Relativistic Fluids Tutorial # 1

In class today we discussed the Tolman-Oppenheimer-Volkoff (TOV) solution for a spherical, static star. We assumed a metric of the form

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + \frac{dr^2}{1 - \dfrac{2m(r)}{r}} + r^2\,d\Omega^2, \tag{1}$$

and the resulting equations for the star are

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \varepsilon(r) \tag{2}$$

$$\frac{dN}{dr} = 4\pi r^2 n(r)\left(1 - \frac{2Gm}{c^2 r'}\right)^{-1} \tag{3}$$

$$\frac{d\Phi}{dr} = \frac{Gm(r) + 4\pi G r^3 P/c^2}{r[r - 2Gm(r)/c^2]} \tag{4}$$

$$\frac{dP}{dr} = -\frac{G}{c^2}\frac{(\varepsilon + P)}{r^2}\left[m(r) + \frac{4\pi r^3 P}{c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]. \tag{5}$$

An equation of state is required to close the system, and we will choose a degenerate gas of Fermions, either electrons or neutrons. A degenerate gas is a gas in its lowest energy state, or $T = 0$. The equation of state is usually in polytropic form

$$P = K\rho_0^{\Gamma}, \tag{6}$$

where ρ_0 is the baryon rest mass density.

1. Before attempting to solve the TOV equations numerically, we notice that the equations contain numbers with a large variation in magnitude. To improve the numerical accuracy, we would like to rescale most numbers to order unity by introducing dimensionless variables. Let

$$r = r_0\bar{r} \qquad m = m_0\bar{m}$$
$$\rho = \rho_0\bar{\rho} \qquad \varepsilon = \rho_0 c^2\bar{\varepsilon} \tag{7}$$
$$P = \rho_0 c^2\bar{P}.$$

It is not convenient to define all of the scales independently. So, for example, we use ρ_0 to set the scale for energy and also pressure. Moreover, the scales for mass and ρ_0 should be related, so we could choose

$$\rho_0 = \frac{m_0}{{r_0}^3}. \tag{8}$$

Finally, define the dimensionless Gravitation constant

$$\bar{G} \equiv \frac{Gm_0}{c^2 r_0} \tag{9}$$

Show that the dimensionless equations become

$$\frac{d\bar{m}}{d\bar{r}} = \frac{4\pi}{c^2}\bar{r}^2\bar{\varepsilon} \tag{10}$$

$$\frac{d\bar{P}}{d\bar{r}} = -\frac{\bar{G}(\bar{\varepsilon} + \bar{P})}{\bar{r}^2}\left[\frac{\bar{m} + 4\pi\bar{r}^3\bar{P}}{1 - 2\bar{G}\bar{m}/\bar{r}}\right]. \tag{11}$$

We recover the physical units simply by choosing m_0 and r_0. The choice of m_0 and r_0 requires some care, and we choose these parameters to correspond to the expected physical properties of the solutions. A reasonable choice for m_0 is one solar mass, $M_\odot = 1.99 \times 10^{33}$ g, since both neutron stars and white dwarfs have masses near $M_\odot$. The choice for r_0 will depend on the type of star. A typical neutron star has a radius of order 10 km, while a white dwarf is about the size of the Earth, or of order 1,000 km.

2. The equation of state for a degenerate gas of relativistic electrons has the limiting form

$$P_{\rm rel} = K_{\rm rel}\,\rho_0^{4/3}, \qquad K_{\rm rel} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Y_e}{m_B}\right)^{4/3}. \tag{12}$$

Here m_B is the average baryon mass, $m_B = 1.66 \times 10^{-24}$ g, and Y_e is the electron fraction, or the number of electrons per baryon. We can approximate $Y_e \simeq 1/2$ for atoms with small Z, like carbon, oxygen, etc. The total energy density consists of the total energy of the electrons and the rest mass of the baryons,

$$\varepsilon = \rho_0 c^2 + \varepsilon_{\rm elec}. \tag{13}$$

where $\rho_0 = m_B n$. For relativistic electrons, the total energy is

$$\varepsilon_{\rm elec} \simeq 3P. \tag{14}$$

Often the rest energy of the baryons is significantly larger than $\varepsilon_{\rm elec}$, even for relativistic electrons, and we can approximate $\varepsilon \simeq \rho_0 c^2$. Using this equation of state, find the maximum mass and corresponding radius for a stable white dwarf. What is the central density of this star?

