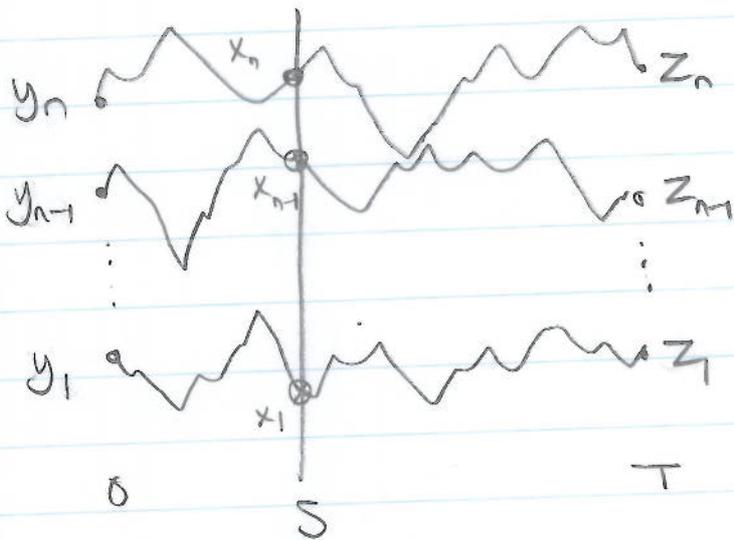


This opens up the possibility of doing an asymptotic analysis.

Before giving some asymptotic results let us discuss one further model

### Non-colliding Brownian motions

Consider  $n$  non-colliding Brownian motions starting at  $y_1, \dots, y_n$  at time 0 and ending at  $z_1, \dots, z_n$  at time  $T$



Let  $x_1, \dots, x_n$  be the positions of the Brownian particles at time  $S$ ,  $0 < S < T$ .

If we let  $T \rightarrow \infty$  the point process  $\{x_j\}_{j=1}^n$  has (up to a rescaling of time/space) the same distribution as Dyson BM starting at  $y_1, \dots, y_n$  at time 0.

$M(t)$  Hermitian matrix at each time  $t$ .

$$M(t) = (x_{jh}(t) + iy_{jh}(t))_{n \times n}$$

$\uparrow \quad \uparrow$   
 independent BM's (OU-processes)

$$M(0) = \text{diag}(y_1, \dots, y_n)$$

$M(t)$  has eigenvalues  $(\lambda_1(t), \dots, \lambda_n(t)) \in \mathbb{R}^n$ .

The analogue of LGV is the (older) Karlin-McGregor theorem. The induced probability measure on  $x_1, \dots, x_n$  is

$$\frac{1}{n! Z_n} \det(P_S(y_j, x_n)) \det(P_{T-S}(z_j, x_n))$$

where

$$P_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

We can also consider particle configurations at several times in analogy with above. This is a kind of continuous version of the models discussed previously.

Compare Random walk  $\rightarrow$  BM

Difference bulk/edge.

We can expect the models to be more similar at the edge.

Set  $z_j = \varepsilon(j-1)$ ,  $1 \leq j \leq n$ . This simplifies the analysis.

$$\begin{aligned} & \det(P_{T-s}(\varepsilon(j-1), x_n)) \\ &= \frac{1}{(\sqrt{2\pi(T-s)})^n} \det\left(e^{-\frac{(\varepsilon(j-1) - x_n)^2}{2(T-s)}}\right) \\ &= \frac{1}{(2\pi(T-s))^{n/2}} \left( \prod_{j=1}^n e^{-\frac{\varepsilon^2(j-1)^2}{2(T-s)} - \frac{x_j^2}{2(T-s)}} \right) \underbrace{\det\left(e^{-\frac{\varepsilon(j-1)x_n}{2(T-s)}}\right)}_{n \times n \text{ Vandermonde det.}} \end{aligned}$$

(Note that  $T \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  has a very similar effect.)

$$Z_n = \det(P_T(\varepsilon(j-1), y_n))$$

In the limit  $T \rightarrow \infty$  we get the following probability measure on  $\{x_j\}_{j=1}^n$

$$\frac{1}{n! Z_n'} \det\left(e^{-\frac{(y_j - x_n)^2}{2s}}\right) \Delta_n(x) \quad (6)$$

