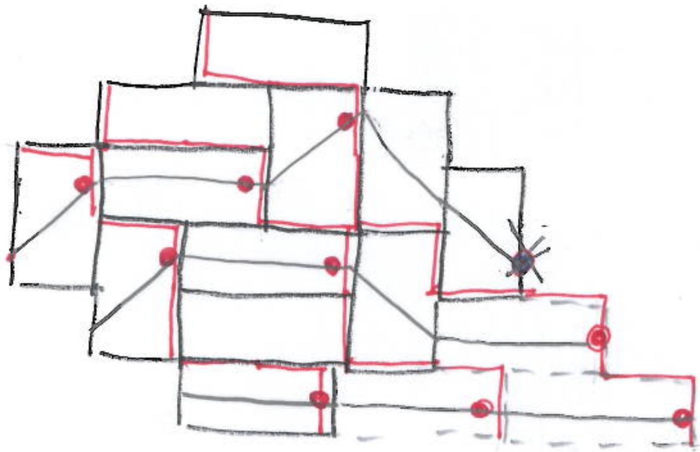
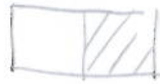
The Aztec diamond

Non-intersecting paths and particles

Zig-zag paths and particles



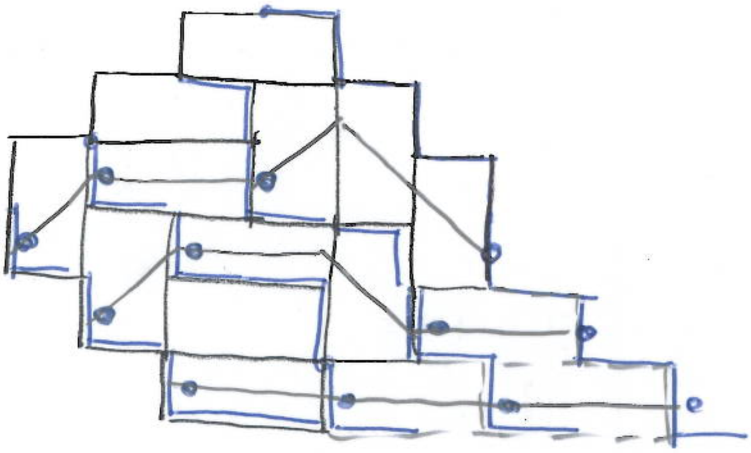


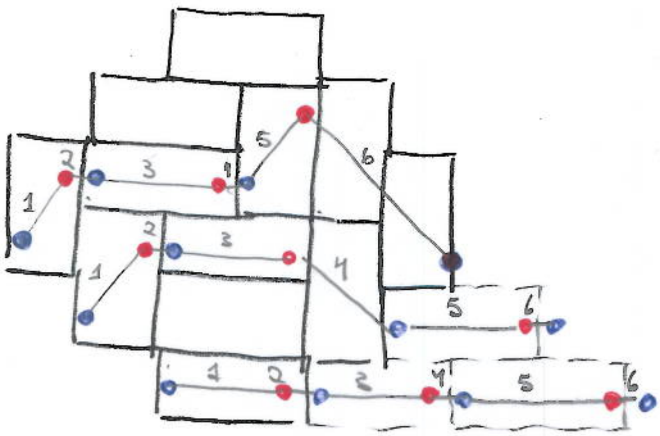
N

S

