Bifurcations and Catastrophe Theory: Physical and Natural Systems

ICTS program on Modern Finance and Macroeconomics: A Multidisciplinary Approach

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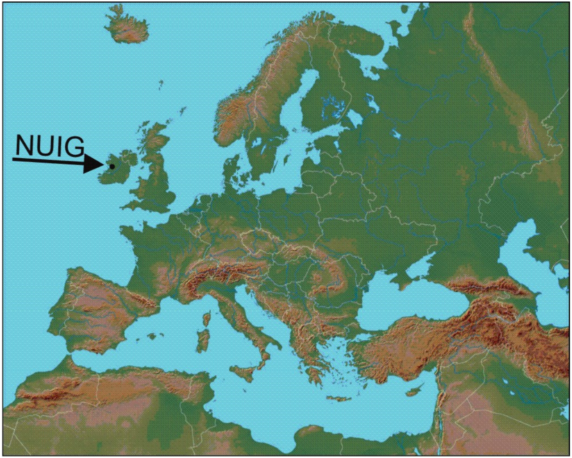
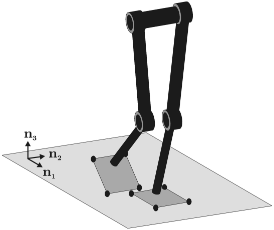
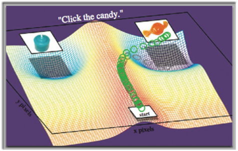
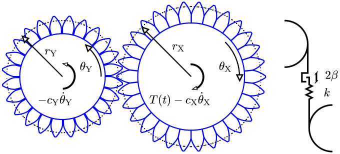
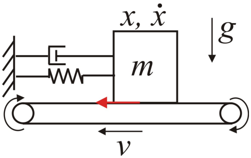


Dynamical systems



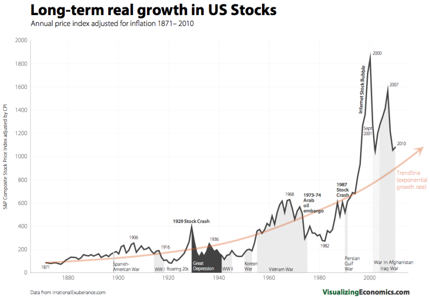


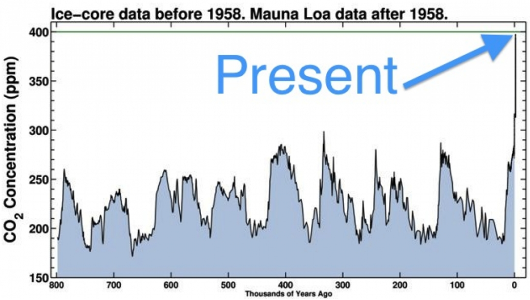




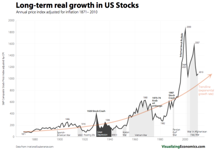
See Piiroinen *et al.* [1-5].

Real data


Long-term real growth in US Stocks
Annual price index adjusted for inflation 1871– 2010
S&P Composite Stock Price Index adjusted by CPI
1929 Stock Crash
Internet Stock Bubble
1987 Stock Crash
1973-74 Arab oil embargo
Trendline (exponential growth rate)
Spanish-American War
WW I
Roaring 20s
Great Depression
WW II
Korean War
Vietnam War
Persian Gulf War
War in Afghanistan
Iraq War
Data from IrrationalExuberance.com
VisualizingEconomics.com


Ice–core data before 1958. Mauna Loa data after 1958.
Present
CO_2 Concentration (ppm)
Thousands of Years Ago

Real data


Long-term real growth in US Stocks
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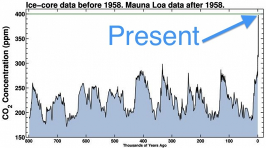

Ice-core data before 1958. Mauna Loa data after 1958.
Present
CO_2 Concentration (ppm)
Thousands of Years Ago


845
745
812
732

Real data

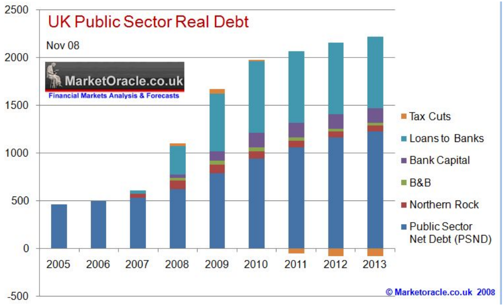
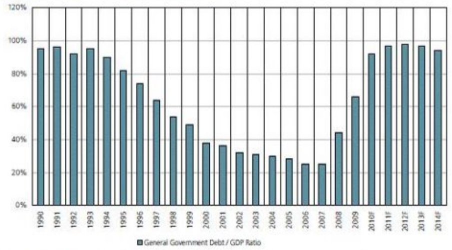


Figure 3: Ireland's debt to GDP ratio

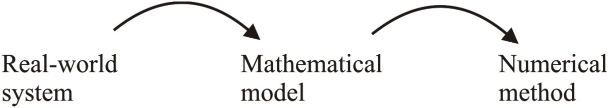


Source: NTMA; Davy

Overview

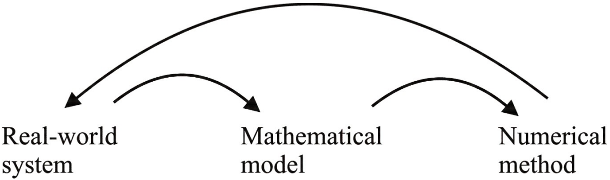
1. Linear Systems
   - Solutions
   - Equilibrium types
2. Nonlinear systems
   - Steady-state solution
   - Transitions
3. Stability
   - Equilibria, fixed points and periodic orbits
4. Bifurcations and transitions
5. Examples

Dynamical System Modelling and Analysis



There is a natural order between real-world systems, modelling and numerical analysis,...

Dynamical System Modelling and Analysis



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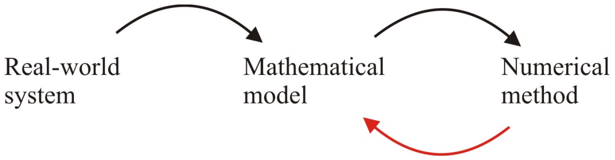
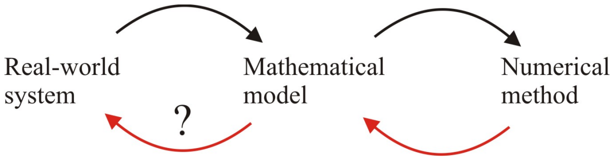
Dynamical System Modelling and Analysis



There is a natural order between real-world systems, modelling and numerical analysis, but that order is not always followed.

Dynamical System Modelling and Analysis



There is a natural order between real-world systems, modelling and numerical analysis, but that order is not always followed.

Notation

Let us first introduce some useful notation [6-10]. We let

$$x = x(t), \quad \dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}.$$

If $x \in \mathbb{R}^n$ then

$$x = (x_1(t), x_2(t), \ldots, x_n(t))^T.$$

For functions we will mostly use the following forms:

$$f = f(x,t), \quad f_x = \frac{\partial f}{\partial x}, \quad f_t = \frac{\partial f}{\partial t}$$

or

$$f = f(x), \quad f_x = \frac{\partial f}{\partial x}.$$

A special function that will be useful is the Jacobian

$$J(x,t) = f_x(x,t) \in \mathbb{R}^{n \times n} \text{ (Jacobian)}.$$

Differential and Difference equations

For time-continuous systems the types of ordinary differential equations we consider are non-autonomous:

$$\dot{x} = f(x,t),\ x(t_0) = x_0, \qquad x, f \in \mathbb{R}^n,\ \ t \in \mathbb{R},$$

or autonomous:

$$\dot{x} = f(x),\ x(t_0) = x_0 \qquad x, f \in \mathbb{R}^n.$$

Similarly, for discrete systems or maps, we consider difference equations of the form

$$x_{k+1} = F(x_k),\ \ x, F \in \mathbb{R}^n, \qquad k = 0, 1, 2, \ldots$$

However, the main focus will be on ODEs.

Differential and Difference equations

For ordinary differential equations *equilibria* x^* are given by

$$\dot{x} = 0 \;\Rightarrow\; f(x^*, t) = 0 \;\text{ or }\; f(x^*) = 0$$

and *limit cycles* are given by

$$x(t_0) = x(t_0 + T) \text{ for some } T > 0..$$

Similarly, for discrete systems or maps, *fixed points* are given by

$$x_k = F(x_k) \;\Rightarrow\; x^* = F(x^*).$$

Poincaré surface and map

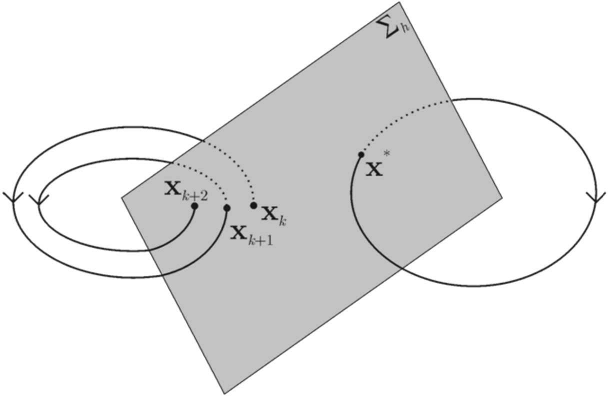


Figure 1: Poincaré surface and map.

Difference and Differential equations

The following two linear one-dimensional examples form the fundament for the analysis of dynamical systems.