— \* —

$$\det\left(e^{-\frac{\varepsilon(j-1)x_n}{2(T-s)}}\right) = \prod_{1 \leq j < k \leq n} \left( e^{-\frac{\varepsilon x_k}{2(T-s)}} - e^{-\frac{\varepsilon x_j}{2(T-s)}} \right)$$

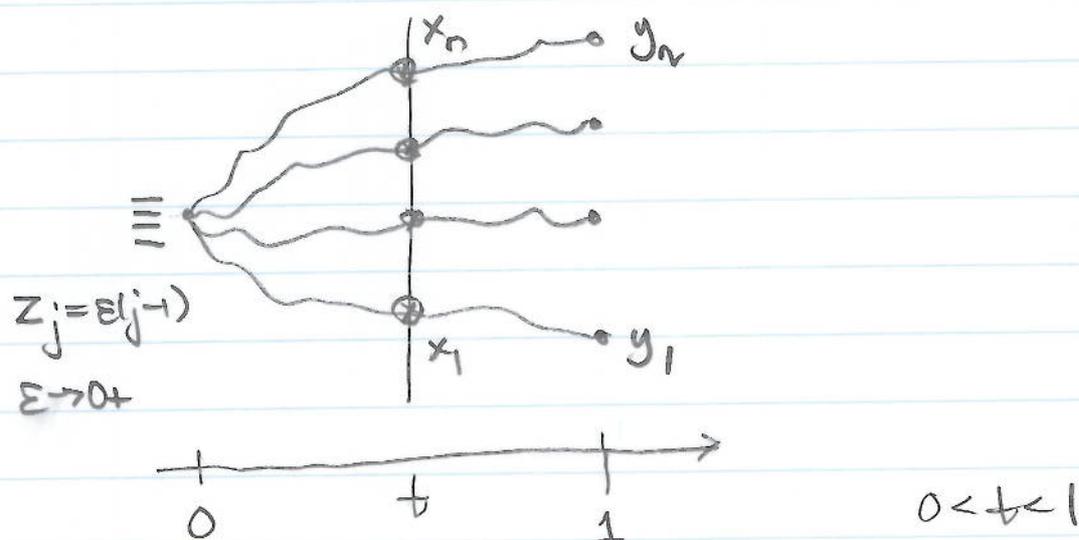
For the matrix model we get the following probability measure on  $M(S)$

$$C_n e^{-\frac{1}{2S} \text{Tr}(M(S) - M(0))^2} dM$$

Integrating out the angular variables using the IZ-formula to get the induced eigenvalue measure gives exactly (6), where  $x_1, \dots, x_n$  are the eigenvalues of  $M(S)$ .

Thus in this situation the non-colliding BM model has a direct relation with a random matrix model. Hence, we can expect some similarities also between our discrete models and random matrix models.

Consider the following situation



The particles at time  $t$  will form a determinantal point process. We can deduce a double integral formula for its kernel.

$$K_{n,\varepsilon}(u,v) = \sum_{j,k=1}^n P_{1-t}(y_k, u) (A^{-1})_{kj} P_t(z_j, v)$$

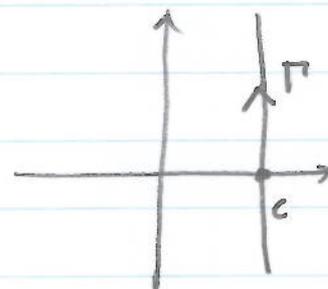
$$\stackrel{\substack{= \\ \uparrow \\ \text{Cramer's rule}}}{=} \sum_{k=1}^n P_{1-t}(y_k, u) \frac{\det(A|p)_k}{\det A} \quad \leftarrow \begin{array}{l} \text{replace column } k \text{ in } A \\ \text{by } p. \end{array}$$

$$A = (P_i(z_j, y_n))_{n \times n}$$

$$p = \begin{pmatrix} P_1(z_1, v) \\ \vdots \\ P_1(z_n, v) \end{pmatrix}$$

We use the Gaussian integral

$$P_t(x, y) = \frac{e^{-\frac{y^2}{2(1-t)}}}{i\sqrt{2\pi(1-t)}} \int_{\Gamma_c} e^{\frac{1}{2(1-t)}(w^2 - 2yw)} P_1(x, w) dw$$



Taking out the integral from  $\det(A|p)_k$  we get the same determinant as in  $\det A$  but with  $y_k$  replaced by  $w$ .