W

E





blue → red → blue → red → blue → red → blue

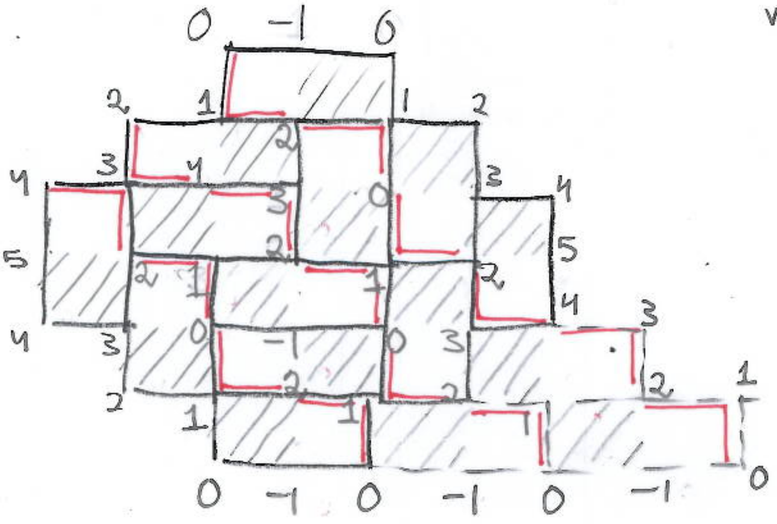
α β α β α β

two types of steps

Map to a better set of paths

Height Function

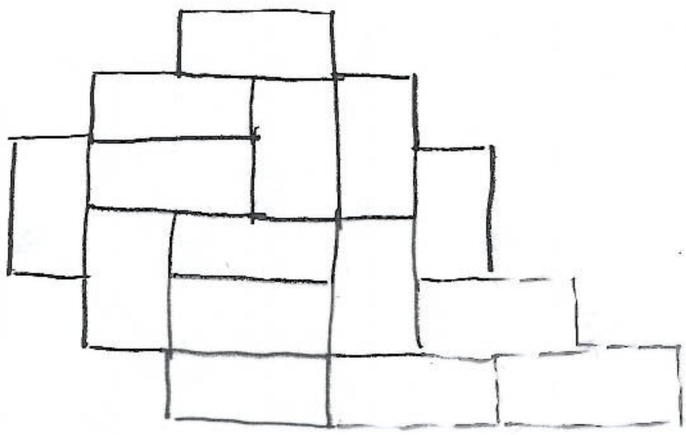
black square on right = +1
white " " " = −1