Ex.

$$\dot{x} = ax, \quad x(t_0) = x_0 \quad \Rightarrow \quad x(t) = x_0 e^{a(t-t_0)}$$

The equilibrium point is given by $x^* = 0$ $(a \neq 0)$.

Ex.

$$x_{k+1} = ax_k, \quad x_0 \text{ known} \quad \Rightarrow \quad x_k = x_0 a^k$$

The fixed point is given by $x^* = 0$ $(a \neq 1)$.

Systems of differential equations

Following this a general linear 2-dim homogeneous system of ODEs can be written as

$$\begin{aligned}\frac{dx_1}{dt} &= ax_1 + bx_2\\ \frac{dx_2}{dt} &= cx_1 + dx_2\end{aligned}$$

where a, b, c, d are real constants, which can be recast as

$$\dot{x} = Ax, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with equilibrium point $(x_1^*, x_2^*)^T = (0, 0)^T$.

Note: Linear systems of different forms can usually be rewritten in this simple form.

Formal solution method for the matrix ODE

For systems of dimension n of the form

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0,$$

where for simplicity we let $t_0 = 0$, a general solution is written as the linear combination of n **linearly independent** vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, so that

$$x(t) = c_1\mathbf{v}_1(t) + \cdots + c_n\mathbf{v}_n(t),$$

where c_1 and c_2 are constants that are determined from the initial condition X_0.

As the linearly independent vectors we can use a combination of eigenvalues and eigenvectors of A (if they exists). Therefore, the first thing we do is to find the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, i.e. solve

$$\det(A - \lambda I) = 0,$$

where I is an identity matrix with the same dimension as A.

Formal solution method for the matrix ODE

For 2-dimensional systems there are three possible combinations of the two eigenvalues, which lead to different types of solutions.

1. λ_1 and λ_2 are distinct real numbers

$$x(t) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{w}_1 e^{\lambda_1 t} + c_2\mathbf{w}_2 e^{\lambda_2 t},$$

with $\mathbf{w}_1$ and $\mathbf{w}_2$ being the corresponding eigenvectors.

2. λ_1 and λ_2 are real and have the same value λ

$$x(t) = (c_1\mathbf{w}_1 + c_2\,(I + t\,(A - \lambda I))\,\mathbf{v})\,e^{\lambda t}$$

with $\mathbf{w}_1$ being one of the eigenvectors and **v** being a second linearly independent vector.

Formal solution method for the matrix ODE

3. λ_1 and λ_2 are distinct complex conjugate numbers

$$x(t) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2,$$

where

$$\mathbf{v}_1 = e^{\alpha t}\left(\cos(\beta t)\mathbf{z}_1 - \sin(\beta t)\mathbf{z}_2\right),$$

$$\mathbf{v}_2 = e^{\alpha t}(\cos(\beta t)\mathbf{z}_2 + \sin(\beta t)\mathbf{z}_1).$$

Here the eigenvalue $\lambda_1 = \alpha + \mathbf{i}\beta$ and the eigenvector $\mathbf{w}_1 = \mathbf{z}_1 + \mathbf{i}\mathbf{z}_2$.

Types of Equilibria and fixed points in phase space

Nodes and Saddle nodes

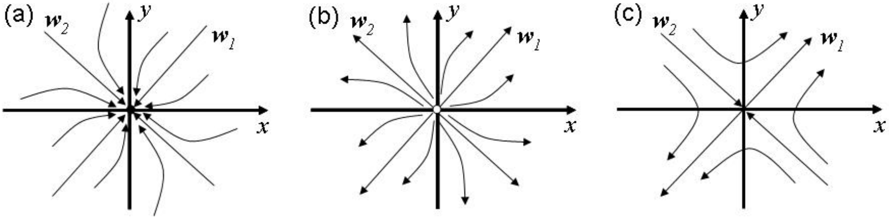


Figure 2: (a) Attractor node ($\lambda_1, \lambda_2 < 0$), (b) Repellor node ($\lambda_1, \lambda_2 > 0$), (c) Saddle node ($\lambda_2 < 0 < \lambda_1$)

Types of Equilibria and fixed points in phase space

Degenerate Nodes

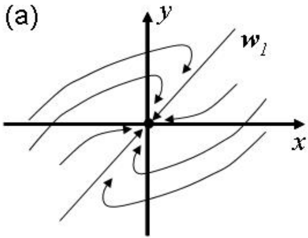


Figure 3: (a) Attractor node ($\lambda < 0$), (b) Repellor node ($\lambda > 0$)

Types of Equilibria and fixed points in phase space

Spirals and Centres

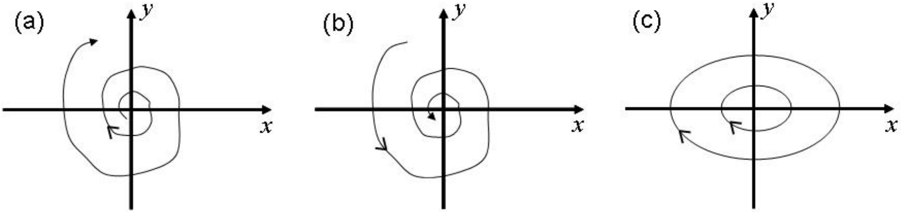


Figure 4: (a) Repellor spiral ($\alpha > 0$), (b) Attractor spiral ($\alpha < 0$), (c) Centre ($\alpha = 0$).

Parameter space of A

The 2-dimensional linear system

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0$$

The eigenvalues of the matrix A can be written in terms of two real parameters

$$\tau = \mathrm{tr}(A) \quad \text{and} \quad \delta = \det(A)$$

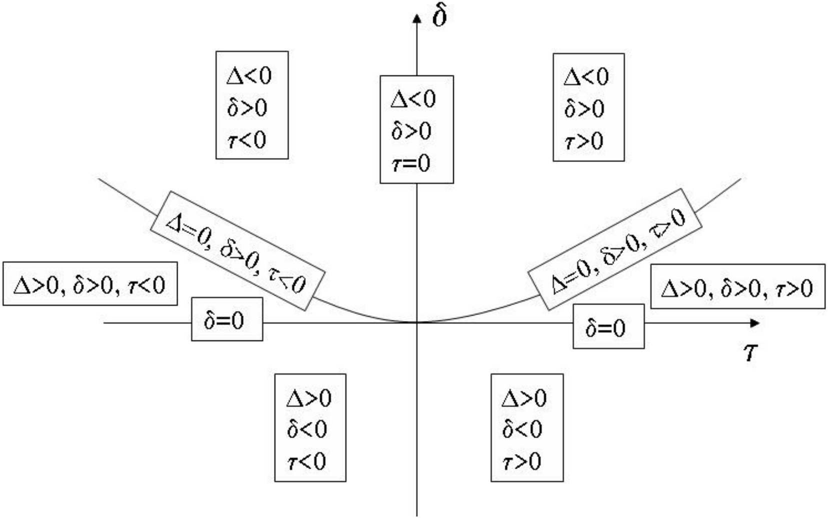
through

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\Delta}}{2} \quad \text{where} \quad \Delta = \tau^2 - 4\delta.$$

We can view this parameter space as a 2-dimensional space with τ as the horizontal axis and δ as the vertical axis. Corresponding to each coefficient matrix A there is a point (τ, δ) in this space (the relationship is not 1-to-1).

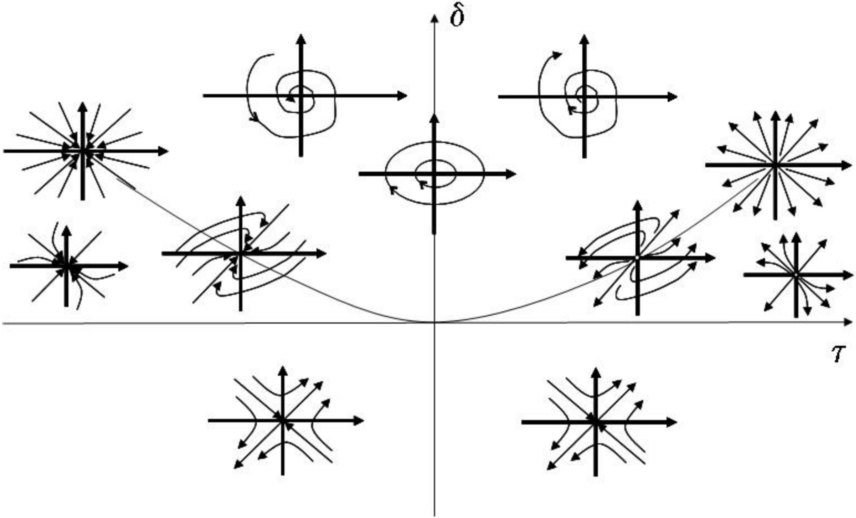
The parameter space is partitioned into regions according to the value of Δ, in each region the associated coefficient matrices have the same properties.

Regions defined by τ, δ and Δ



The properties of the equilibrium point at $(0,0)$ can be deduced by noting the region to which A belongs.

Nature of equilibrium points



Nonlinear Systems of ODEs

Guckenheimer, Holmes , Springer,

Number of variables →

Nonlinearity ↓	$n = 1$	$n = 2$	$n \geq 3$	$n >> 1$	Continuum
Linear	*Growth, decay, or equilibrium* Exponential growth RC circuit Radioactive decay	*Oscillations* Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	*Collective phenomena* Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	*Waves and patterns* Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
			The frontier		
Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	*Chaos* Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos ?	Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization Neural networks Immune system Ecosystems Economics	*Spatio-temporal complexity* Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation Epilepsy Turbulent fluids (Navier-Stokes) Life

What can we expect?

