

Lectures Random Matrix School Bangalore

January 2012

Discrete polynuclear growth (PNG)

○ $h(x, t)$ = the height at x at time t , a one-dimensional interface

○ $h(x, 0) = 0$, $x \in \mathbb{Z}$

Growth rule:

$$h(x, t+1) = \max(h(x-1, t), h(x, t), h(x+1, t)) + w(x, t+1) \quad (1)$$

$w(x, t)$, $(x, t) \in \mathbb{Z} \times \{0, 1, \dots\}$ independent random variables

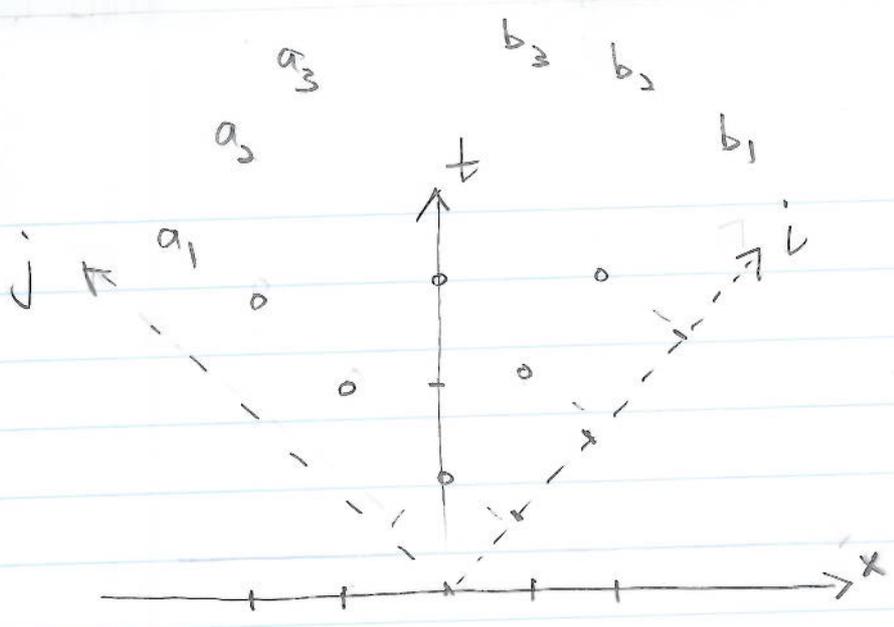
○ Special case: $w(x, t) = 0$ if $t-x$ is even or if $|x| > t$ and they are independent geometric random variables:

$$w(x, t) = w\left(\frac{t+x+1}{2}, \frac{t-x+1}{2}\right)$$

$$\mathbb{P}[w(i, j) = m] = (1 - a_i b_j)(a_i b_j)^m, \text{ independent}$$

$$m \geq 0$$

(Can also consider exponential r.v.'s instead.)