But, as we saw above, when  $z_j = \varepsilon(j-1)$ ,  $\det A$  is a Vandermonde determinant

$$\det \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\varepsilon(j-1) - y_h)^2} \right)$$

$$= \frac{1}{(2\pi)^{n/2}} \prod_{j=1}^n e^{-\frac{1}{2}(\varepsilon(j-1)^2 + y_j^2)} \prod_{1 \leq j < h \leq n} (e^{\varepsilon y_h} - e^{i\varepsilon y_j})$$

Write

$$y_j^{(h)} = \begin{cases} y_j & \text{if } j \neq h \\ w & \text{if } j = h \end{cases}$$

Then

$$K_{n,\varepsilon}(u,v) = \frac{e^{\frac{v^2}{2(1-t)}}}{2\pi i(1-t)} \int_{\Gamma_c} e^{\frac{1}{2(1-t)}(w^2 - 2vw)} \sum_{h=1}^n e^{-\frac{(u-y_h)^2}{2(1-t)}}$$

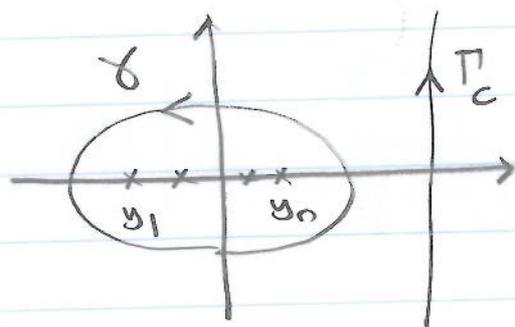
$$\cdot e^{\frac{1}{2}(y_h^2 - w^2)} \prod_{j=1, j \neq h}^n \frac{e^{\varepsilon w} - e^{\varepsilon y_j}}{e^{\varepsilon y_h} - e^{\varepsilon y_j}} dw$$

Here we can take the limit  $\varepsilon \rightarrow 0+$  and we can also rewrite the  $h$ -sum using the residue theorem. This gives

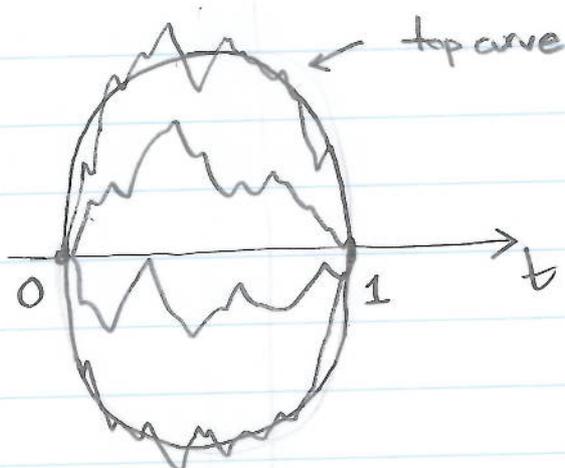
$$K_n(u,v) = \frac{e^{\frac{1}{2(1-t)}(v^2 - u^2)}}{(2\pi i)^2(1-t)} \int_{\Gamma_c} dw \int_{\gamma} dz e^{\frac{1}{2(1-t)}(w^2 - z^2) + \frac{1}{1-t}(uz - vw)}$$

$$\cdot \prod_{j=1}^n \frac{w - y_j}{z - y_j} \frac{1}{w - z}$$

(6:2)



When all  $y_j = 0$  we have the following picture



At each time we have a rescaled version of GUE  
(We start with  $M(0) = 0$  in the matrix model above.)

The top curve in the discrete models should be similar to the top curve here in the limit as the number of paths  $\rightarrow \infty$ .

Consider  $t = 1/2$ .

$$K_n(w, v) = \frac{2e^{v^2 - w^2}}{(2\pi i)^2} \int_{\Gamma_c} dw \int_{\gamma} dz e^{\frac{1}{2}(w^2 - z^2) + 2(vz - vw)} \left(\frac{w}{z}\right)^n \frac{1}{w - z} \quad (7)$$

(This is the GUE-kernel. It can also be rewritten in terms of Hermite polynomials.)