What steady-state behaviours can we expect in a nonlinear system?

ODEs:

- Equilibria
- Limit cycles
- Quasi-periodic attractors
- Chaotic/strange attractors

Maps:

- Fixed points
- Periodic orbits
- Quasi-periodic attractors
- Chaotic/strange attractors

In general, it is important to realise that in linear systems there can only one equilibrium/fixed point, while in nonlinear systems there can be any number of possible steady-state solutions.

Nonlinearities - nonlinear vector field

Let us again consider

$$\dot{x} = f(x, t).$$

Nonlinear smooth vector fields $f(x, t)$ can have any characteristics that fulfills this property. Consider in 1D for instance:

Polynomial functions

$$f(x, t) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

Trigonometric and exponential functions

$$f(x, t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + c_3 e^t + c_4 k^x$$

Nonlinearities - Nonsmooth/switching

Systems of the form

$$\dot{x} = f(x,t)$$

can also have jumps, switches and/or discontinuous vector fields.

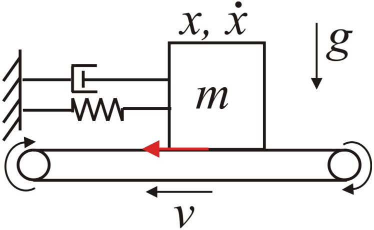
In recent years Filippov systems of the form

$$f(x,t) = \begin{cases} f_1(x,t), x \in S_1 \\ f_2(x,t), x \in S_2 \end{cases}$$

has been widely studied [11-13] in many areas science and engineering, but not that often in economics, but there are some cases for discrete systems, see Gardini *et al.*

Modelling - Friction models

To explain the concept let us think of a simple block-on-a-surface model:



The equations of motion are

$$m\ddot{x} + d\dot{x} + cx = F_{fric}(v_{rel}, m, g)$$

or if we let $(x, \dot{x}) = (y_1, y_2)$ then

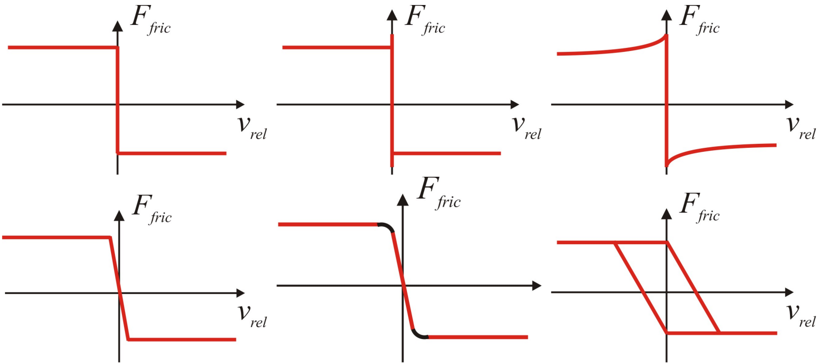
$$\dot{y}_1 = y_2, \quad (1)$$

$$\dot{y}_2 = \frac{1}{m}\left(-dy_2 - cy_1 + F_{fric}(v_{rel}, m, g)\right), \quad (2)$$

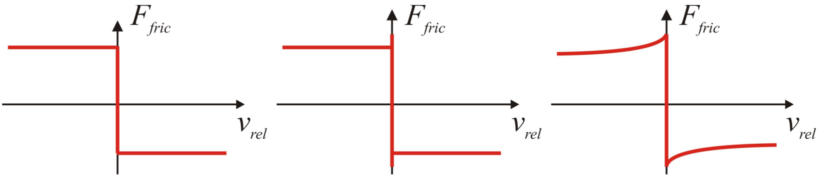
where $v_{rel} = \dot{x} - v = y_2 - v$.

Modelling - Friction models

For such systems one can consider a number of different models:



Nonlinearities - Nonsmooth/switching



For these three examples we can extend the Filippov vector field to

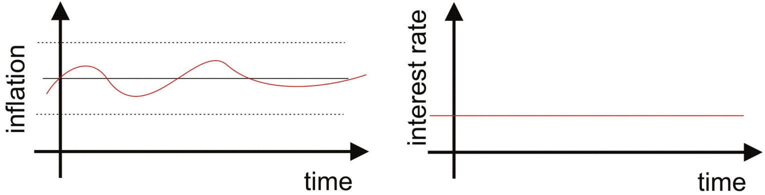
$$f(x,t) = \begin{cases} f_1(x,t), & x \in S_1 \\ f_s(x,t), & x \in \Sigma \\ f_2(x,t), & x \in S_2 \end{cases}$$

When $x \in \Sigma$ for some positive time period we say that we have a *sliding* solution.

Dynamics - Discontinuities

Ex. Consider some model from economics where the interest rate r is considered as a parameter that changes discretely, then

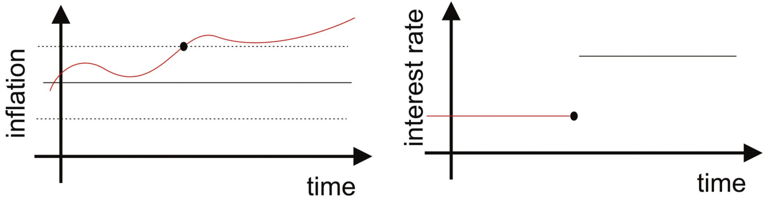
$$\dot{x} = \begin{cases} f_1(x, r_1), & \text{inflation} < i^* \\ f_2(x, r_2), & i_* < \text{inflation} < i^* \\ f_3(x, r_3), & \text{inflation} < i_* \end{cases}$$



In this case the interest rate stays constant as the inflation lie between i_* and i^* and vector field f_2 is used at all times.

Dynamics - Discontinuities

Ex. If instead the inflation takes another path we may have



and we have to swap from vector field f_2 to f_1 at some time.

$$\dot{x} = \begin{cases} {\color{red} f_1(x, r_1)}, & \text{inflation} < i^* \\ f_2(x, r_2), & i_* < \text{inflation} < i^* \\ f_3(x, r_3), & \text{inflation} < i_* \end{cases}$$

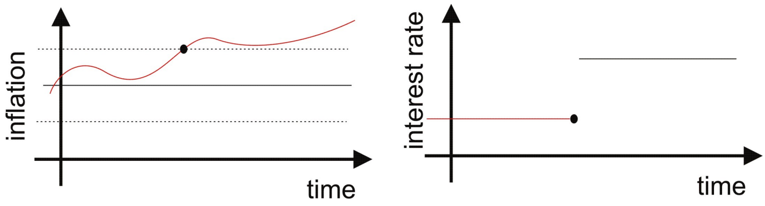
This is a transition I will refer to as being *event driven*.

Dynamics - Discontinuities

Ex. However, if the interest is considered as a variable in the system we can think of the same system as

$$\begin{aligned} \dot{x} &= f(x,r) \\ \dot{r} &= g(x,r) \\ r &\mapsto r \overset{+}{\scriptstyle(-)} \Delta r, \text{ when inflation reaches } i^* \ (i_*). \end{aligned}$$

We still get the same behaviour



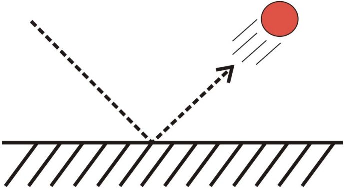
but now we can see the system as a system with discrete changes (or impacts) [14].

Modelling - Impact models

Ex. Let x be height above the impact surface and v the normal relative velocity between the surface and the ball so that

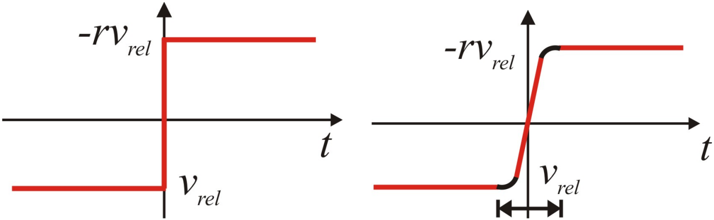
$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -g \\ v_{rel} &\mapsto -ev_{rel}, \text{ when } x = x_s, \end{aligned}$$

where $0 \leq e \leq 1$ and x_s is the position of the impacting surface and $v_{rel} = \dot{x}_s - v$.



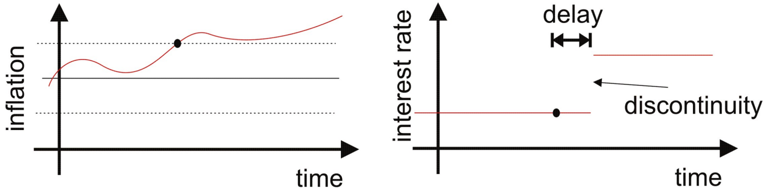
Modelling - Impact models

The time it takes to make the rapid change sometimes matter and sometimes it does not and should be modelled on a case-by-case basis. The hard part is to find out when one or another situation applies, i.e. the modelling.



Nonlinearities - Delay

Ex. There are many other complications we can see in dynamical systems. If we again look at the simple example form economics we can imagine the following situation.



We see that a decision to change the interest rate is delayed a time τ after i^* (i_*) has been reached.

$$\begin{aligned}\dot{x} &= f(x,r)\\ \dot{r} &= g(x,r)\\ r &\mapsto r \overset{+}{\underset{(-)}{}} \Delta r, \text{ when inflation}(t-\tau) = i^* \ (i_*).\end{aligned}$$

This is an example of a *delay differential equation* (not discuss further).