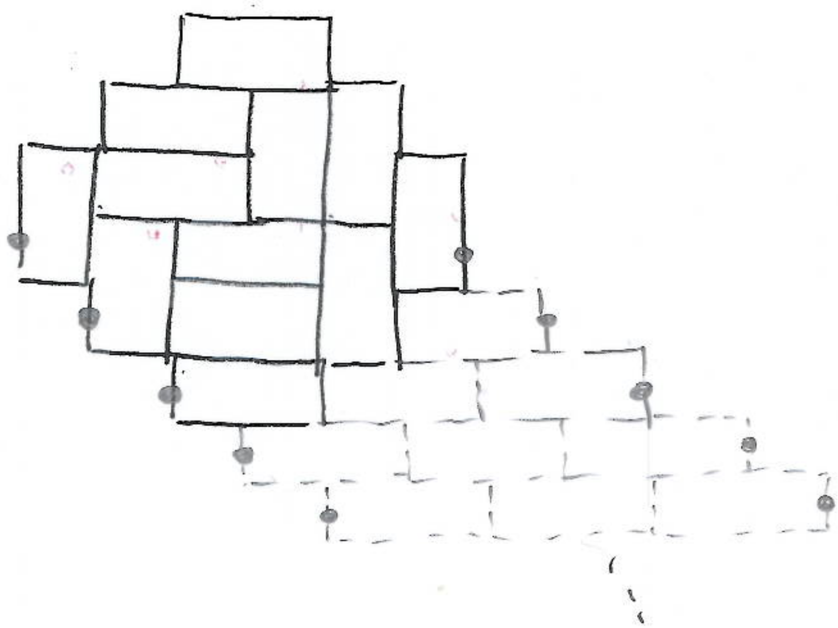


┐ = −2

└ = +2

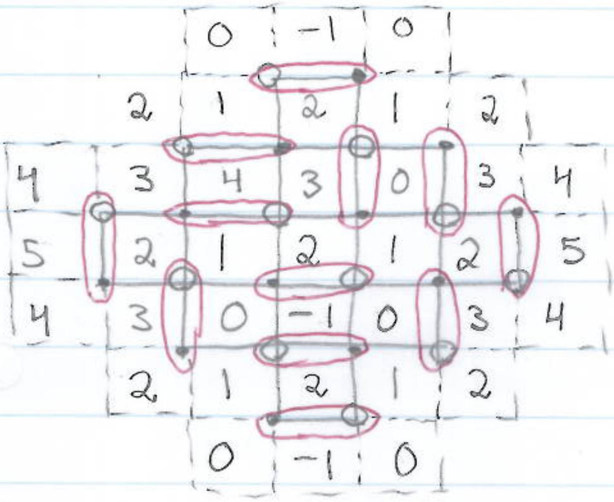
boundary values fixed





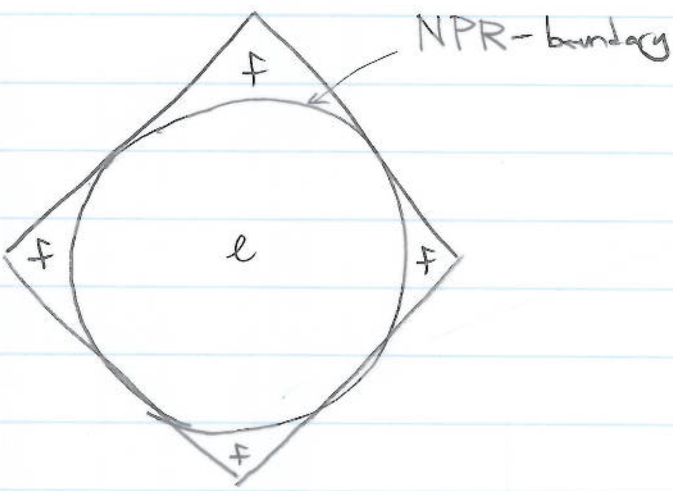
14:2
15:2

Dimer model



Dimer representation of the same tiling and the height function.

Arctic circle

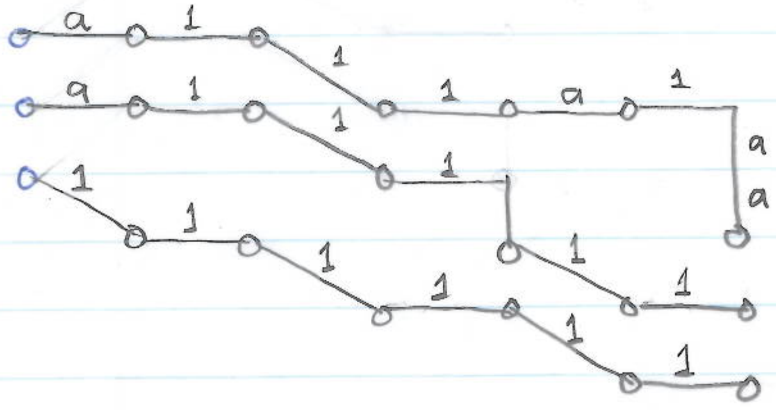


NPR boundary process

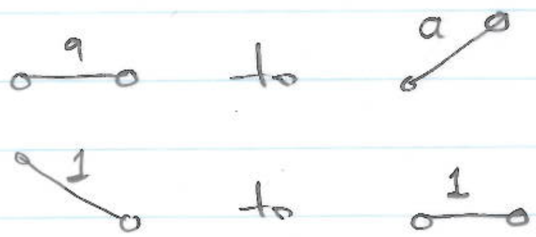
$X_n(t)$, $|t| \le n$

Analogue in the present problem of the height process in the previous problem.
The NPR-process is given by the top particles on each vertical line.