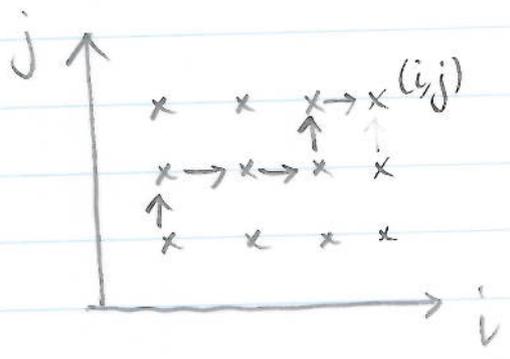


$$G(i,j) = h(i-j, i+j-1)$$

The growth rule translates into

$$G(i,j) = \max(G(i-1,j), G(i,j-1)) + w(i,j)$$

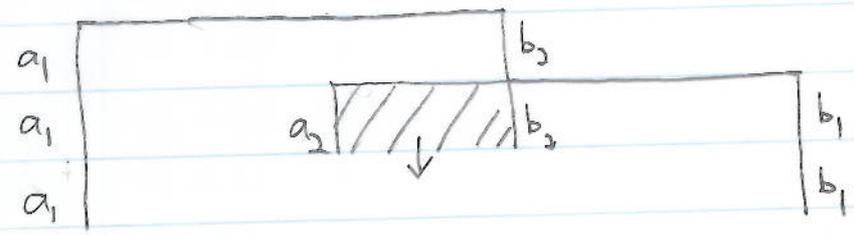
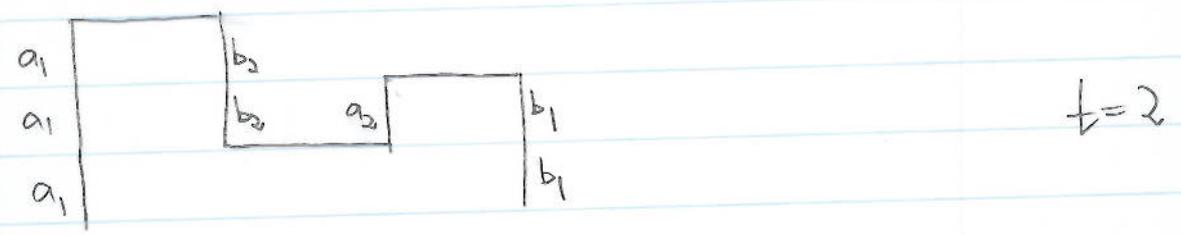
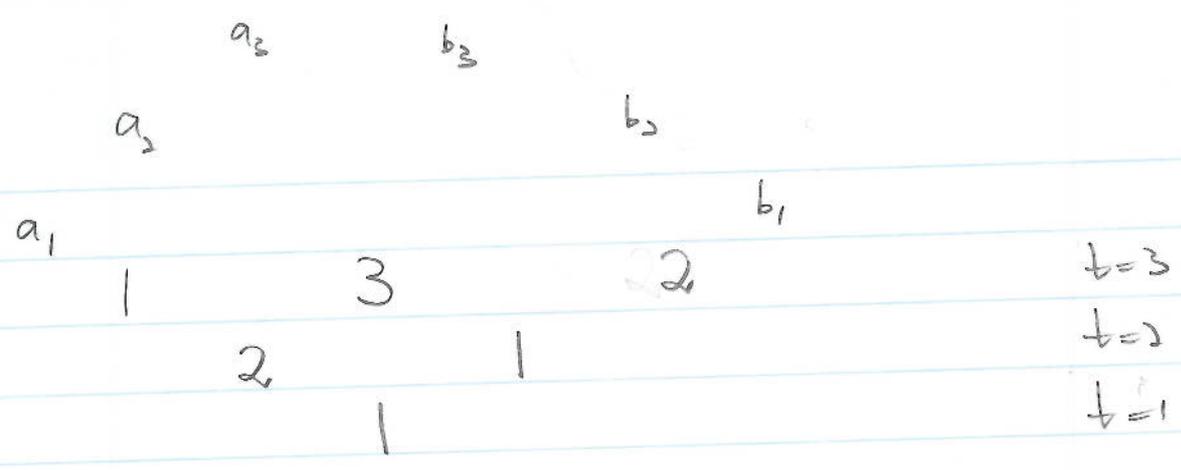
Directed last-passage time