3. The equation of state for degenerate gas of relativistic neutrons has the limiting form

$$P_{\rm rel} = K_{\rm rel}\,\rho_0^{4/3}, \qquad K_{\rm rel} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2}{m_n}\right)^{4/3} \tag{15}$$
$$\varepsilon_{\rm rel} = 3P_{\rm rel}$$

Using this equation of state, find the maximum mass and corresponding radius for a stable neutron star. What is the central density of this star?

Note. A problem occurs if we assume the simple relationship $\varepsilon \approx 3P_{\rm rel}$, so use the full expression

$$\varepsilon = \rho_0 c^2 + \frac{P}{\Gamma - 1}. \tag{16}$$

If you have time, you can investigate this problem and determine the cause.

4. More realistic nuclear equations of state at $T = 0$ are also usually written in polytropic form, but now Γ and K will vary with ρ_0. One such nuclear EOS is

$$\frac{P}{c^2} = K\rho_0^\Gamma \tag{17}$$

where

$$K = \begin{cases} 3.9987 \times 10^{-8} & \rho_0 \le 1.4173 \times 10^{14}\ \mathrm{g/cm^3} \\ 2.2387 \times 10^{-31} & \rho_0 > 1.4173 \times 10^{14}\ \mathrm{g/cm^3} \end{cases} \tag{18}$$

and

$$\Gamma = \begin{cases} 1.3569 & \rho_0 \le 1.4173 \times 10^{14}\ \mathrm{g/cm^3} \\ 3 & \rho_0 > 1.4173 \times 10^{14}\ \mathrm{g/cm^3} \end{cases} \tag{19}$$

The total energy density is

$$\varepsilon = \rho_0(1 + a) + \frac{P}{\Gamma - 1}, \tag{20}$$

with

$$a = \begin{cases} 0 & \rho_0 \le 1.4173 \times 10^{14}\ \mathrm{g/cm^3} \\ 0.0104 & \rho_0 > 1.4173 \times 10^{14}\ \mathrm{g/cm^3} \end{cases} \tag{21}$$

Here ρ_0 has units of g/cm^3, and the constants K have units such P/c^2 also has units of g/cm^3. Find the maximum masses for stable neutron stars using this nuclear equation of state.

5. An interesting question in the study of neutron stars is what is the maximum neutron star mass? If we knew the exact nuclear equation of state for matter at all densities, then this would be an easy question to answer using the TOV solutions and stability arguments. Unfortunately, we can only perform laboratory experiments at densities close to the nuclear density, so there is no experimental verification of equation of state models at higher densities. The stiffest possible equation of state would be one where ρ_0 is constant in the star, and fortunately we can solve the TOV equations analytically for this case. If we let $\rho_0 = $ const, then show that the solution is

$$m(r) = \frac{4}{3}\pi r^3 \rho_0 \tag{22}$$

$$P(r) = \rho_0 c^2 \frac{(1 - 2GMr^2/c^2R^3)^{1/2} - (1 - 2GM/c^2R)^{1/2}}{3(1 - 2GM/c^2R)^{1/2} - (1 - 2GMr^2/c^2R^3)^{1/2}}, \tag{23}$$

where M and R are the star's mass and radius.

The central pressure (the pressure at $r = 0$) is

$$P_c = \rho_0 c^2 \frac{(1 - 2GM/c^2R)^{1/2} - 1}{1 - 3(1 - 2GM/c^2R)^{1/2}} \tag{24}$$

$P_c \to \infty$ if the denominator vanishes, so we require

$$\frac{2GM}{c^2R} < \frac{8}{9} \tag{25}$$

This gives a condition on M and R for possible stars. Assuming that we know the proper equation of state up to 4.6×10^{14} g/cm^3, this solution limits the maximum allowed neutron star mass to be

$$M_{\rm max} = \frac{8}{27}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} \sim 5.3\left(\frac{4.6 \times 10^{14}\mathrm{g/cm^3}}{\rho_0}\right)^{1/2} M_\odot \tag{26}$$