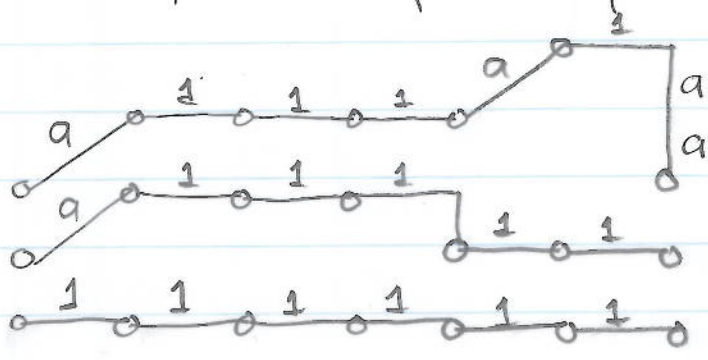
$\alpha \quad \beta \quad \alpha \quad \beta \quad \alpha \quad \beta$



α-step: change



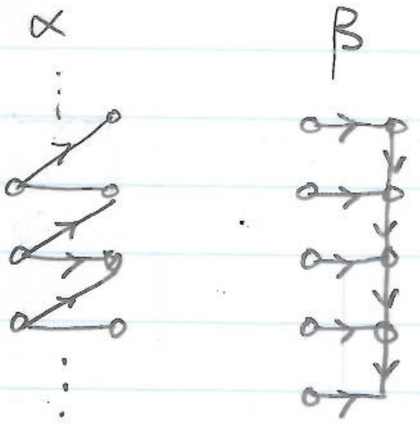
$\alpha \quad \beta \quad \alpha \quad \beta \quad \alpha \quad \beta$



β-step

Vertical dominos weight a
Horizontal " " 1

Non-intersecting paths on a weighted directed graph



Transition functions for the random walk

$\varphi_{r,r+1}(x,y)$, $0 \le r < 2n$. Fixed initial and final point.

$$\varphi_{2r,2r+1}(x,y) = \begin{cases} a & \text{if } y-x=1 \\ 1 & \text{if } y-x=0 \\ 0 & \text{otherwise} \end{cases} = \hat{f}_{y-x}.$$

$f(z) = 1+az$

$$\varphi_{2r+1,2r+2}(x,y) = \begin{cases} a^{-(y-x)} & \text{if } y-x \le 0 \\ 0 & \text{otherwise} \end{cases} = \hat{g}_{y-x}$$

$g(z) = (1-a/z)^{-1}$

We want to analyze the induced probability measure on non-intersecting paths/particle system = point process

Probability measure induced on the particle configuration:

$$\frac{1}{(n!)^{n-1} Z_n} \prod_{r=0}^{2n-1} \det(\varphi_{r,r+1}(x_j^r, x_k^{r+1}))_{n\times n}$$

$(x_j^r)_{j=1}^n$ configuration on the r:th line

$$x_j^0 = 1-j = x_j^{2n}$$

We have fixed inital and final positions. Note that we can extend the process by adding further initial and final points. This does not change the probability measure on the n top paths since the remaining $N-n$, $N \geq n$, paths are fixed.

$$x^r = (x_j^r)_{j=1}^N \quad , \quad N \geq n \quad , \quad \underline{x} = (x^r)_{0 \leq r \leq 2n}.$$

(This will be used later.)

<u>Point processes</u>

Consider a <u>point process</u> on X ($= \mathbb{R}, \mathbb{Z}, \{0,\dots,M\} \times \mathbb{Z}$)

$\{x_i\}$ point configuration in X (finite or infinite)

$|\{x_i\} \cap C| < \infty$ for any compact (bounded) set $C \subseteq X$

(<u>locally finite</u>)

$$h \in C_c^+(X) \quad , \quad g = 1 - e^{-h} \in C_c^+(X) \, , \quad 0 \leq g \leq 1.$$

$$\mathbb{E}\left[\prod_i e^{-\sum h(x_i)}\right] = \mathbb{E}\left[\prod_i (1-g(x_i))\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{X^m} \prod_{j=1}^{m} g(x_j)\, \rho_m(x_1,\dots,x_m)\, d\lambda^n(x)$$

correlation functions

reference measure on X

We have a <u>determinantal point process</u> if

$$\rho_m(x_1,\dots,x_m) = \det\left(K(x_i,x_j)\right)_{m\times m}$$

<u>correlation kernel</u>

$$K \in L^2_{loc}(X\times X)$$

We have that (under suitable assumptions)

$$\mathbb{E}\left[\prod_i (1-g(x_i))\right] = \det(I - Kg)_{L^2(X)}$$

$$= \det(I - g^{1/2} K g^{1/2})_{L^2(X)}$$

a Fredholm determinant.