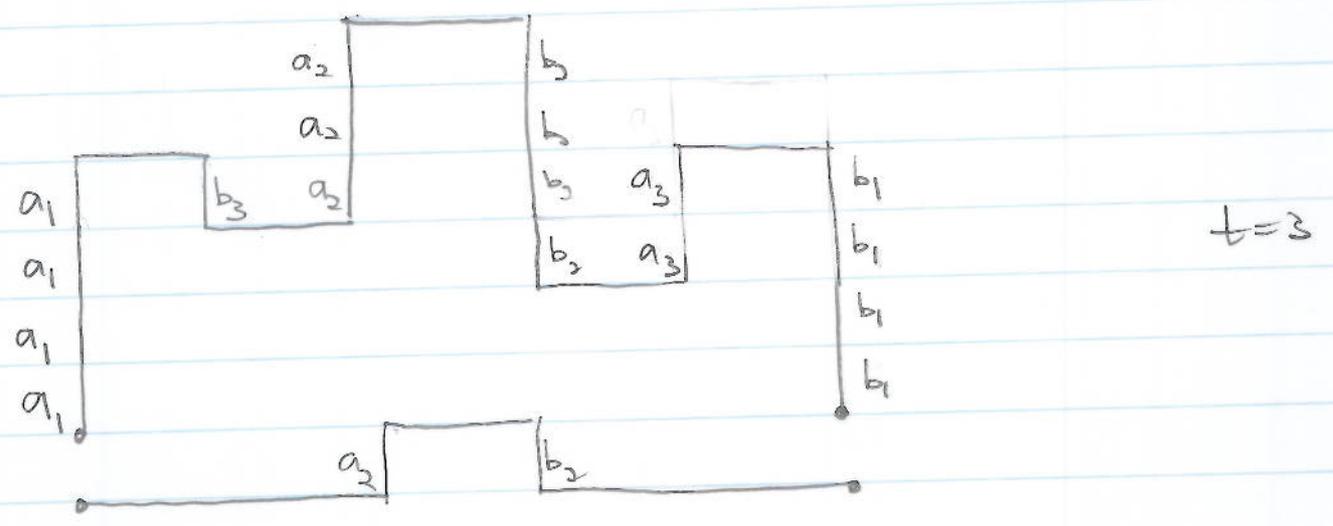
π up/right path
(random walk paths)

$$G(i,j) = \max_{\pi: (1,1) \rightarrow (i,j)} \sum_{(i,j) \in \pi} w(i,j) \quad (\text{zero-temp. directed polymer})$$

