invariant line $\mathcal{L}_1$ is now elliptic, not loxodromic. Hence the orbits of points in $\mathcal{L}_1$, others than the two fixed points e_1, e_2, move rotating along circles. All other points approach e_3 when travelling forwards, doing "spirals", and they approach the line $\mathcal{L}_1$ when moving backwards. These transformations are therefore called **screws**, and they are also loxodromic as elements in PSL$(3, \mathbb{C})$.

When $|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$ the situation is quite different. Now the restriction of g to each of the three lines $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ is an elliptic transformation, and g carries the elements of the pencil into elements of the pencil in an "elliptic way", that we will make precise. Notice one has in this case that

$$T^{(1)}(r) = \{[z_1 : z_2 : z_3] \in P^2_{\mathbb{C}} : |z_2|^2 + |z_3|^2 = r|z_1|^2\}, \quad r > 0\,, \tag{3.2}$$

is a family of 3-spheres, each of these being invariant under the action of g. These transformations are **elliptic**.

Let us envisage now the second case considered above, that is matrices of the form:

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad ,$$

with $\lambda_3 = (\lambda_1)^{-2}$. Notice that the top Jordan block determines a projective line $\mathcal{L}_1$ on which the transformation is parabolic. As a Möbius transformation in $\mathcal{L}_1$ this map is:

$$z \mapsto z + \frac{1}{\lambda_1}\,.$$

So the map in $\mathcal{L}_1$ is parabolic. Now observe that the points e_1 and e_3 are the only fixed points of g. As before, we have the invariant pencil $\{\mathcal{L}_y\}_{e_1}$. Notice there are two cases: $|\lambda_1| = 1$ or $|\lambda_1| \neq 1$. In the first case, g carries each element in the pencil into another element in the pencil in an "elliptic way". One can show too that in this case there is a family of 3-spheres in $\mathbb{CP}^2$ which are invariant by g and they all meet at the point e_3. These type of maps belong to the class of **parabolic elements** in PSL$(3, \mathbb{C})$, and they belong to the sub-class of **ellipto-parabolic** transformations.

If we now take $|\lambda_1| \neq 1$, then the dynamics in $\mathcal{L}_1$ is as before, but away from this invariant line the dynamics is dominated by the eigenvalue λ_3. If we assume $|\lambda_1| > 1$, then all points in $\mathbb{CP}^2 \setminus \mathcal{L}_1$ escape towards e_3 when moving forwards, and they accumulate in the line $\mathcal{L}_1$ when moving backwards. If $|\lambda_1| < 1$ the dynamics just reverses and the backwards orbits accumulate at e_3. These maps are **loxodromic elements**, of the type called **loxo-parabolic**.

Finally consider the case:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now one has that all eigenvalues are equal to 1. There is only one fixed point, e_1, and an invariant line, $\mathcal{L}_1 := \overleftrightarrow{e_1, e_2}$, in which the transformation is parabolic. Moreover, one has in this case the following family of g-invariant 3-spheres, which are all tangent to the line $\mathcal{L}_1$ at the point e_1

$$T_r = \{[z_1 : z_2 : z_3] \mid |z_2|^2 + r|z_3|^2 - (\overline{z_1}z_3 + z_1\overline{z_3}) - \frac{1}{2}(\overline{z_2}z_3 + z_2\overline{z_3}) = 0\},\ r \in \mathbb{R}.$$

These maps are all **parabolic**, of the type called **unipotent**.

Now we give several characterizations of each of the three types of transformations one has in $\mathrm{PSL}(3,\mathbb{C})$, generalizing those described above for the elements in $\mathrm{PSL}(3,\mathbb{C})$. For this we consider the complex polynomial,

$$F(x,y) = x^2y^2 - 4(x^3 + y^3) + 18xy - 27\,,$$

which extends the complex polynomial used by Goldman in [16] to classify the elements in $PU(2,1)$ by the trace.