## Structure of the argument so far

- Combinatorial part / Derivation of the probability measure / point process. Integrating out to get an eigenvalue measure.
- Algebraic part (measure  $\rightarrow$  det. p.p.)
- Asymptotic analysis (scaling limit)

The last part remains to be done. Let us consider the edge asymptotics of (7). We will only give a sketch.

$$u = an^{1/2} + cn^{-1/6} \xi$$

$$v = an^{1/2} + cn^{-1/6} \eta$$

Take  $w \rightarrow \sqrt{n} w$ ,  $z \rightarrow \sqrt{n} z$  in (7). We can remove  $e^{v^2 - u^2}$  since that does not affect the correlation functions.

$$\frac{2cn^{1/3}}{(2\pi i)^2} \int_c \int_\gamma dw dz e^{-n(\frac{1}{2}z^2 - 2az + \log z) + n(\frac{1}{2}w^2 - 2aw + \log w)}$$

$$= e^{2cn^{1/3}(\xi z - \eta w)} \frac{1}{w-z} = cn^{-1/6} K_n(u, v) \quad (8)$$

For a saddle-point argument we have to analyze

$$f(z) = \frac{1}{2} z^2 - 2az + \log z$$

Edge behaviour will correspond to a double zero.

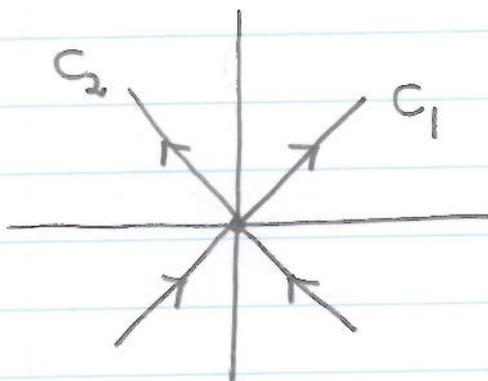
$$f'(z) = z - 2a + \frac{1}{z}$$

$$f''(z) = 1 - \frac{1}{z^2} = 0 \Rightarrow z = \pm 1$$

$z_c = 1$  and  $f'(z_c) = 0$  gives  $a = 1$ .

$$f^{(3)}(z) = \frac{2}{z^3}, \quad f^{(3)}(1) = 2$$

$$f(z) = \frac{1}{3} (z-1)^3 + \dots, \quad z \text{ close to } 1$$



Local analysis  
around the saddle  
point

$$z = 1 + dn^{-1/3} \zeta, \quad w = 1 + dn^{-1/3} \omega$$

In the limit we obtain (skipping many details)

$$\lim_{n \rightarrow \infty} c n^{-1/6} K_n(\sqrt{n} + c n^{-1/6} \xi, \sqrt{n} + c n^{-1/6} \eta)$$

$$= \frac{2cd}{(2\pi i)^2} \int_{C_1} d\omega \int_{C_2} d\zeta e^{-\frac{d^3}{3}\zeta^3 + \frac{d^3}{3}\omega^3 + 2cd\xi\zeta - 2cd\eta\omega} \frac{1}{\omega - \zeta}$$

Choose  $d=1$  and  $2cd=1$ ,  $c=1/2$ .

$$\lim_{n \rightarrow \infty} \frac{1}{2} n^{-1/6} K_n(\sqrt{n} + \frac{1}{2} n^{-1/6} \xi, \sqrt{n} + \frac{1}{2} n^{-1/6} \eta)$$

$$= \frac{1}{(2\pi i)^2} \int_{C_1} d\omega \int_{C_2} d\zeta e^{\frac{1}{3}(\omega^3 - \zeta^3) + \xi\zeta - \eta\omega} \frac{1}{\omega - \zeta}$$

$$= K_{\text{Ai}}(\xi, \eta) \quad \left( = \int_0^{\infty} \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda \right)$$

the Airy kernel

The point process at the edge converges to the Airy kernel point process. We have that

$$\mathbb{E} \left[ \prod_{j=1}^n (1 - g(2n^{1/6}(x_j - \sqrt{n}))) \right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^m g(2n^{1/6}(x_j - \sqrt{n})) \det(K_n(x_j, x_h)) d^m x$$