Example of PNG



- top-curve = PNG
- multi layer PNG



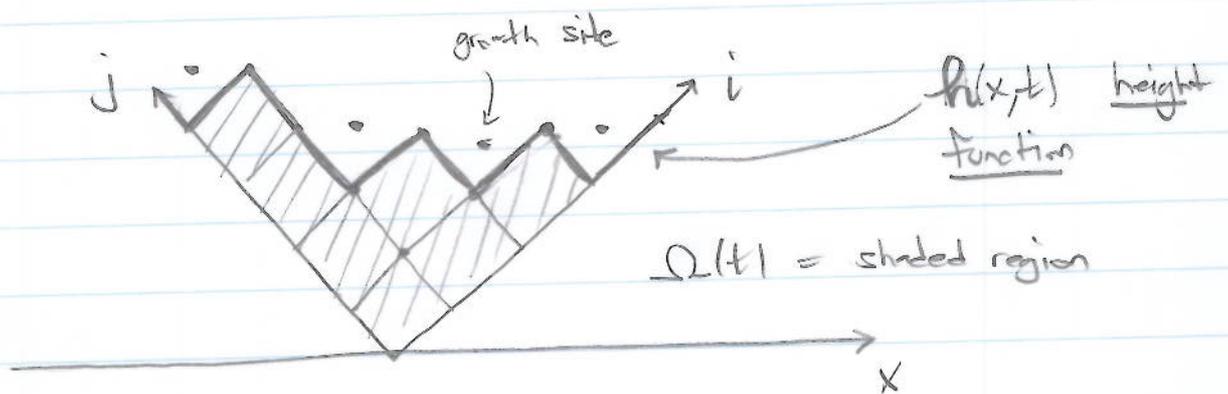
Corner growth model

$$w^*(i,j) = w(i,j) + 1$$

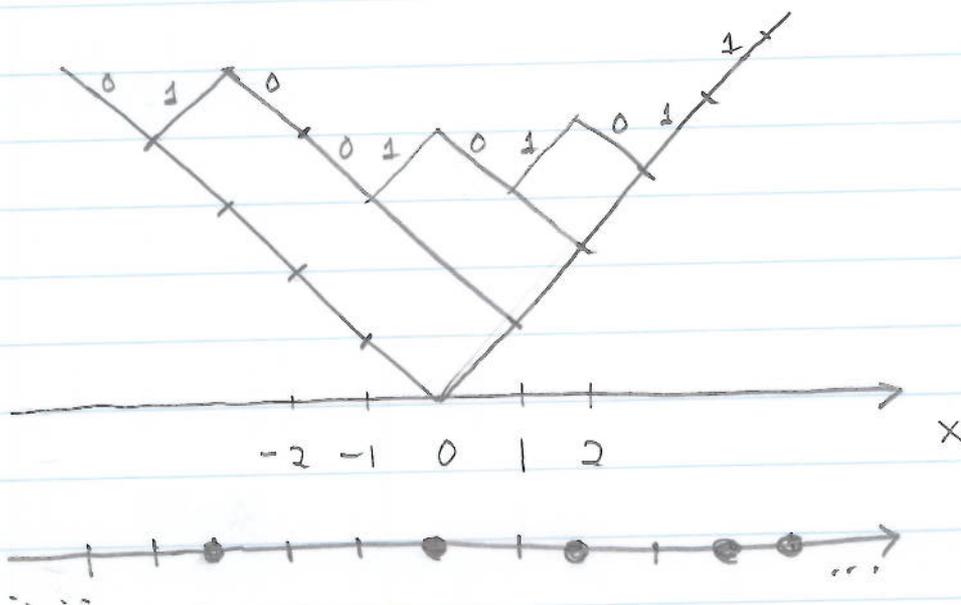
$$G^*(i,j) = \max(G^*(i-1,j), G^*(i,j-1)) + w^*(i,j)$$

$$G^*(i,j) = G(i,j) + i + j - 1$$

$$\Omega(t) = \{(i,j); G^*(i,j) \leq t\}$$

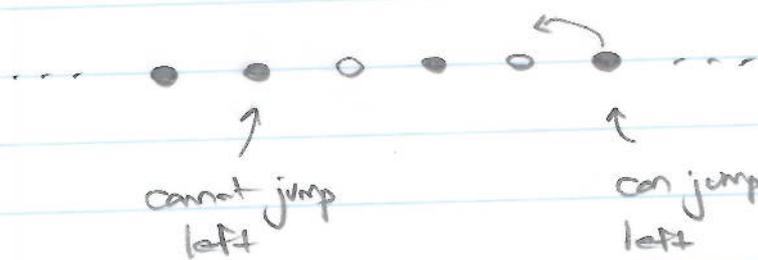


TASEP



$$h(0,x) = |x|$$

totally asymmetric
simple exclusion
process

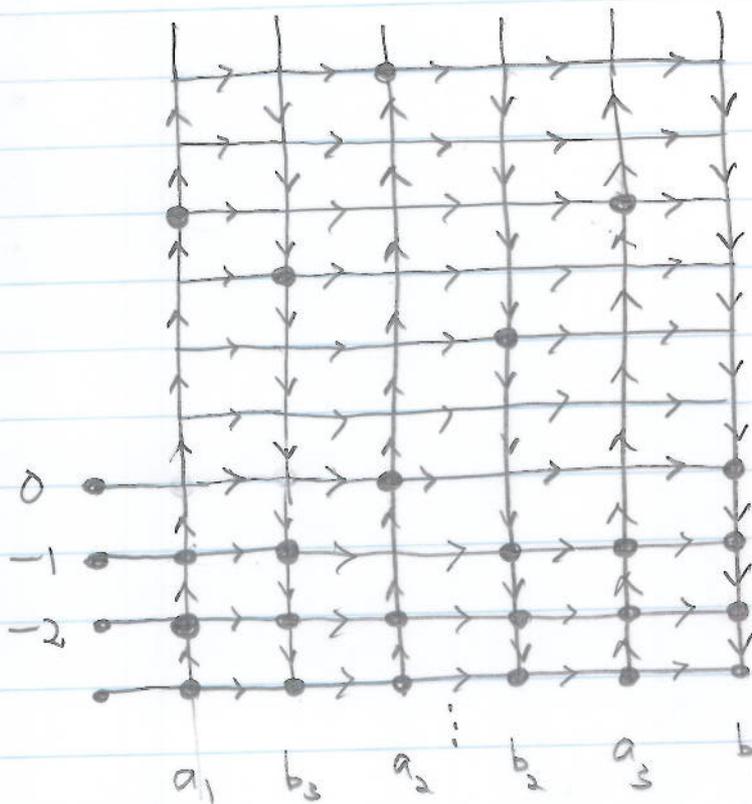


each particle has a geometric waiting time

(Can also consider exponential r.v. instead.)

We start with all the particles on \mathbb{Z}^+ , step initial condition.