Let g be an element in $\mathrm{PSL}(3,\mathbb{C})$, $\widehat{g}$ a lifting to $\mathrm{SL}(3,\mathbb{C})$, and denote by $\tau(g)$ the trace of $\widehat{g}$, which is invariant under conjugation.

The following theorems are essentially contained in [25], with some refinements coming from [8]. We begin with an algebraic characterization in terms of the trace. We denote by C_3 the cubic roots of unity.

Theorem 3.3 *The transformation g is:*

i) Elliptic if and only if $\overline{\tau(g)} = \tau(g^{-1})$ and $F(\tau(g), \overline{\tau(g)}) < 0$, or else $\widehat{g}$ is diagonalizable, $\tau(g) \notin 3C_3$, $\overline{\tau(g)} = \tau(g^{-1})$ and $F(\tau(g), \overline{\tau(g)}) = 0$.

iii) Parabolic if and only if $\overline{\tau(g)} = \tau(g^{-1})$ and either $\tau(g) \in 3C_3$ and $\hat{g}$ is not the identity element, or else $\widehat{g}$ is not diagonalizable and $F(\tau(g), \overline{\tau(g)}) = 0$, $\tau(g) \notin 3C_3$.

ii) Loxodromic if and only if $\overline{\tau(g)} \neq \tau(g^{-1})$ or $\overline{\tau(g)} = \tau(g^{-1})$ and $F(\tau(g), \overline{\tau(g)}) > 0$.

For instance, the following matrices have the same trace, but the first of these is parabolic (of the type called ellipto-parabolic):

$$\begin{pmatrix} e^{i\theta} & 1 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix}$$

while the following element is elliptic:

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix}$$

The next three theorems give, respectively, four different characterizations of each type of transformations: elliptic, parabolic and loxodromic. The first three of these characterizations in each of these theorems comes from [25], while the fourth characterization comes from [8].

Theorem 3.4 *The transformation* $g \in \mathrm{PSL}(3,\mathbb{C})$ *is elliptic if and only if one of the following conditions is satisfied:*

i. There is family of g*-invariant spheres in* $\mathbb{CP}^2$*, conjugate to those given in equation (3.2), which determine a foliation of* $\mathbb{C}^2 \setminus \{0\}$*, where we are thinking of* $\mathbb{CP}^2$ *as being the compactification of* $\mathbb{C}^2$ *obtained by attaching to it the line at* ∞*.*

ii. The set of accumulation points of the $\langle g\rangle$*-orbits of points in* $\mathbb{CP}^2$ *is either empty or the whole space* $\mathbb{CP}^2$*.*

iii. The equicontinuity set of $\langle g\rangle$ *is all of* $\mathbb{CP}^2$*.*

iv. For each pair of fixed points $x, y \in \mathbb{CP}^2$*, the restriction of* g *to the invariant line* $\overleftrightarrow{x,y} \cong \mathbb{CP}^1$ *is an elliptic element in* $\mathrm{PSL}(2,\mathbb{C})$*.*

Theorem 3.5 *The transformation* $g \in \mathrm{PSL}(3,\mathbb{C})$ *is parabolic if and only if one of the following conditions is satisfied:*

i. There is a g*-invariant projective line* $\mathcal{L} \subset \mathbb{CP}^2$ *and a family of* g*-invariant 3-spheres in* $\mathbb{CP}^2$*, tangent to* $\mathcal{L}$ *at a point* p *which is fixed by* g*, such that the set* $\overline{\bigcup \mathcal{F}}$ *is a closed round ball.* (Here, a round ball means the image by an element in $\mathrm{PSL}(3,\mathbb{C})$ of the ball consisting of points in $\mathbb{CP}^2$ whose homogeneous coordinates satisfy $|z_1|^2+|z_2|^2 < |z_3|^2$.)