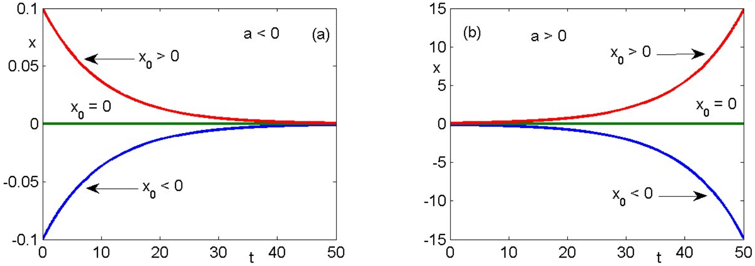
Stability for equilibria of 1-dimensional ODEs

The notion of stability we deal with is **stability with respect to small disturbances**.

The stability of the solution

$$x = x(t) = x_0 e^{at} \quad \text{of} \quad \frac{dx}{dt} = ax,$$

where $x(t)$ is some function of t satisfying the equation, is determined by examining what happens when the solution is disturbed by an arbitrarily small amount.



Stability for equilibria of ODEs

There are two commonly used concepts of stability

Asymptotic stability (AS): disturbed solution $\rightarrow$ undisturbed solution as $t \rightarrow \infty$,

Lyapunov stability (LS): A disturbed solution remains "close" to the undisturbed solution for all future times.

Stability of equilibria for one-dimensional ODEs

Consider

$$\frac{dx}{dt} = f(x),$$

where $f(x)$ is usually a known non-linear function of x.

We find the equilibrium points x_{eq} by solving

$$f(x_{eq}) = 0$$

and their stability is found by considering

$f'(x_{eq}) < 0$: $x = x_{eq}$ is a **stable equilibrium**, and thus $x = x_{eq}$ is an attractor that attracts "nearby" non-equilibrium solutions.

$f'(x_{eq}) > 0$: $x = x_{eq}$ is an **unstable equilibrium**, and thus $x = x_{eq}$ is a repellor that repels "nearby" non-equilibrium solutions.

$f'(x_{eq}) = 0$: To determine the stability the first non-zero derivative of f at x_{eq} has to be found.

Stability for general 1-dim non-linear ODEs

First we perturb the equilibrium solution slightly, such that

$$x(t) = x_{eq} + \varepsilon(t), \quad |\varepsilon(t)| \ll 1,$$

and substitute $x(t)$ into the non-linear ODE. Thus for the left-hand side we get

$$\frac{dx}{dt} = \frac{d}{dt}\left(x_{eq} + \varepsilon(t)\right) = \frac{d\varepsilon}{dt}$$

and for the right-hand side

$$f(x) = f\left(x_{eq} + \varepsilon\right) = \underbrace{f\left(x_{eq}\right)}_{=0} + f'\left(x_{eq}\right)\varepsilon + \frac{1}{2}f''\left(x_{eq}\right)\varepsilon^2 + \ldots,$$

which gives us the following non-linear equation for $\varepsilon(t)$:

$$\frac{d\varepsilon}{dt} = f'\left(x_{eq}\right)\varepsilon + \ldots$$

Compare this to the the result for linear one-dimensional ODEs above,

Stability for general n-dim non-linear ODEs

Similarly for systems of ODEs ($x, \varepsilon \in \mathbb{R}^n$) we first perturb the equilibrium solution slightly, such that

$$x(t) = x_{eq} + \varepsilon(t), \quad |\varepsilon_i(t)| \ll 1,$$

and substitute $x(t)$ into the non-linear ODE, which gives gives us the following non-linear equation for $\varepsilon(t)$:

$$\frac{d\varepsilon}{dt} = \frac{\partial f}{\partial x}(x_{eq})\,\varepsilon + \ldots = J(x_{eq})\varepsilon + \ldots$$

Now, the eigenvalues of the Jacobian $J(x_{eq})$ determine the stability of and local behaviour about the equilibrium.

Compare this to the 2-dimensional linear ODEs discussed above.

Stability for general n-dim non-linear ODEs

Similarly for maps ($x, \varepsilon \in \mathbb{R}^n$) we first perturb the equilibrium solution slightly, such that

$$x_k = x_{fp} + \varepsilon_k, \quad |\varepsilon_0| \ll 1,$$

and substitute x_k into the map, which gives gives us the following non-linear equation for $\varepsilon_k(t)$:

$$\varepsilon_k = (F_x(x_{fp}))^k \varepsilon_0 + \dots$$

Now, the eigenvalues of $F_x(x_{fp})$ determine the stability of and local behaviour about the equilibrium. If $|\lambda_i| < 1$ for all i the fixed point is stable, otherwise unstable.

Stability for general n-dim non-linear ODEs

Similarly, the stability of limit cycles for systems of ODEs can be found by solving the *first variational equations*

$$\dot{\Phi}(t) = J(x)\Phi(t), \quad \Phi(t_0) = Id.$$

Recall that limit cycles are given by

$$x(t_0) = x(t_0 + T).$$

The stability is determined in the same fashion as for maps, i.e. by analysing the eigenvalues of $\Phi(t_0 + T)$.

Typically boundary value problems of this type are solved numerically using shooting methods or collocation methods [15-16].

Basin/Domain of attraction

The **basin of attraction** is the set of all initial conditions whose orbits converge to a sink or another attractor (e.g. chaotic attractor).

The basin of attraction of a *source* is usually a set of discrete points (at most).

Basin/Domain of attraction

Ex. Consider the ODE

$$\dot{x} = f(x) = 1 - x^2$$

with equilibrium points $x_{eq} = \pm 1$. It is easy to see that

$$f'(-1) = 2, \qquad f'(1) = -2$$

and thus $x_{eq} = 1$ is stable with basin of attraction $(-1, \infty)$. The basin of attraction for $x_{eq} = -1$ is exactly -1 and the basin of attraction for $x = -\infty$ is $(-\infty, -1)$.

Bifurcations in n-dim non-linear ODEs

Consider next one-dimensional ODEs of the type

$$\dot{x} = f(x, a)$$

a where $a \in \mathbb{R}$ is a real parameter. Suppose the ODE has an equilibrium at $(x, a) = (x^*, a)$, i.e.,

$$f(x_0, a) = 0.$$

Two questions immediately arise:

1. Is the equilibrium point stable or unstable?
2. How is the stability or instability affected as a is varied?

These questions can be answered answered from the stability analysis we introduced earlier [6-10,17,18].

Bifurcations in 1-dim non-linear ODEs

There are a number of ways in which the qualitative behaviour of the dynamics about equilibrium points can be changed, namely, though

- saddle-node or flip bifurcations,
- transcritical bifurcations,
- pitch-fork bifurcations.

Bifurcations in 1-dim non-linear ODEs

Branches of equilibrium points for which $f(x, a) = 0$. The solid curvees correspond to stable equilibria and the dashed curvee to unstable equilibria.

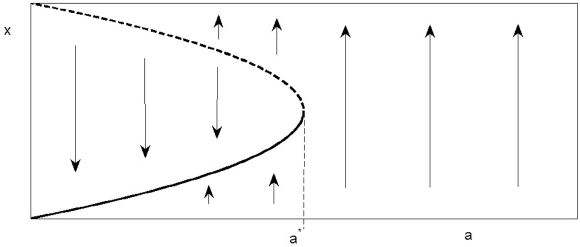


Figure 5: Saddle-node or fold bifurcation

Bifurcations in 1-dim non-linear ODEs

Branches of equilibrium points for which $f(x,a) = 0$. The solid curvees correspond to stable equilibria and the dashed curvee to unstable equilibria.

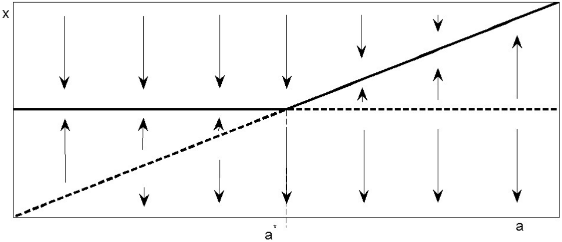


Figure 6: Transcritical bifurcation

Bifurcations in 1-dim non-linear ODEs

Branches of equilibrium points for which $f(x, a) = 0$. The solid curvees correspond to stable equilibria and the dashed curvee to unstable equilibria.

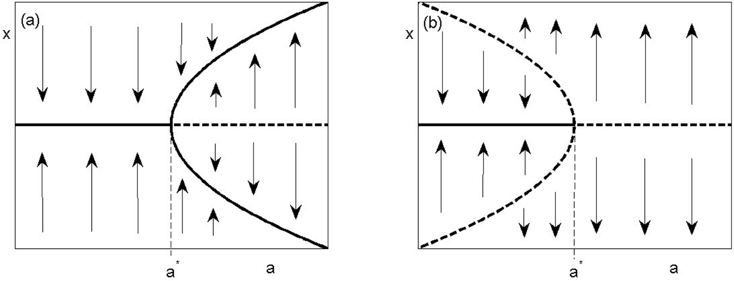


Figure 7: (a) Supercritical pitchfork bifurcation. (b) Subcritical pitchfork bifurcation

General theorems – bifurcations

Frequently in applications we will not know the the explicit form of the non-linear function

A lot of attention has been given to determining the qualitative nature of the equilibrium and non-equilibrium solutions when the non-linear function in the equation satisfies certain specified conditions.

The following three theorems asserts the occurrence of bifurcations of the type we met in the beginning of this lecture under certain specified conditions.

They all involve an ODE

$$\frac{dx}{dt} = f(x, a)$$

involving a (real) parameter a.