Analysis of the multi-layer PNG



- non-int. paths in a directed graph

- particles

- random point process

horizontal edges have weight 1

← we can add further horizontal lines

$\Gamma: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 = 2 \cdot 3 = 2 \cdot \# \text{ time steps}$
 $= 2(2n-1) \text{ if } t = 2n-1$

Particle configuration

X_j^r = position of the j :th particle on line r

$$X_1^r > X_2^r > X_3^r > \dots, \quad 0 \leq r \leq M = 2m \quad (\text{2-time})$$

$$X_j^0 = 1 - j = X_j^M, \quad j = 1, 2, \dots$$

Can also be described by a partition

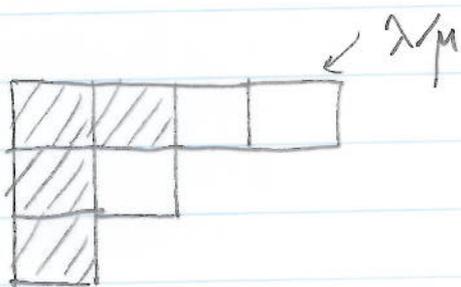
$$\lambda = (\lambda_1, \lambda_2, \dots), \quad \lambda_j \geq \lambda_{j+1}, \quad \lambda_j \geq 0$$

$$|\lambda| = \sum_j \lambda_j < \infty, \quad l(\lambda) = \text{number of } \neq 0 \text{ entries}$$

length of the partition

$$\mu_j^s = X_j^{2s} + j - 1, \quad \mu^0 = \emptyset = (0, 0, \dots) = \mu^m$$

$$\lambda_j^s = X_j^{2s-1} + j - 1$$



$$\lambda = (4, 2, 1, 0, \dots)$$



$$\mu = (2, 1, 1, 0, \dots)$$

$$\mu \subset \lambda, \quad \lambda \supset \mu.$$

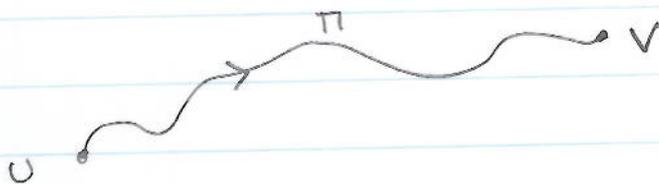
$$\emptyset = \mu^0 \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \dots \subset \lambda^m \supset \mu^m = \emptyset \quad (2)$$

$\underline{x} = (x^0, \dots, x^M)$ total particle configuration

We get an induced measure on the sequence of partitions (2) or equivalently a point process in $\{0, \dots, M\} \times \mathbb{Z}$.

Lindström - Gessel - Viennot formula

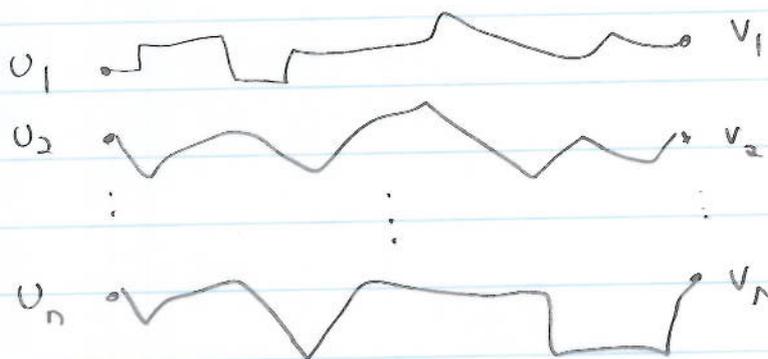
weight of a path in a weighted directed path π
the product of the weights along the path = $w(\pi)$



$$q(u, v) = \sum_{\pi: \pi(u, v)} w(\pi)$$

↑ all paths
from u to v.