ii. The set of accumulation points of the $\langle g\rangle$*-orbits of points in* $\mathbb{CP}^2$ *is a single point or a single projective line, and in either case the Kulkarni limit set* $\Lambda_{\mathrm{Kul}}(\langle g\rangle)$ *is a projective line.*

iii. The equicontinuity set $\mathrm{Eq}(\langle g\rangle)$ *is the complement of a projective line, and in this case* $\mathrm{Eq}(\langle g\rangle)$ *coincides with the Kulkarni region of discontinuity* $\Omega_{\mathrm{Kul}}(\langle g\rangle)$*.*

iv. There is only one invariant line and the restriction of g *to this line is a parabolic element in* $\mathrm{PSL}(2,\mathbb{C})$*.*

Theorem 3.6 *The transformation* $g \in \mathrm{PSL}(3,\mathbb{C})$ *is loxodromic if and only if one of the following conditions is satisfied:*

i. There is an open set $U \subset \mathbb{CP}^2$ *such that* $g(\overline{U}) \subset U$*.*

ii. The set of accumulation points of the $\langle g\rangle$-orbits of points in $\mathbb{CP}^2$ is a union of either two projective lines or a projective line and a point, and these two spaces form the Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\langle g\rangle)$.

iii. The equicontinuity set $\mathrm{Eq}(\langle g\rangle)$ is the complement of two projective subspaces, and in this case $\mathrm{Eq}(\langle g\rangle)$ coincides with the Kulkarni region of discontinuity $\Omega_{\mathrm{Kul}}(\langle g\rangle)$.

iv. There exist two distinct fixed points $x, y \in \mathbb{CP}^2$ such that the restriction of g to the invariant line $\overleftrightarrow{x, y} \cong \mathbb{CP}^1$ is a loxodromic element in $\mathrm{PSL}(2, \mathbb{C})$.

We remark that every elliptic and every parabolic element in $\mathrm{PSL}(3, \mathbb{C})$ is conjugate to, respectively, an elliptic or a parabolic element in $\mathrm{PU}(2, 1)$. Also, every loxodromic element in $\mathrm{PU}(2, 1)$ is loxodromic in $\mathrm{PSL}(3, \mathbb{C})$, but not conversely: the loxodromic elements in $\mathrm{PSL}(3, \mathbb{C})$ which are of the types called "screws" or "homotheties", and also some "strongly-loxodromic" elements, are not conjugate to elements in $\mathrm{PU}(2, 1)$ (see [25, 10]).

As we explain below, in higher dimensions, every elliptic element in $\mathrm{PSL}(n+1, \mathbb{C})$ is conjugate to an elliptic element in $PU(n, 1)$, but this is not so regarding the parabolic and loxodromic elements.

3.3 Classification of the elements in $\mathrm{PSL}(n+1, \mathbb{C})$

We now discuss the classification in [8] of the elements in $\mathrm{PSL}(n+1, \mathbb{C})$. We start with the definition.

Definition 3.7 Consider an element $g \in \mathrm{PSL}(n+1, \mathbb{C})$. Then:

i. g is *elliptic* if it has a lifting to $\mathrm{SL}(n+1, \mathbb{C})$ which is diagonalizable with unitary eigenvalues.

ii. g is *parabolic* if it has a lifting to $\mathrm{SL}(n+1, \mathbb{C})$ which is non-diagonalizable with unitary eigenvalues.

iii. g is *loxodromic* if it has a lifting to $\mathrm{SL}(n+1, \mathbb{C})$ which has at least one eigenvalue which is not unitary.