Theorem: Saddle-node/fold bifurcation

If at $x = x_{eq}, a = a^*$ an equilibrium occurs then

$$f(x_{eq}, a^*) = 0$$

and the following conditions are satisfied:

$$\frac{\partial f}{\partial x}(x_{eq}, a^*) = 0, \quad \frac{\partial^2 f}{\partial x^2}(x_{eq}, a^*) \neq 0, \quad \frac{\partial f}{\partial a}(x_{eq}, a^*) \neq 0$$

then

- no equilibrium occur either for $a < a^*$ or for $a > a^*$ depending on the signs of the non-zero derivatives above,
- two equilibria occur, one attractor and one repellor, for the "other" values of a ($a > a^*$ or $a < a^*$).

Theorem: Saddle-node/fold bifurcation

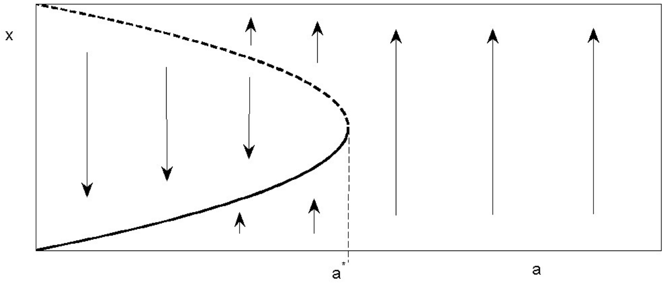


Figure 8: Saddle-node/fold bifurcation

Theorem: Transcritical bifurcation

If at $x = x_{eq}, a = a^*$ an equilibrium occurs then

$$f(x_{eq}, a^*) = 0$$

and the following conditions are satisfied:

$$\frac{\partial f}{\partial x}(x_{eq}, a^*) = 0, \quad \frac{\partial^2 f}{\partial x^2}(x_{eq}, a^*) \neq 0, \quad \frac{\partial f}{\partial a}(x_{eq}, a^*) = 0$$

$$\left(\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial a^2} - \left(\frac{\partial^2 f}{\partial x \partial a} \right)^2 \right) (x_{eq}, a^*) < 0$$

then

- an equilibrium at x_{eq} exists for a range of values of a around a^*,
- a second equilibrium occurs at $\hat{x}_{eq}$ (for a range of values of a around a^*), which coincides with x_{eq} when $a = a^*$,
- the stability properties of the equilibria x_{eq} and $\hat{x}_{eq}$ changes as a passes through a^*,
- the stability properties of the equilibria x_{eq} and $\hat{x}_{eq}$ are opposite to one another.

Theorem: Transcritical bifurcation

The signs of the non-zero derivatives listed above determine the detailed stability properties of the two equilibria.

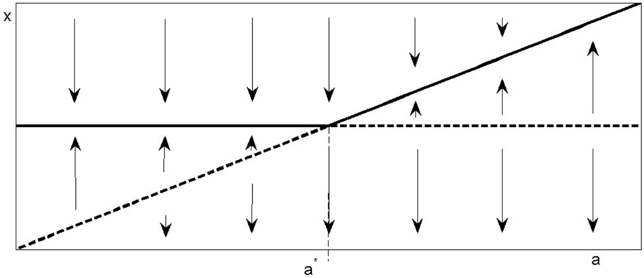


Figure 9: Transcritical bifurcation

Theorem: Pitchfork bifurcation

If at $x = x_{eq}, a = a^*$ an equilibrium occurs then

$$f(x_{eq}, a^*) = 0$$

and the following conditions are satisfied:

$$\frac{\partial f}{\partial x}(x_{eq}, a^*) = 0, \quad \frac{\partial^2 f}{\partial x^2}(x_{eq}, a^*) = 0, \quad \frac{\partial^3 f}{\partial x^3}(x_{eq}, a^*) \neq 0,$$

$$\frac{\partial f}{\partial a}(x_{eq}, a^*) = 0, \quad \frac{\partial^2 f}{\partial x \partial a}(x_{eq}, a^*) \neq 0 \quad \text{then}$$

- an equilibrium at x_{eq} exists for a range of values of a around a^*,
- the stability properties of the equilibria x_{eq} changes as a passes through a^*,
- two branches of equilibrium points occur for either $a < a^*$ or $a > a^*$ (depending on the sign of the third derivative above, see further below),
- the equilibrium x_{eq} and two extra branches have opposite stability properties.

Theorem: Pitchfork bifurcation

To decide on which side of a^* the two extra branches occur we can use the following conditions (compare this with the figures below).

$$\frac{\partial^3 f}{\partial x^3}(x_{eq}, a^*) < 0 \quad \text{– supercritical pitchfork bifurcation}$$

$$\frac{\partial^3 f}{\partial x^3}(x_{eq}, a^*) > 0 \quad \text{– subcritical pitchfork bifurcation}$$

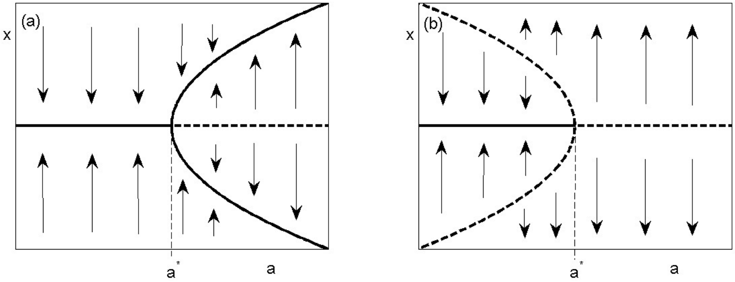


Figure 10: (a) Supercritical. (b) Subcritical.

Andronov-Hopf bifurcation

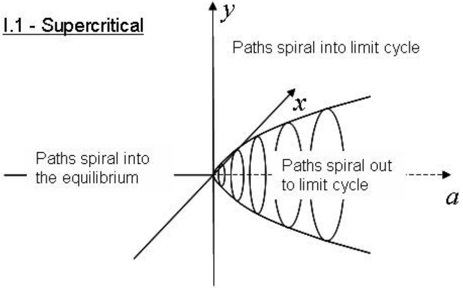
For systems of ODEs

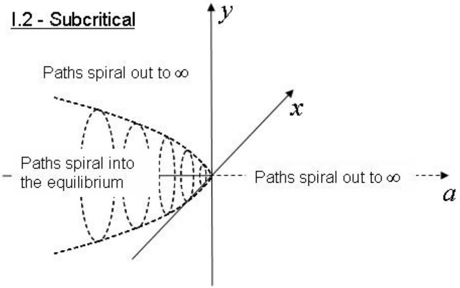
$$\dot{x} = f(x, a),$$

where $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$ with $n \geq 2$ it is possible that limit cycles are born in, so called, **Andronov-Hopf bifurcations** (or Hopf bifurcations), which

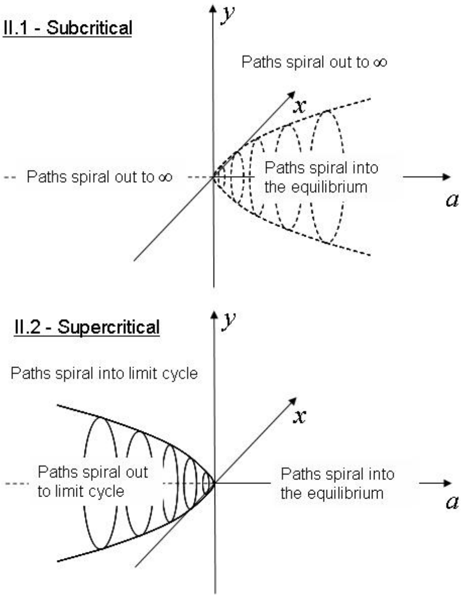
- are 2-dimensional version of a pitchfork bifurcation,
- usually involves a limit cycle "appearing" suddenly,
- appears in n-dim non-linear ODE systems with a parameter.

Possible Hopf bifurcations





Possible Hopf bifurcations



Theorem: Hopf bifurcation

The following is an example of a Hopf-bifurcation theorem in 2-dim ODE systems: Consider for $i = 1, 2$

$$\frac{dx_i}{dt} = f_i(x, a), \; x = (x_1, x_2), \; a \text{ is a real parameter.}$$

If
- $f_i(x, a)$ are smooth functions of x_1, x_2 and a,
- an equilibrium point occurs at x_{eq} for all a, i.e. $f_i(x_{eq}, a) = 0$,
- the Jacobian matrix

  $$J(x_1, x_2) = \left.\frac{\partial f_i}{\partial x_j}\right|_{x=x_{eq}}$$

  has a pair of complex eigenvalues $\lambda_{\pm} = \alpha(a) \pm \mathbf{i}\beta(a)$ with the property that

  $$\alpha(a_c) = 0 \quad \text{with} \quad \left.\frac{d\alpha}{da}\right|_{a=a_c} \neq 0,$$

  i.e. at $a = a_c$ the two eigenvalues are imaginary,

then a Hopf bifurcation occurs at the critical value $a = a_c$.

Theorem: Hopf bifurcation

The local behaviour of the non-equilibrium solutions will depend on the sign of the real part of the eigenvalue:

I. $\alpha(a_c) = 0$ and $\frac{d\alpha}{da}\Big|_{a=a_c} > 0$

1. $a < a_c, \alpha(a) < 0$: Trajectories spiral into the equilibrium point.
2. $a > a_c, \alpha(a) > 0$: Trajectories spiral away from the equilibrium point.