$(y_j = 2n^{1/6}(x_j - \sqrt{n}))$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^m g(y_j) \det\left(\frac{1}{2n^{1/6}} K_n\left(\sqrt{n} + \frac{y_j}{2n^{1/6}}, \sqrt{n} + \frac{y_h}{2n^{1/6}}\right)\right) d^m y$$

$$\xrightarrow{n \rightarrow \infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{\mathbb{R}^m} \prod_{j=1}^m g(y_j) \det(K_{Ai}(y_j, y_h))_{m \times m} d^m y$$

$$= \det(I - K_{Ai}g)_{L^2(\mathbb{R})} \quad (*) \quad (\text{I am omitting technical details.})$$

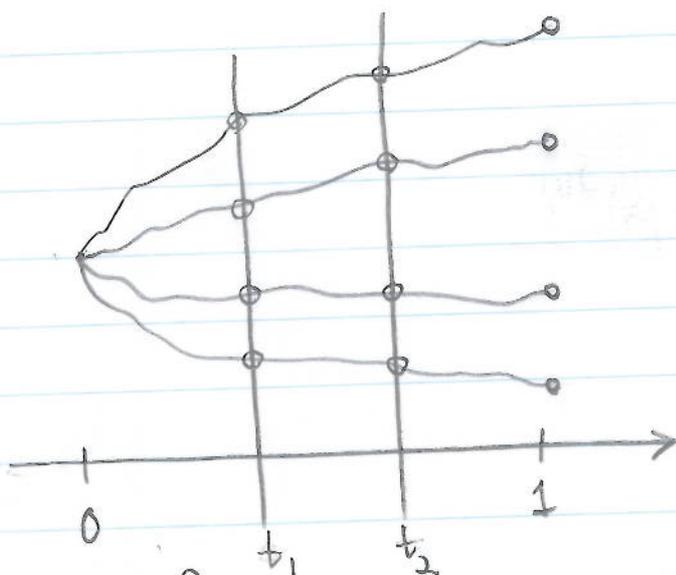
In particular we get the Tracy-Widom distribution

$$\lim_{n \rightarrow \infty} \mathbb{P}[2n^{1/6}(x_{\max} - \sqrt{n}) \leq t] = F_{TW}(t)$$

$$= \det(I - K_{Ai})_{L^2[t, \infty)}.$$

We can extend this analysis to several lines to get the extended Airy kernel in the limit.

(\*) This proves weak convergence of the rescaled point process to the Airy kernel point process.



$\{X_j^r\}_{j=1}^n$  positions on the  $r$ -th line (time  $t_r$ )

Determinantal p.p. with kernel (extended kernel)

$$K_n(s, u; t, v) = \frac{e^{\frac{v^2}{2(1-t)} - \frac{u^2}{2(1-s)}}}{(2\pi i)^2 \sqrt{(1-s)(1-t)}} \int_{\Gamma_c} dw \int_{\gamma} dz$$

time space  
↓ ↓

$$e^{\frac{t}{2(1-t)} w^2 - \frac{s}{2(1-s)} z^2 + \frac{1}{1-s} u z - \frac{1}{1-t} v w} \prod_{j=1}^n \frac{w - y_j}{z - y_j} \frac{1}{w - z}$$

$$= P_{t \rightarrow s}(u, v)$$

(9)

↑ Gaussian transition function

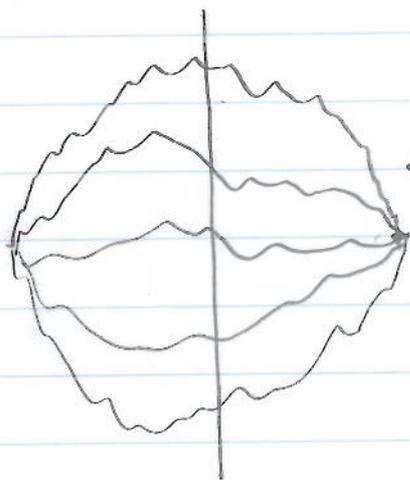
(Compare (4.2) above, Eynard-Mehta theorem)