The following determinantal identity is very useful in connection with determinantal point processes.

(Generalized) Cauchy-Binet identity

$$\frac{1}{n!}\int_{X^n} \det\left(\varphi_j(x_h)\right)_{n\times n} \det\left(\psi_j(x_h)\right) d\lambda^n(x)$$

$$= \det\left(\int_X \varphi_j(x)\psi_h(x)\,d\lambda(x)\right)_{n\times n}$$

— * —

Let us go back to the problems we are studying. By the GCB-identity (λ = counting measure on $\mathbb{Z}$)

$$\sum_{x_1^{r+1}<\dots<x_n^{r+1}} \det\left(\varphi_{r,r+1}(x_j^r, x_h^{r+1})\right)_{n\times n} \det\left(\varphi_{r+1,r+2}(x_j^{r+1}, x_h^{r+2})\right)_{n\times n}$$

$$= \det\Big(\underbrace{\sum_{y\in\mathbb{Z}} \varphi_{r,r+1}(x_j^r, y)\,\varphi_{r+1,r+2}(y, x_h^{r+2})}_{:=\ \varphi_{r,r+2}(x_j^r,\, x_h^{r+2})}\Big)_{n\times n}$$

Set

$$\varphi * \psi(x,y) = \sum_{x\in\mathbb{Z}} \varphi(x,z)\,\varphi(z,y)$$

$$\varphi_{r,s}(x,y) = \varphi_{r,r+1} * \dots * \varphi_{s-1,s}(x,y) \quad , r<s$$

$$\varphi_{r,s} \equiv 0 \quad \text{if } r>s.$$

The probability measure on the particles on a single line $r = R$, then has the form $(x_j^R = x_j)$

$$\frac{1}{Z_{N,M} N!} \underbrace{\det\left(\varphi_{0,R}(x_j^0, x_k)\right)_{N\times N}}_{\doteq\, \varphi_j(x_k)} \underbrace{\det\left(\varphi_{R,M}(x_k, x_j^M)\right)_{N\times N}}_{\psi_j(x_k)}$$

By GCB:

$$Z_{N,M} = \det\Big(\underbrace{\sum_{x\in\mathbb{Z}} \varphi_j(x)\, \psi_k(x)}_{A}\Big)$$

<u>Proposition</u> A probability measure of the form

$$\frac{1}{Z_N N!} \det\left(\varphi_j(x_k)\right)_{N\times N} \det\left(\psi_j(x_k)\right)_{N\times N} d^N\lambda(x)$$

defines a determinantal p.p. with correlation kernel

$$K_N(x,y) = \sum_{j,k=1}^{N} \psi_j(x)\, (A^{-1})_{jk}\, \varphi_k(y),$$

where

$$A = \left(\int_X \varphi_j(x)\psi_h(x)\, d\lambda(x) \right)_{N\times N}$$

<u>Proof:</u> (Sketch)

$$\mathbb{E}\left[\prod_{j=1}^{N} (1-g(x_j))\right]$$

$$= \frac{1}{Z_N N!} \int_{X^N} \prod_{j=1}^{N} (1-g(x_j)) \det(\varphi_j(x_h)) \det(\psi_j(x_h))\, d^N\lambda(x)$$

$$= \frac{1}{\det A} \det\left(\int_X \varphi_j(x)\psi_h(x)(1-g(x))\, d\lambda(x) \right)$$

$$= \frac{\det\left(a_{jh} - \int_X \varphi_j(x)\psi_h(x) g(x)\, d\lambda(x)\right)}{\det(a_{jh})}$$

$$= \det\left(\delta_{jh} - \int_X \underbrace{\left(\sum_{i=1}^{N} (A^{-1})_{ji}\varphi_i(x) g(x)\right)}_{b(j,x)} \underbrace{\psi_h(x)}_{c(x,h)}\, d\lambda(x)\right)$$