II. $\alpha(a_c) = 0$ and $\frac{d\alpha}{da}\Big|_{a=a_c} < 0$

1. $a < a_c, \alpha(a) > 0$: Trajectories spiral away from the equilibrium point.
2. $a > a_c, \alpha(a) < 0$: Trajectories spiral into the equilibrium point.

Whether the Hopf bifurcation is subcritical or supercritical depends on the results of other analysis, such as, asymptotic analysis far from the equilibrium point and a search for limit-cycle solutions for either $a < a_c$ or $a > a_c$.

Normal forms

For the bifurcations we have seen it is possible to find the simplest form of vector field for them to occur. Such form is known as *normal forms* of a bifurcations. The normal forms are

1D

$$\begin{aligned}
\dot{x} &= a \pm x^2 \qquad \text{(saddle-node/fold)} \\
\dot{x} &= ax \pm x^2 \qquad \text{(transcritical)} \\
\dot{x} &= ax \pm x^3 \qquad \text{(pitch-fork)}
\end{aligned}$$

2D

$$\begin{aligned}
\dot{x}_1 &= ax_1 - x_2 \pm \left(x^2 + y^2\right) \qquad \text{(Hopf bifurcation)} \\
\dot{x}_2 &= x_1 + ax_2 \pm \left(x^2 + y^2\right)
\end{aligned}$$

By normalising and changing variables it is may be possible to transform a system about a bifurcation point to one of the normal forms above and thus highlight a specific bifurcation.

Other transitions

Period-doubling/fold bifurcations. A bifurcation where a periodic orbit or limit cycle changes stability and a new periodic orbit or limit cycle with twice the period or period time is born. See further the logistic map discussed later.

Neimark-Sacker bifurcation. A bifurcation where a periodic orbit or limit cycle changes stability and a quasi-periodic solution is born.

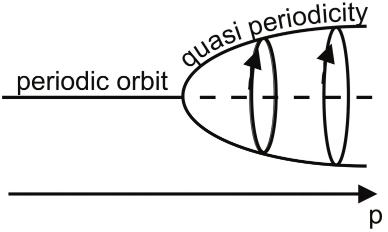
Discontinuity Induced Bifurcations (DIBs). A non-standard bifurcation where the qualitative change occurs doe to some discontinuity [19-23].

Cusp bifurcations. A two-parameter bifurcation where two branches of saddle-node bifurcations come together.

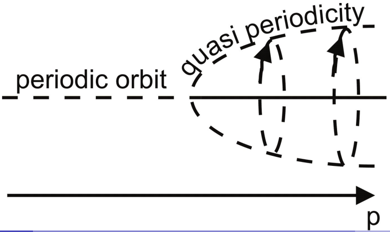
Other catastrophes and bifurcations. Swallow-tail, Butterfly, Canards, etc.

Possible Neimark-Sacker bifurcations

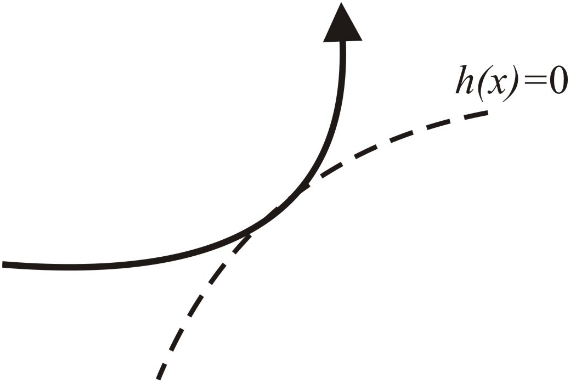
Supercritical Neimark-Sacker:



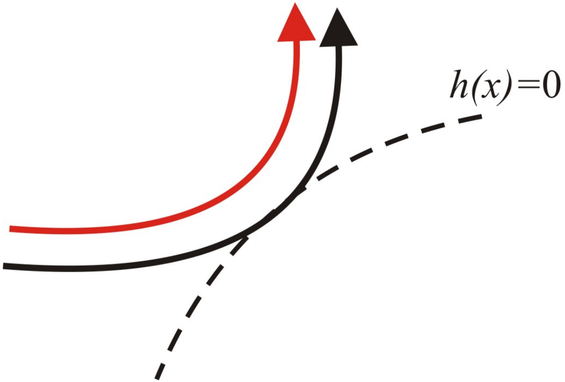
Subcritical Neimark-Sacker:



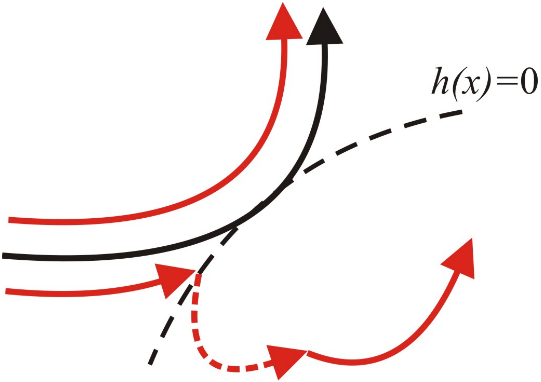
Discontinuity Induced Bifurcations - Grazing



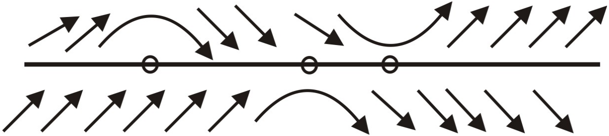
Discontinuity Induced Bifurcations - Grazing



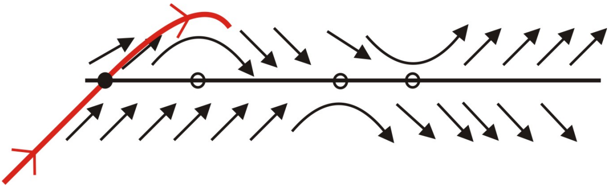
Discontinuity Induced Bifurcations - Grazing



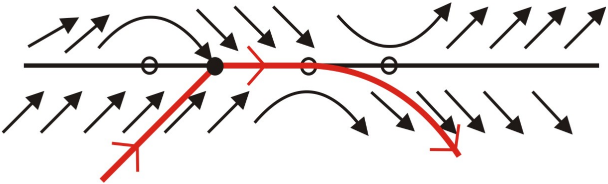
Discontinuity Induced Bifurcations - Sliding



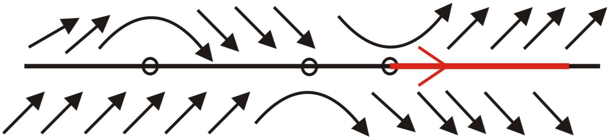
Discontinuity Induced Bifurcations - Sliding



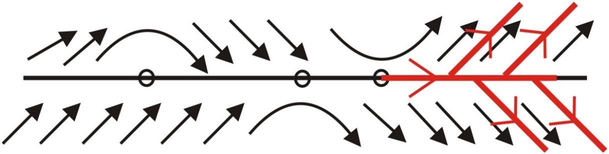
Discontinuity Induced Bifurcations - Sliding



Discontinuity Induced Bifurcations - Sliding



Discontinuity Induced Bifurcations - Sliding



Hysteresis

Consider

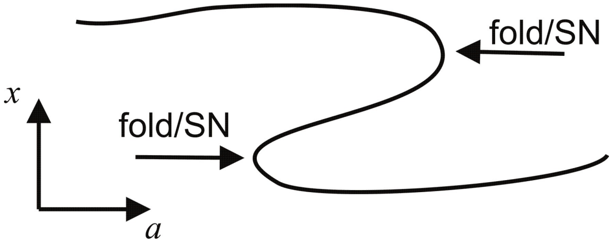
$$\dot{x} = f(x, a), \quad f(x, a) = a + bx - x^3$$

for some constant b.

For an equilibrium we want

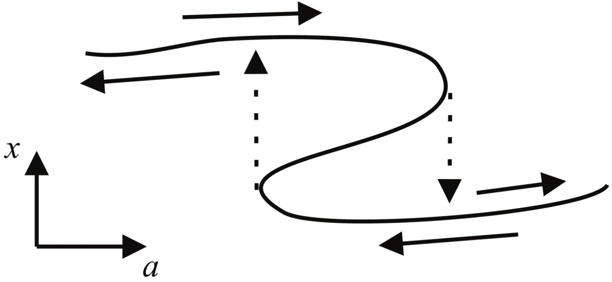
$$\dot{x} = f(x^*, a) = 0.$$

The figure below shows a schematic of what this may look like for $b > 0$.



Hysteresis

Assuming that the upper and lower parts are stable and letting $a = a(t)$ be allowed to vary in time it is possible for this system to have a *hysteresis*.