"transition weight from u to v"

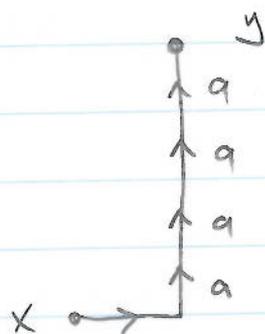


The total transition weight for all non-intersecting paths from (u_1, \dots, u_n) to (v_1, \dots, v_n) is

$$\det(q(u_i, v_j))_{i,j=1}^n$$

Transition functions in PNG

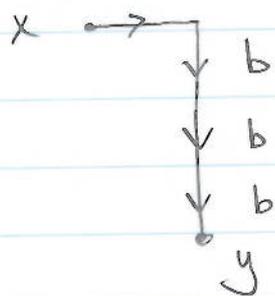
up-step



$$P_{2r, 2r+1}(x, y) = \begin{cases} a_{r+1}^{y-x}, & y \geq x \\ 0, & y < x \end{cases} = \hat{f}_{y-x}$$

$$0 \leq r < m$$

down-step



$$P_{2r+1, 2r+2}(x, y) = \begin{cases} b_{m-r}^{-(y-x)}, & y \leq x \\ 0, & y > x \end{cases} = \hat{g}_{y-x}$$

$$f(z) = \frac{1}{1-az}, \quad z = e^{i\theta}, \quad 0 \leq a < 1$$

$$= \sum_{k=0}^{\infty} a^k z^k = \sum_{k \in \mathbb{Z}} \hat{f}_k z^k$$

$$g(z) = \frac{1}{1-b/z} = \sum_{k=0}^{\infty} \frac{b^k}{z^k} = \sum_{k \in \mathbb{Z}} \hat{g}_k z^k$$

The probability of seeing a configuration \underline{x} in an m -layer PNG becomes ($n \geq$ number of active lines)

$$\frac{1}{Z_{n,m}} \prod_{r=0}^{m-1} \det(\varphi_{r,r+1}(x_j^r, x_k^{r+1}))_{j,k=1}^n, \quad (3)$$

where $Z_{n,m}$ is a normalization constant.

(remove ordering)

We can also write this as a prob. measure on the partitions (2).

Let $h_k(a_1, \dots, a_N)$ denote the k th complete, symmetric polynomial,

$$\sum_{k \in \mathbb{Z}} h_k(a_1, \dots, a_N) z^k = \prod_{j=1}^N \frac{1}{1-za_j}$$

The skew Schur polynomial $S_{\lambda/\mu}(a)$ is then

given by

$$S_{\lambda/\mu}(a) = \det(h_{\lambda_j - \mu_k + k - j}(a)) \quad \left(\begin{array}{l} \mu = \emptyset \\ \text{Schur} \\ \text{polynomial} \end{array} \right)$$

where the size of the determinant is $\geq \ell(\lambda)$.

($S_{\lambda/\mu}(a) = 0$ if $\mu \not\leq \lambda$). Note that if

$$\text{We } F(z) = \prod_{j=1}^N \frac{1}{1 - zc_j},$$

$$\text{then } h_k(a) = \hat{F}_k,$$

$$\text{So } S_{\lambda/\mu}(a) = \det(\hat{F}_{\lambda_j - \mu_k + k - j})$$

If we have just one variable $c_1 = a_{r+1}$, then we see that $F = f$ and

$$\det(\varphi_{2r, 2r+1}(X_j^{2r}, X_k^{2r+1}))_{n \times n} = \det(\hat{f}_{X_k^{2r+1} - X_j^{2r}})$$

$$= \det(\hat{f}_{\lambda_k^{r+1} - \mu_j^r + j - k}) = S_{\lambda^{r+1}/\mu^r}(a_{r+1}).$$

Similarly,

$$\det(\varphi_{2r-1, 2r}(X_j^{2r-1}, X_k^{2r})) = S_{\lambda^r/\mu^r}(b_{m-r}).$$

Hence, the induced probability measure on partitions becomes

$$\frac{1}{Z_{n,m}} S_{\lambda^1/\mu^0}(a_1) S_{\lambda^1/\mu^1}(b_m) S_{\lambda^2/\mu^1}(a_2) S_{\lambda^2/\mu^2}(b_m) \dots$$

$$\dots S_{\lambda^m/\mu^m}(b_1) \quad (4)$$

This is an instance of the so called Schur process introduced by Okounkov and Reshetikhin. (We can take more general Schur functions and different specializations.)