$\ulcorner$ $b : L^2(X) \to \ell^2(\{1,\dots,N\}) \cong \mathbb{R}^N$

$c : \ell^2(\{1,\dots,N\}) \to L^2(X)$ $\lrcorner$

$$= \det(I - bc)_{\ell^2(\{1,\dots,N\})} = \det(I - cb)_{L^2(X)}$$

$$\left\lceil (cb)(x,y) = \sum_{i,j=1}^{N} \psi_j(x)(A^{-1})_{ji}\,\varphi_i(y)g(y) = K_N(x,y)g(y). \right\rfloor$$

$$= \det(I - Kg)_{L^2(X)}.$$

$\blacksquare$

- A^{-1} is the difficult part

Can be extended to the case when we consider particles on several lines.

Consider a probability measure on $\{1,\dots,M-1\} \times X$ of the form

$$\frac{1}{Z_{N,M}(N!)^{M-1}} \prod_{r=0}^{M-1} \det\left(\varphi_{r,r+1}(x_j^r, x_j^{r+1})\right)_{N\times N} d^{N(M-1)}\lambda(x) \qquad (*)$$

where $\{x_j^0\}_{j=1}^N$ and $\{x_j^M\}_{j=1}^N$ are fixed.

<u>Eynard-Mehta theorem</u>: $(*)$ gives a determinantal p.p. on $\{1,\dots,M-1\} \times X$ with correlation kernel

$$K_{N,M}(r,x;s,y) = -\varphi_{r,s}(x,y) + \tilde{K}_{N,M}(r,x;s,y), \qquad (4.2)$$

$$\tilde{K}_{N,M}(r,x;s,y) = \sum_{j,k=1}^{N} \varphi_{r,M}(x,x_j^M)(A^{-1})_{jk}\,\varphi_{0,s}(x_k^0,y),$$

where

$$A = \left(\varphi_{0,M}(x_j^0,x_k^M)\right)_{N\times N}$$

— * —

Let us return to our examples. $a_j = a$, $b_j = b$

$$x_j^0 = x_j^M = 1-j, \quad 1\le j\le N, \quad M = 2m^{(*)}$$

$$\varphi_{2r,2r+1}(x,y) = \hat{f}_{y-x} \quad ; \quad \varphi_{2r+1,2r+2}(x,y) = \hat{g}_{y-x}$$

It follows that

$$\varphi_{0,M}(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})^m g(e^{i\theta})^m e^{-i(y-x)\theta}\,d\theta,$$

which gives

(*) $m=n$ for the Aztec diamond

$$\varphi_{0,M}(1-j,1-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})^m\, g(e^{i\theta})^m\, e^{-i(j-k)\theta}\, d\theta$$

A is a Toeplitz matrix.

In both of our examples we can let $N \to \infty$ without changing the nature of the problem. In the limit we want to invert an infinite Toeplitz matrix.

Wiener–Hopf factorization

$$f(z) = f_+(z)\, f_-(z) \quad , \quad g(z) = g_+(z)\, g_-(z)$$

f_+ analytic, in $|z|<1$, $\neq 0$ in $|z| \le 1$, $f_+(0)=1$

f_- " in $|z|>1$, $\neq 0$ in $|z| \geq 1$, $f_-(\infty)=1$

For PNG: $f_+(z) = \dfrac{1}{1-az}$, $f_-(z) = 1$

$g_+(z) = 1$, $g_-(z) = \dfrac{1}{1-b/z}$.