The above classification theorems for the elements in $\mathrm{PSL}(3, \mathbb{C})$ extend with almost no changes to higher dimensions in the cases of elliptic and loxodromic elements. One has:

Theorem 3.8 *The transformation $g \in \mathrm{PSL}(n+1, \mathbb{C})$ is elliptic if and only if one of the following equivalent conditions is satisfied:*

i. Up to conjugation, it leaves invariant each leaf of a foliation of $\mathbb{C}^n \setminus \{0\}$ by concentric $(2n-1)$-spheres, where $\mathbb{C}^n$ is being regarded as $\mathbb{C}^n \cong \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$.

ii. *The set of accumulation points of the* $\langle g\rangle$*-orbits of points in* $\mathbb{CP}^n$ *is either empty or the whole space* $\mathbb{CP}^n$.

iii. *The equicontinuity set of* $\langle g\rangle$ *is all of* $\mathbb{CP}^n$.

iv. *Each lifting to* $SL(n+1,\mathbb{C})$ *is diagonalizable and for each pair* $x,y\in\mathbb{CP}^n$ *of distinct fixed points, the restriction of* g *to the invariant line* $\overleftrightarrow{x,y}\cong\mathbb{CP}^1$ *is an elliptic element in* $\mathrm{PSL}(2,\mathbb{C})$.

Theorem 3.9 *The transformation* $g\in\mathrm{PSL}(n+1,\mathbb{C})$ *is loxodromic if and only if one of the following conditions is satisfied:*

i. *There is an open set* $U\subset\mathbb{CP}^n$ *such that* $g(\overline{U})\subset U$.

ii. *The set of accumulation points of the* $\langle g\rangle$*-orbits of points in* $\mathbb{CP}^n$ *is a union of two disjoint projective subspaces of dimensions* $<n$.

iii. *The equicontinuity set* $\mathrm{Eq}(\langle g\rangle)$ *is the complement of two projective subspaces of dimensions* $<n$.

iv. *The Kulkarni limit set* $\Lambda(\langle r\rangle)$ *is the complement of a union of two projective subspaces of dimensions* $<n$.

v. *There exist two distinct fixed points* $x,y\in\mathbb{CP}^n$ *such that the restriction of* g *to the invariant line* $\overleftrightarrow{x,y}\cong\mathbb{CP}^1$ *is a loxodromic element in* $\mathrm{PSL}(2,\mathbb{C})$.

To state the equivalent classification theorem for parabolic elements we need to introduce some notation. Recall that in the $\mathrm{PSL}(3,\mathbb{C})$-case, the 3-spheres used to characterize the parabolic elements are all equivalent to the sphere in $\mathbb{CP}^2$ defined by the quadratic form

$$Q(z_1,z_2,z_3)=|z_1|^2+|z_2|^2-|z_3|^2=0\,,$$

which corresponds to a bilinear form of signature $(2,1)$. The null vectors in $\mathbb{C}^3$ for this quadratic form are a complex cone over a 3-sphere $\mathbb{S}^3$, whose projectivization is the boundary $\partial\mathbb{H}^2_{\mathbb{C}}$ in $\mathbb{CP}^2$ of a ball that serves as model for the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$.

We know from the previous discussion that a parabolic element in $\mathrm{PSL}(3,\mathbb{C})$ is by definition an element that has a fixed point p, a unique invariant line ℓ containing p, and it leaves invariant a family of spheres $\{\mathcal{S}_r\}$, each being a copy of $\partial\mathbb{H}^2_{\mathbb{C}}$ by an element in $\mathrm{PSL}(3,\mathbb{C})$, which are all tangent to ℓ at p. The union of all these spheres is the image of $\mathbb{H}^2_{\mathbb{C}}$ by an element in $\mathrm{PSL}(3,\mathbb{C})$, where $\mathbb{H}^2_{\mathbb{C}}$ is the 4-ball in $\mathbb{CP}^2$ of points that correspond to negative vectors for the quadratic form Q.

Furthermore, we know also that every parabolic element in $\mathrm{PSL}(3,\mathbb{C})$ is conjugate to a parabolic element in $\mathrm{PU}(2,1)$ and these are, by definition, the projective transformations that leave invariant the ball:

$$\{(z_1:z_2:z_3)\in\mathbb{CP}^2\,\big|\,|z_1|^2+|z_2|^2<|z_3|^2\,\}\,,$$

and have a unique fixed point in the boundary of this ball.