We will concentrate on the version (3) and $a_i = a, b_j = b, q = ab \in (0,1)$

Note that the height process $t \rightarrow h(x,t)$ is given by the positions of the last particles on each vertical line. Examples of questions:

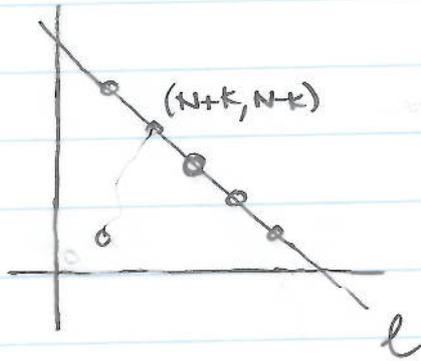
We would like to investigate a scaling limit like

$$\lim_{t \rightarrow \infty} \frac{h([c_1 \xi t^{2/3}], t) - c_2 t}{c_3 t^{1/3}} = \mathcal{J}_\xi(\xi)$$

^ limiting process
(log gamma)

with appropriate constants c_1, c_2, c_3 . Recall (KPZ scaling exponents)

$$h(2K, 2N-1) = G(N+K, N-K)$$

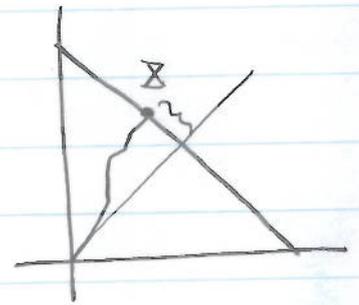


The height function is the values of $G(m, n)$ along the line l .

$G(m, n)$ is a point to point last-passage time.

We can also consider a point-to-line last-passage time:

$$G_{PL}(N) = \max_{|k| \leq N} G(N+k, N-k)$$



(= max. of the height curve) and ask for the position of the X for which the maximum is attained (transversal fluctuation)

I will come back to the analysis of the asymptotic questions. We have to understand more about the measure (3).

$$S_{\lambda^1/\lambda^1}(a_1) S_{\lambda^2/\lambda^1}(a_2) \dots S_{\lambda^n/\lambda^{n-1}}(a_n) S_{\lambda^1/\lambda^{n+1}}(b_n) S_{\lambda^{n+1}/\lambda^{n+2}}(b_{n+1})$$

$$\dots S_{\lambda^{2n-1}/\lambda^{2n}}(b_1)$$

Summation over $\lambda^1, \dots, \lambda^{n-1}$ and $\lambda^{n+1}, \dots, \lambda^{2n-1}$ gives
(Generalized Cauchy-Binet identity, see below) the following
probability measure on $\lambda = \lambda^n$:

$$\frac{1}{Z_n} S_{\lambda}(a_1, \dots, a_n) S_{\lambda}(b_1, \dots, b_n) \quad \text{Schur measure}$$

where

$$S_{\lambda}(a_1, \dots, a_n) = \det(h_{\lambda_i + \delta_{ij}}(a_1, \dots, a_n))_{1 \leq i, j \leq n}$$

is the Schur polynomial ($l(\lambda) \leq n$) and

$$\frac{1}{Z_n} = \prod_{i,j=1}^n (1 - a_i b_j).$$

Thus,

$$\begin{aligned} \mathbb{P}[G(n, n) \leq s] &= \mathbb{P}[\lambda_1 \leq s] \\ &= \frac{1}{Z_n} \sum_{\lambda; \lambda_1 \leq s} S_{\lambda}(a_1, \dots, a_n) S_{\lambda}(b_1, \dots, b_n). \end{aligned}$$

If $m \leq n$, we can get the distribution of $G(m, n)$ by

setting $a_{m+1} = \dots = a_n = 0$ since then $G(m,n) = G(n,n)$.