Thus,

$$F(z) = f(z)^m g(z)^m = g_-(z)^m\, f_+(z)^m = F_-(z)\, F_+(z)$$

Let $T[F(z)]$ denote the infinite Toeplitz matrix with symbol F

$$T[F(z)] = \det\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\theta})e^{-i(j-k)\theta}\,d\theta\right)_{j,k=1}^{\infty}$$

Then,

$$T[F_-F_+]^{-1} = (T[F_-]T[F_+])^{-1}$$

$$= T[F_+]^{-1}T[F_-]^{-1} = T[F_+^{-1}]T[F_-^{-1}]$$

Thus,

$$\tilde{K}_M(r,x;s,y) = \lim_{N\to\infty} \tilde{K}_{N,M}(r,x;s,y)$$

$$= \sum_{j,k=1}^{\infty} \varphi_{r,M}(x,1-j)\left(T(F_+^{-1})T(F_-^{-1})\right)_{jk}\varphi_{0,s}(1-k,y)$$

Consider the generating function at even numbered vertical lines

$$\sum_{x,y\in\mathbb{Z}} \tilde{K}_M(2r,x;2s,y)\,z^{-x}w^{y}$$

Note that,

$$\sum_{x\in\mathbb{Z}} \varphi_{2r,2m}(x,1-j)\,z^{-x} = f(z)^{m-r}\,g(z)^{m-r}\,z^{j-1}$$

and

$$\sum_{y\in\mathbb{Z}} \varphi_{0,2s}(1-h,y)\,w^{y} = f(w)^{s}\, g(w)^{s}\, w^{1-h}$$

Also,

$$\left(T(F_+^{-1})T(F_-^{-1})\right)_{jh} = \sum_{\ell=1}^{\infty} T(F_+^{-1})_{j\ell}\; T(F_-^{-1})_{\ell h}\,.$$

Now,

$$\sum_{j=1}^{\infty} z^{j}\, T(F_+^{-1})_{j\ell} = z^{\ell} \sum_{j=1}^{\infty} z^{j-\ell}\, T(F_+^{-1})_{j\ell}$$

$$= \frac{z^{\ell}}{F_+(z)} = \frac{z^{\ell}}{f(z)^{m}}$$

and similarly,

$$\sum_{h=1}^{\infty} T(F_-^{-1})_{\ell h}\, w^{-h} = \frac{w^{-\ell}}{g(w)^{m}}$$

Thus,

$$\sum_{x,y\in\mathbb{Z}} \tilde{K}_M(2r,x;2s,y)\, z^{-x} w^{y}$$

$$= \sum_{\ell=1}^{\infty} f(z)^{m-r} g(z)^{m-r} z^{-1} \frac{z^{\ell}}{f(z)^{m}} \frac{1}{w^{\ell} g(w)^{m}} f(w)^{s} g(w)^{s}\, w$$

$\ulcorner$ Assume $|z| = 1 - \varepsilon < |w| = 1$ say, ε small. $\lrcorner$

$$= \frac{f(w)^s \, g(w)^{s-m}}{f(z)^r \, g(z)^{r-m}} \; \frac{w}{w-z} .$$

Thus, we get the following double-integral formula for $\tilde{K}_M$, which we can apply both for the PNG and the Aztec diamond (and other models of the same structure). We get

$$\tilde{K}_M (2r, x; 2s, y) =$$

$$= \frac{1}{(2\pi i)^2} \int_{|z|=r_1} \frac{dz}{z} \int_{|w|=r_2} \frac{dw}{w} \frac{z^x}{w^y} \frac{f(w)^s \, g(w)^{s-m}}{f(z)^r \, g(z)^{r-m}} \frac{w}{w-z} , \qquad (5)$$

where $a < r_1 < 1/a$, $r_1 < r_2$.

For PNG: $f(z) = \dfrac{1}{1-az}$, $g(z) = \dfrac{1}{1-b/z}$ and hence

$$\tilde{K}_M (2r, x; 2s, y)$$

$$= \frac{1}{(2\pi i)^2} \int_{|z|=r_1} \frac{dz}{z} \int_{|w|=r_2} \frac{dw}{w} \frac{z^x}{w^y} \frac{(1-az)^r \, (1-b/w)^{m-s}}{(1-b/z)^{m-r} (1-aw)^s} \frac{w}{w-z} \qquad (6)$$

We get a similar formula for the Aztec diamond.