Recall that in (3.7) we defined an element in $\mathrm{PSL}(n+1,\mathbb{C})$ to be parabolic if it has a non-diagonalizable lifting to $\mathrm{SL}(n+1,\mathbb{C})$ whose eigenvalues are all unitary. Even so, it is natural to ask the following question:

Question 3.10 *What ought to be a parabolic element in* $\mathrm{PSL}(n+1,\mathbb{C})$*?*

Of course that parabolic elements in $PU(n,1)$ must be parabolic in $\mathrm{PSL}(n+1,\mathbb{C})$; the question is: are there any other parabolics in this latter group? This is answered in [8]. For this we must consider bilinear forms of signature (k,l) for all possible positive integers k,l such that $k+l=n+1$. The corresponding groups are the projectivized Lorentz groups $\mathrm{PU}(k,l)\subset\mathrm{PSL}(k+l)$. That is, we denote by $\mathbb{C}^{k,l}$ the complex vector space $\mathbb{C}^{k+l}$ equipped with the Hermitian form:

$$\prec u,v\succ_{k,l}:=u_1\bar{v}_1+\cdots+u_k\bar{v}_k-u_{k+1}\bar{v}_{k+1}-\cdots-u_{k+l}\bar{v}_{k+l}\,.$$

Let $\mathbb{S}_{k,l}$ be the projectivization of the set of null vectors for this Hermitian form, *i.e.*, these are the points in $\mathbb{CP}^n$ whose homogeneous coordinates $(u_1:\cdots:u_{n+1l})$ satisfy:

$$|u_1|^2+\cdots+|u_k|^2=|u_{k+1}|^2+\cdots+|u_{n+1}|^2\,.$$

We call $\mathbb{S}_{k,l}$ a (k,l)-projective sphere. In fact we call **a** (k,l)**-sphere** the images of $\mathbb{S}_{k,l}$ under the elements in $\mathrm{PSL}(n+1)$.

Notice that $(n,1)$-spheres are usual $(2n-1)$-spheres while a (k,l)-sphere in general is the projectivization of a product of spheres $\mathbb{S}^{2k-1}\times\mathbb{S}^{2l-1}$, diffeomorphic to a quotient $(\mathbb{S}^{2k-1}\times\mathbb{S}^{2l-1})/\mathbb{S}^1$.

Similarly, by a (k,l)**-ball** $\mathbb{B}_{k,l}$ we mean the image under an element in $\mathrm{PSL}(n+l,\mathbb{C})$ of the projectivization of the set $\mathbb{H}^{k,l}_{\mathbb{C}}$ of negative vectors for the corresponding quadratic form, *i.e.*, the points whose homogeneous coordinates satisfy:

$$|u_1|^2+\cdots+|u_k|^2<|u_{k+1}|^2+\cdots+|u_{n+1}|^2\,.$$

Notice that the boundary of (k,l)-ball is a (k,l)-sphere.

Now we can define:

Definition 3.11 Let g be an element in some group $\mathrm{PU}(k,l)$ with $1\le l\le k$. Then:

i. g is $(k,l)-$elliptic if it has at least one fixed point in the (k,l)-ball $\mathbb{H}^{k,l}_{\mathbb{C}}$.

ii. g is $(k,l)-$loxodromic if its fixed points in $\overline{\mathbb{H}}^{k,l}_{\mathbb{C}}$ are all in the (k,l)-sphere $\partial\mathbb{H}^{k,l}_{\mathbb{C}}$ and there exist at least two such points x,y such that the action of g on the invariant line $\overleftrightarrow{x,y}$ is loxodromic.

iii. g is $(k,l)-$parabolic if its fixed points in $\overline{\mathbb{H}}_{\mathbb{C}}^{k,l}$ are all in $\partial\mathbb{H}_{\mathbb{C}}^{k,l}$ and for each pair of two such points x, y, the action of g on the invariant line $\overleftrightarrow{x,y}$ is elliptic.