The structure is very similar to that in (6.2) so we can do a very similar asymptotic analysis in the case  $y_1 = \dots = y_n = 0$

as above. In this case we take the scaling limit

$$s = \frac{1}{2}(1 + \tau_1 n^{-1/3}), \quad t = \frac{1}{2}(1 + \tau_2 n^{-1/3})$$

$$u = \sqrt{n} + \frac{1}{2}(\xi_1 - \tau_1^2)n^{-1/6}, \quad v = \sqrt{n} + \frac{1}{2}(\xi_2 - \tau_2^2)n^{-1/6}$$



top curve  $h_n(t)$ ,  $0 \leq t \leq 1$ ,  
a stochastic process

$$\lim_{n \rightarrow \infty} K_n(s, u; t, v) \cdot \frac{1}{2} n^{-1/6} = K_{\text{ext. Airy}}(\tau_1, \xi_1, \tau_2, \eta)$$

$$= \begin{cases} \int_0^{\infty} e^{-\lambda(\tau_1 - \tau_2)} \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda, & \text{if } \tau_1 \geq \tau_2 \\ - \int_{-\infty}^0 e^{-\lambda(\tau_1 - \tau_2)} \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda, & \text{if } \tau_1 < \tau_2 \end{cases}$$

(Can also be written using double contour integrals.)

The top curve has a scaling limit known as the Airy-2 process

Thm. There exists a continuous, stationary stochastic process  $\tau \rightarrow A_2(\tau)$  with finite-dimensional distribution

$$\mathbb{P}[A_2(\tau_1) \leq a_1, \dots, A_2(\tau_m) \leq a_m] \\ = \det(\mathbf{I} - f^{1/2} K_{\text{ext. Airy}} f^{1/2})_{L^2(\{\tau_1, \dots, \tau_m\} \times \mathbb{R})},$$

where

$$f(\tau_j, x) = 1_{(a_j, \infty)}(x).$$

Note that

$$\mathbb{P}[A_2(\tau) \leq a] = F_{\text{TW}}(a),$$

so the Airy-2-process extends the TW-distribution.

Thm

$$2n^{1/6} \left( h_n \left( \frac{1}{2} (1 + \tau n^{-1/3}) \right) - \sqrt{n} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} A_2(\tau) - \tau^2$$

(Also functional limit theorem.)

Let us return to the PNG-process studied above.

$x \rightarrow h(x, t)$  height curve at time  $t$

The extended kernel describing the multi-layer PNG-process has a double contour integral formula that can be analyzed asymptotically in a very similar way

$$G(N+k, N-k) = h(2k, 2N-1)$$

Thm

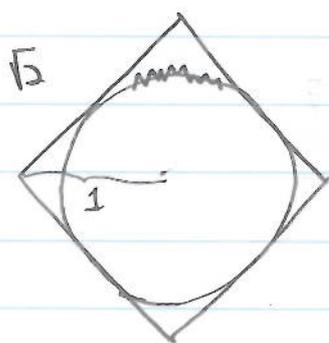
$$\frac{h(c_1(q)t^{2/3}\xi, t) - c_2(q)t}{c_3(q)t^{1/3}} \xrightarrow{\mathcal{D}} A_2(\xi) - \xi^2$$

as  $t \rightarrow \infty$ .

$$c_1(q) = 2^{1/3} q^{-1/6} (4\sqrt{q})^{2/3}$$

$$c_2(q) = \sqrt{q} (1-\sqrt{q})^{-1}, \quad c_3(q) = 2^{-1/3} q^{1/6} (1+\sqrt{q})^{1/3} (4\sqrt{q})^{-1}$$

We also have an analogous result for the Aztec diamond.



rescaled by  $n$ .

$t \rightarrow X_n(t)$ ,  $-n \leq t \leq n$ , NPR-boundary process

$$\frac{X_n(2^{-1/6} n^{2/3} \tau) - n/\sqrt{2}}{2^{-5/6} n^{1/3}} \xrightarrow{D} A_2(\tau) - \tau^2$$

as  $n \rightarrow \infty$ .

The Airy-2-process occurs in 3 different contexts:

- limit of top-eigenvalue in PBM  
(non-colliding BM's) (coupled matrix models)
- asymptotical limit of 1-dimensional interface in a local random growth model
- boundary of non-frozen region in random tiling (dimer-) model

Universality problem.