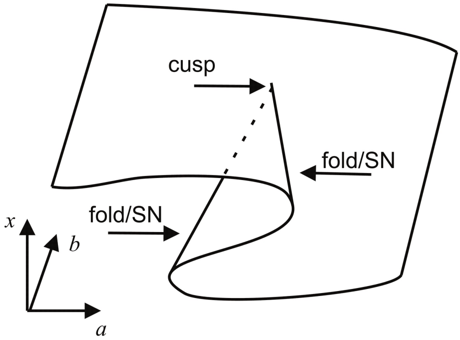
Cusp catastrophe

We can now allow ourselves to vary the parameter b as well [27, 28]. Thus the equilibrium condition

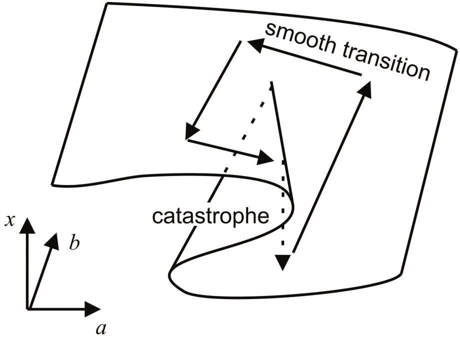
$$\dot{x} = f(x^*, a, b) = 0$$

has to be fulfilled, where

$$f(x, a, b) = a + bx - x^3.$$

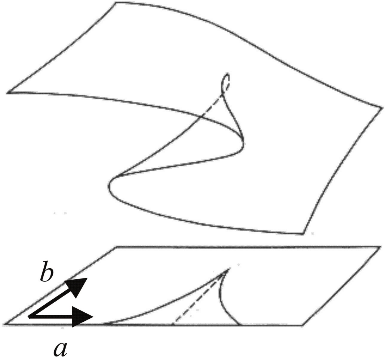


Cusp catastrophe



Allowing both a and b vary with time it is possible to have both smooth and catastrophic transitions.

Cusp catastrophe



The projection of the branches of saddle-node bifurcations onto the $a - b$ plane gives the characteristic cusp curve.

Safe and dangerous bifurcations

After going through some of all the possible transitions that can occur in nonlinear dynamical systems we can characterise them as *safe* or *dangerous* [26].

The terms *safe* or *dangerous* are to be taken as technical. Whether a bifurcation is *actually* safe or dangerous has to be assessed in the specific context where it occurs.

Safe bifurcations

Safe:

- Supercritical Hopf
- Supercritical Neimark-Sacker
- Supercritical period-doubling bifurcations
- Some DIBs

Typical Behaviour:

- Continuous growth of new attractor
- No fast jumps to new attractors
- Determinacy under perturbations
- No hysteresis

Dangerous bifurcations

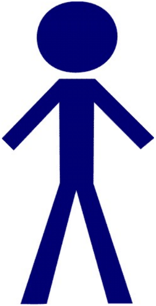
Dangerous:

- Saddle-node/fold of equilibria and periodic orbits
- Subcritical Hopf
- Subcritical Neimark-Sacker
- Subcritical period-doubling bifurcations
- Some DIBs

Typical Behaviour:

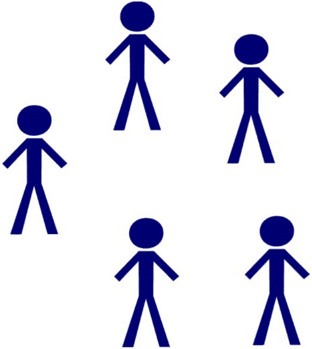
- Sudden disappearance of attractor
- Sudden jump to new attractor
- Indeterminacy under perturbations
- Basin of attraction tend to zero
- Critical slowing down

Networks with dynamics



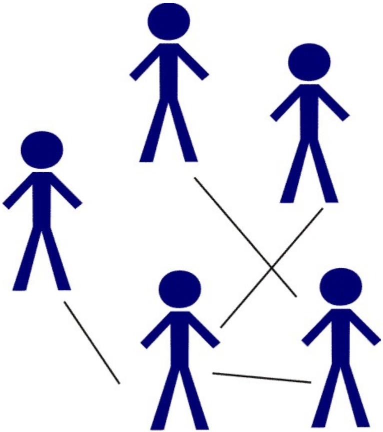
$$\dot{x} = f(x)$$

Networks with dynamics



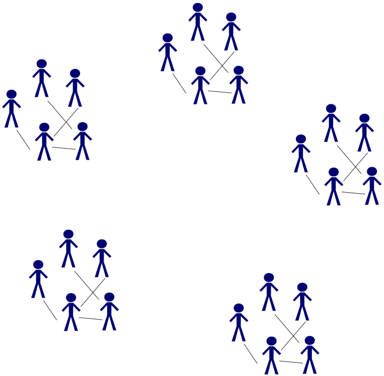
$$\dot{x}_i = f_i(x_i),\ i = 1, \ldots, N$$

Networks with dynamics



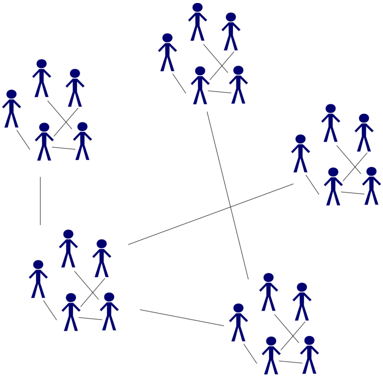
$$\dot{x}_i = f_i(x_i) + h_i(x_j),\ i, j = 1, \ldots, N$$

Networks with dynamics



$$\dot{x}_i^k = f_i^k(x_i^k) + h_i^k(x_j^k),\ \ k = 1, \ldots, M$$

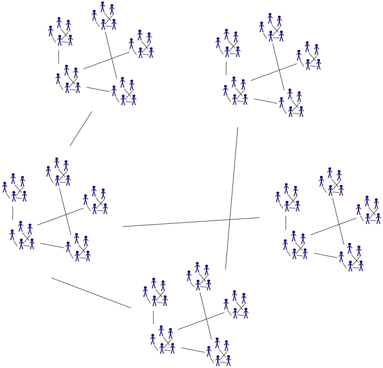
Networks with dynamics



$$\dot{\bar{x}}^k = g^k(\bar{x}^s), \;\; k, s = 1, \dots, M$$

$$\bar{x}^k = \mu(x_1^k, \dots, x_n^k)$$

Networks with dynamics



. . . and so on . . .

Modelling of dynamics of and on large-scale network are still areas where much research is ongoing. For instance it is not clear in many cases how local bifurcations and large-scale phase transitions are linked together.

Example: Logistic map

Consider the logistic map

$$x_{i+1} = f(x_i), \quad f(x) = ax(1-x), \quad a > 0,$$

for which the fixed points are

$$x = ax(1-x) \quad \Rightarrow \quad x = ax - ax^2 \quad \Rightarrow$$

$$ax\left(\frac{a-1}{a} - x\right) = 0 \quad \Rightarrow \quad x = 0,\ x = \frac{a-1}{a}.$$

$\underline{a < 1:}$ If $a < 1$ the second fixed point is **less than** 0, which is unphysical (in some sense), i.e. for $0 \leq x \leq 1$ only one fixed point exists at $x = 0$.

The stability of the fixed points can be found from

$$f'(x) = a(1-2x).$$

Example: Logistic map

$x = 0$**:**

$$f'(0) = a < 1$$

and thus $x = 0$ is a sink for $0 < a < 1$.

$\underline{a > 1}$: There are two fixed points, $x = 0$ and $x = \frac{a-1}{a}$ and their stabilities are

$$f'(0) = a > 1 \quad \Rightarrow \quad \text{unstable fixed point}$$

and

$$f'\left(\frac{a-1}{a}\right) = a\left(1 - 2\frac{a-1}{a}\right) = a - 2(a-1) = 2 - a.$$

The second fixed point is stable when

$$-1 < 2 - a < 1 \quad \Rightarrow \quad 1 < a < 3$$

and unstable when $a > 3$.

Example: Logistic map

The period-2 points can be located by solving

$$\begin{aligned} x &= f^2(x) = f(f(x)) = f(ax - ax^2) = a\left(ax - ax^2\right)\left(1 - ax + ax^2\right) \\ &= a^2 x\,(1-x)\left(1 - ax + ax^2\right), \qquad \Rightarrow \\ 0 &= x\left(1 + a^2(x-1)\left(1 - ax + ax^2\right)\right) \end{aligned}$$

The factor x is explicit and we also know that $\left(x - \frac{a-1}{a}\right)$ must also be a factor. Thus

$$a^3\,\underbrace{x\left(x - \frac{a-1}{a}\right)}_{\text{fixed points}}\underbrace{\left(x^2 - \frac{a+1}{a}x + \frac{a+1}{a^2}\right)}_{\text{period-2 points}} = 0$$

and the period-2 points are

$$x_{\pm} = \frac{a+1}{2a}\left(1 \pm \sqrt{\frac{a-3}{a+1}}\right) = \frac{1}{2a}\left(a + 1 \pm \sqrt{(a-3)(a+1)}\right).$$

The **existence** of the period-two points requires $a > 3$.

Example: Logistic map

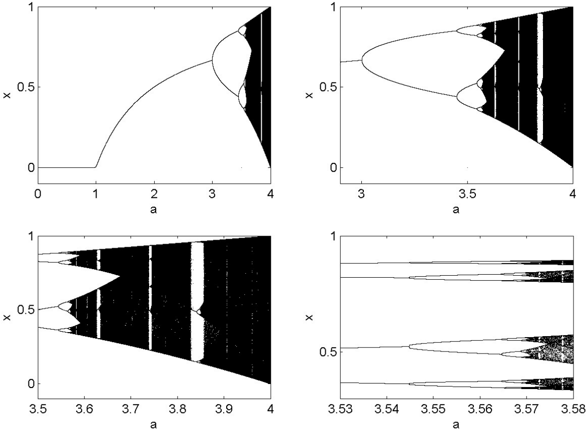


Figure 11: Bifurcation diagrams for the logistic map for $0 \leq a \leq 4$.