It is proved in [8] that these three types of transformations determine a partition of $\mathrm{PU}(k,l)$, and this coincides with the classical definition in the case of $\mathrm{PU}(2,1)$. Moreover we have:

Theorem 3.12 *The elliptic, loxodromic and parabolic elements in* $\mathrm{PU}(k,l)$ *are elliptic, loxodromic and parabolic in* $\mathrm{PSL}(n+1,\mathbb{C})$*, respectively. Furthermore, a transformation* $g \in \mathrm{PSL}(n+1,\mathbb{C})$ *is parabolic if and only if one of the following conditions is satisfied:*

i. The set of accumulation points of the $\langle g\rangle$*-orbits of points in* $\mathbb{CP}^n$ *is a single projective space of dimension* $< n$.

ii. There exist positive integers k, l *with* $k+l = n+1$ *such that* g *is parabolic in* $\mathrm{PU}(k,l)$.

iii. The equicontinuity set $\mathrm{Eq}(\langle g\rangle)$ *and the Kulkarni region of discontinuity* $\Omega_{\mathrm{Kul}}(\langle g\rangle)$ *are the complement of a single projective subspace of* $\mathbb{CP}^n$.

One also has:

Proposition 3.13 *If* g *is parabolic, then there exist a family* $\{\mathcal{S}_\alpha\}$ *of* g*-invariant* (k,l)*-spheres and a* g*-invariant proper projective subspace* $\mathcal{Z} \subsetneq \mathbb{CP}^n$*, such that:*

- *The fixed points of* g *are all contained in the Hermitian orthogonal complement of* $\mathcal{Z}$ *(for the corresponding quadratic form), i.e.,* $Fix(g) \subset \mathcal{Z}^\perp$.
- *The intersection of every two distinct elements in the family* $\{\mathcal{S}_\alpha\}$ *is contained in* $\mathcal{Z}^\perp$*; and*
- *The union of all* $\{\mathcal{S}_\alpha \setminus \mathcal{Z}^\perp\}$ *fills out the whole space* $\mathbb{CP}^n \setminus \mathcal{Z}^\perp$.

Remark 3.14 It is worth saying that an element $g \in \mathrm{PSL}(n+1,\mathbb{C})$ is elliptic if and only if there exists some pair (k,l) with $1 \le k \le n$ and $k+l = n+1$, such that g has a fixed point inside the (k,l)-ball, and this implies that for each pair (k,l) with $1 \le k \le n$ and $k+l = n+1$ one has that g has a fixed point inside the (k,l)-ball.

4 THE KULKARNI LIMIT SET IN DIMENSION 2

Recall that for discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ we know that the limit set is a closed invariant set in $\mathbb{CP}^1$ which consists of 1 point, two points or else it has infinite cardinality. There is also an interesting theorem stating that given any closed subset $\mathbb{C} \subset \mathbb{CP}^1$, there exists a Kleinian subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ whose limit set contains C. Philosophically this means that the limit set can be as rich and complicated as we wish. The idea to show this theorem is to start by considering a finite number of points in C; for each of these points we consider a small circle centered at the point, so that all these circles are pairwise disjoint, and then we look at the group generated by the inversions in these circles. This is Schottky group in $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ and its limit set contains the chosen points. Now we take more points in C and repeat the process, and so on. In the limit we arrive to a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ whose limit set contains C.

In this section we shall make similar considerations for the Kulkarni limit set Λ_{Kul} of discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$.