Consider the case $a_1 = \dots = a_m = \sqrt{q} = b_1 = \dots = b_n$.

$$S_{\lambda}(\underbrace{\sqrt{q}, \dots, \sqrt{q}}_m) = (\sqrt{q})^{|\lambda|} S_{\lambda}(\underbrace{1, \dots, 1}_{=1^m})$$

$$\mathbb{P}[G(m,n) \leq s] = (1-q)^{mn} \sum_{\lambda: \lambda \leq s} q^{|\lambda|} S_{\lambda}(1^m) S_{\lambda}(1^n)$$

We have that

$$S_{\lambda}(1^n) = \prod_{1 \leq i < j \leq n} \frac{q^{\lambda_i - \lambda_j + j - i}}{j - i}$$

┌ This follows from the classical formula for the Schur polynomial

$$S_{\lambda}(a) = \frac{\det(a_j^{\lambda_i + n - i})}{\det(a_j^{n - i})}$$

$$S_{\lambda}(1, q, \dots, q^{n-1}) = \frac{\det((q^{j-1})^{\lambda_i + n - i})_{n \times n}}{\det((q^{j-1})^{n - i})_{n \times n}}$$

$$= \frac{\det((q^{\lambda_i + n - i})^{j-1})_{n \times n}}{\det((q^{n - i})^{j-1})_{n \times n}} = \prod_{1 \leq i < j \leq n} \frac{q^{\lambda_j + n - j} - q^{\lambda_i + n - i}}{q^{mj} - q^{mi}} \quad ; q > 1$$

We obtain

$$P[G(m,n) \leq s] = (tq)^m \sum_{\lambda_j \leq s} q^{|\lambda|} \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

$S_\lambda(1^m) \neq 0$ requires $l(\lambda) \leq m$, so $\lambda_j = 0$ for $m < j \leq n$.

Define $t_j = \lambda_j + mj$ so that $0 \leq t_m < \dots < t_1$.
A computation gives

$$P[G(m,n) \leq s] = \frac{1}{m! Z_{m,n,q}} \sum_{t \in \{0, 1, \dots, sm\}^m} \Delta_m(t)^2 \prod_{j=1}^m \binom{t_j + nm}{t_j} q^{t_j},$$

where

$$\Delta_m(t) = \prod_{1 \leq i < j \leq m} (t_i - t_j)$$

(Meixner OPE)
($\beta=2$)

and $Z_{m,n,q}$ can be computed explicitly.

The Laguerre limit

Let $U(i,j)$, $i,j \geq 1$ be independent exponential random variables with mean 1,

$$\mathbb{P}[U(i,j) \leq y] = 1 - e^{-y}$$

$$H(m,n) = \max_{\pi} \sum_{(i,j) \in \pi} U(i,j), \quad \pi: (1,1) \rightarrow (m,n)$$

(Last-passage time for geometric random variables.)

We can consider the corner growth model for geometric instead of exponential random variables.

Let $W_L(i,j)$ be geometric r.v.'s with parameter $1/L$.
Then

$$\frac{1}{L} W_L(i,j) \xrightarrow{\mathcal{D}} U(i,j) \quad \text{as } L \rightarrow \infty.$$

Set

$$G_L(m,n) = \max_{\pi} \sum_{(i,j) \in \pi} W_L(i,j).$$

Then,

$$\frac{1}{L} G_L(m,n) \xrightarrow{\mathcal{D}} H(m,n), \quad L \rightarrow \infty.$$

Thus,

$$\mathbb{P}[H(m,n) \leq s] = \lim_{L \rightarrow \infty} \mathbb{P}\left[\frac{1}{L} G_L(m,n) \leq s\right]$$

$$= \lim_{L \rightarrow \infty} \frac{1}{m! Z_{m,n,L}} \sum_{t \in \{0,1,\dots,[Ls]+m-1\}^m} \Delta_m(t)^2 \prod_{j=1}^m \binom{t_j+n-1}{t_j} \left(1 - \frac{1}{L}\right)^{t_j}$$

$$= \frac{1}{m! Z_{m,n}} \int_{[0,s]^m} \Delta_m(x)^2 \prod_{j=1}^m x_j^{n-1} e^{-x_j} dx$$

$$= \mathbb{P}_{\text{Laguerre}}[\lambda_{\max} \leq s]$$

⌊ If M is an $n \times m$ matrix with independent complex $N(0, \frac{1}{2})$ entries then λ_{\max} is the maximal eigenvalue of the sample covariance matrix M^*M . ⌋

Here we see a direct contact with random matrix theory.