Recall that in the previous section we gave examples of groups in $\mathrm{PSL}(3,\mathbb{C})$ where the limit set Λ_{Kul} consists of:

- One line;
- one line and one point;
- two lines;
- three lines;

Before we move forward, let us give a construction that originates in [26] and was later refined and extended in [24, 12]: The suspension construction. We start with a discrete group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ and look at its inverse image $\widetilde{\Gamma} \subset \mathrm{SL}(2,\mathbb{C})$. Now take the natural inclusion of $\mathrm{SL}(2,\mathbb{C})$ in $\mathrm{SL}(3,\mathbb{C})$, which is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives a subgroup $\widetilde{\Gamma} \subset \mathrm{PSL}(3,\mathbb{C})$ with a fixed point at e_3 and an invariant line $\overleftrightarrow{e_1, e_2} \cong \mathbb{CP}^1$. It is then an exercise to show that the Kulkarni limit set of $\widetilde{\Gamma}$ is the union of all the complex lines passing through e_3 and a point in the limit set $\Lambda(\Gamma) \subset \overleftrightarrow{e_1, e_2}$. Hence, if the limit set $\Lambda(\Gamma)$ is not all of $\overleftrightarrow{e_1, e_2}$, then $\widetilde{\Gamma}$ has non-empty region of discontinuity in $\mathbb{CP}^2$ and therefore it is complex Kleinian.

Notice that from the classical theory of Kleinian groups we know that if Γ is not a finite group, then $\Lambda(\Gamma)$ consists of one point, two points, or infinitely many points. Hence the Kulkarni limit set of this type of groups consists of one line, two lines or infinitely many lines, which are all concurrent, since they pass through the point e_3.

One has the following theorem proved by W. Barrera, A. Cano and J. P. Navarrete:

Theorem 4.1 *Let Γ be an infinite discrete subgroup of* $\mathrm{PSL}(3,\mathbb{C})$ *and let $\Lambda_{\mathrm{Kul}} \subset \mathbb{P}^2_{\mathbb{C}}$ be its Kulkarni limit set. Then:*

i. The set Λ_{Kul} *always contains at least one projective line.*

ii. The number of lines in Λ_{Kul} *is either 1, 2, 3 or infinite.*

iii. The number of lines in Λ_{Kul} *lying in general position is either 1, 2, 3, 4 or infinite.*

The first statement in this theorem follows easily from the aforementioned classification results in [25] of the cyclic groups in $\mathrm{PSL}(3,\mathbb{C})$. Yet, it is worth saying that in [11] is proved that this statement holds also in higher dimensions: The limit set of every complex Kleinian group in $\mathrm{PSL}(n+1,\mathbb{C})$ contains at least one projective line. The second and third statements in this theorem are proved in [5].

Notice that we already gave examples of limits set with one, two and three lines in their limit set. The complete classification of the groups in $\mathrm{PSL}(3,\mathbb{C})$ with at most three lines in their limit set is given in [9]. In [4] the authors give a complete classification of the complex Kleinian groups in $\mathrm{PSL}(3,\mathbb{C})$ with exactly four lines in general position in their limit set.

Notice too that the suspension groups described above give examples of groups with infinitely many lines in their limit set, but only two of them are in general position, since they all are concurrent. Yet, these suspension groups show that the structure of the limit set can be at least as rich as that of the classical Kleinian subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

In [6] the authors give examples of groups in $\mathrm{PSL}(3,\mathbb{C})$ with infinitely many lines in general position in their limit set. In fact the following theorem from [6] shows that the limit set Λ_{Kul} can actually be as rich and complicated as we want:

Theorem 4.2 *Let* $\mathcal{L}$ *be an arbitrary collection of lines in* $\mathbb{CP}^2$*, such that the topological closure of their union is not the whole space, i.e.,* $\overline{\bigcup \mathcal{L}} \neq \mathbb{CP}^2$*. Then there exists a complex Kleinian group* G *such that* $\overline{\bigcup \mathcal{L}} \subset \Lambda_{Kul}(G) \neq \mathbb{CP}^2$*. Moreover,* G *can be chosen so that it does not have neither fixed points nor invariant lines, neither it is complex hyperbolic.*

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