Example: Logistic map

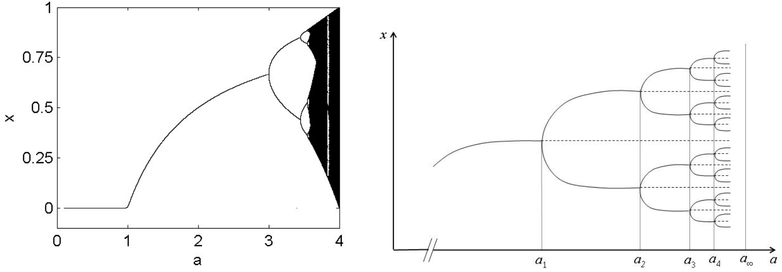


Figure 12: Real and schematic bifurcation diagrams for the logistic map for $0 \leq a \leq 4$.

Example: Logistic map

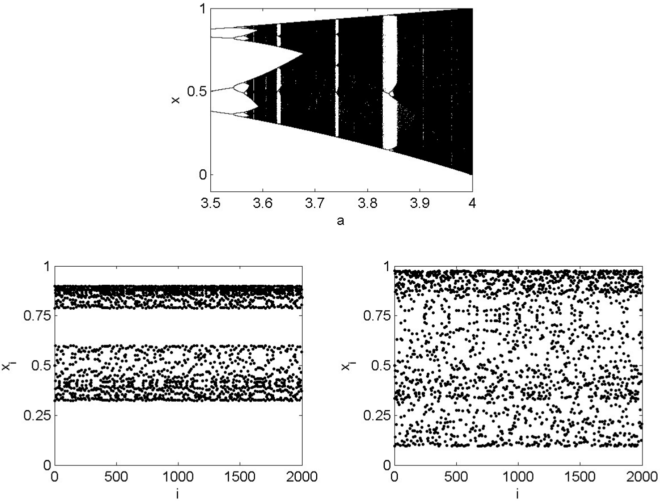


Figure 13: Iterations of the logistic map at $a = 3.6$ and $a = 3.9$.

Example: Logistic map with additive noise

In [25] the logistic map with small additive noise was considered so that

$$x_{i+1} = rx_i(1 - x_i) + \sigma\xi_i, \quad r > 0,$$

where σ is the noise intensity and ξ_i is a the stochastic variable with mean 0 and variance 1.

Example: Logistic map with additive noise

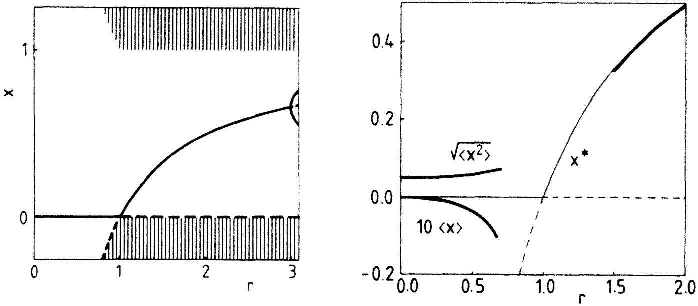


Figure 14: (Left) Bifurcation diagrams for logistic map without noise including the basin of attraction for $-\infty$. (Right) Bifurcation diagrams showing the stationary means for the logistic map without noise (thin lines) and with noise (think lines) as well as the order parameter. Here the noise intensity $\sigma = 0.05$. The figures are from [25].

Example: Predator-Pray dynamics with migration

Here we will analyse a model describing predator-prey dynamics with migration [24] and use the following methodology.

- A continuous-time approach leads to the study of differential equations.
- The overall population is split into a set of subpopulations each residing on a particular habitat.
- A differential equation that represents the dynamics of one of these subpopulations can be written in the form

$$\text{total rate of change} = \text{birth rate} - \text{death rate} + \text{net immigration rate}.$$

Migration types

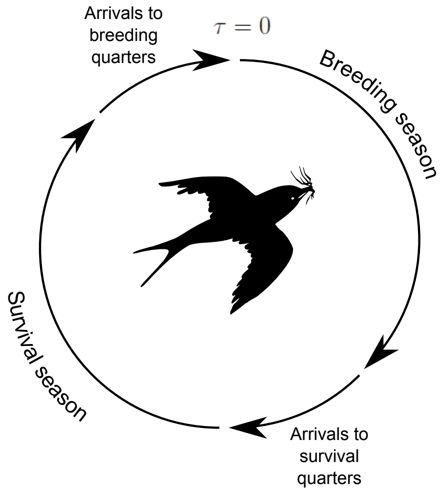
Migrations are macro-level trends in population movement and can usually be classified as:

- **Continuous** – dispersal of individuals leads to an ongoing evolution of the overall *range* of the population (N large).
- **Irregular** – large numbers of individuals sporadically undergo mass movement, in response to overcrowding or food shortages (N large).
- **Regular/seasonal** – individuals exhibit a regular pattern of movement with respect to time and location ($N = 2$).

where $N > 0$ is the number of distinct habitats considered.

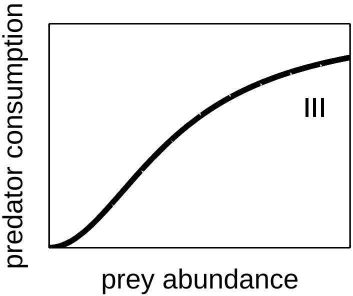
Seasonal migrations are usually triggered by abiotic factors such as day-length or temperature.

Model timeline



$$\tau := t \bmod 365$$

Generalist predation



- *A generalist predator is one that does not rely upon any one energy source to ensure its survival.*

Generalist model

The biomass of sub-populations inhabiting a breeding range and non-breeding range is given by A_1 and A_2, respectively. The breeding of A_1 results in a newborn population with biomass N. The variation in the size of the population is modelled using the hybrid dynamical-system

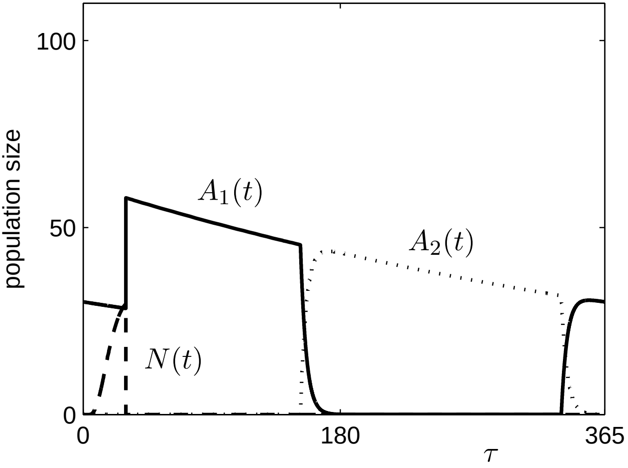
$$
\begin{aligned}
\frac{dA_1}{dt} &= \alpha_v k_v f_4(\tau) A_2 - k_v f_2(\tau) A_1 - c_1 A_1 - \frac{e{A_1}^2}{1 + he{A_1}^2 + heN^2} P, \\
\frac{dA_2}{dt} &= \alpha_v k_v f_2(\tau) A_1 - k_v f_4(\tau) A_2 - c_2 A_2, \\
\frac{dN}{dt} &= f_1(\tau) r A_1 \left(1 - \frac{N}{K}\right) - c_N N - \frac{eN^2}{1 + he{A_1}^2 + heN^2} P, \\
\frac{d\tau}{dt} &= 1,
\end{aligned}
$$

with switches

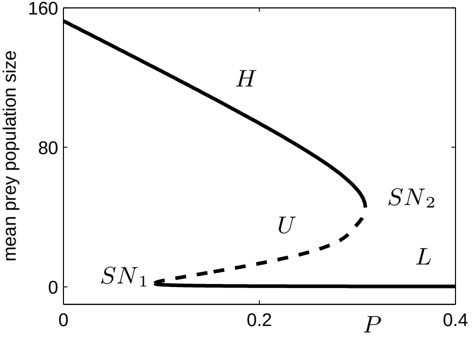
$$
\begin{aligned}
A_1 &\mapsto A_1 + N && \text{at } \tau = t_1, \\
N &\mapsto 0 && \text{at } \tau = t_1, \\
\tau &\mapsto 0 && \text{at } \tau = 365,
\end{aligned}
$$

where t_1 is the time at which the breeding season comes to an end.

Zero/low predation

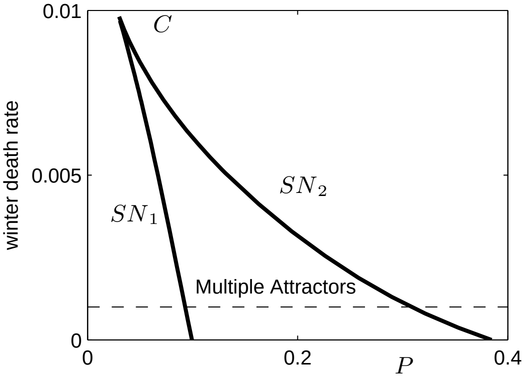


Variation in predator abundance



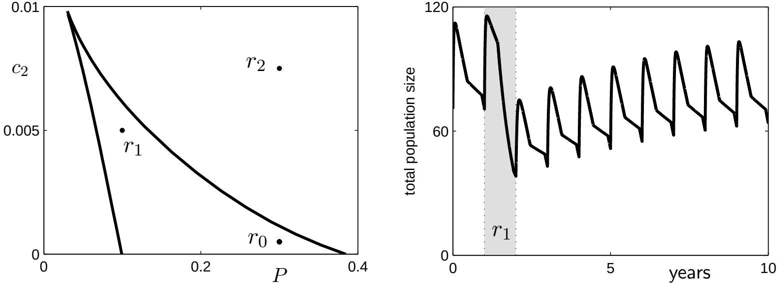
Mean steady-state population sizes as functions of the predator population size P. All limit cycles are of period one year and the values plotted are averaged over the annual cycle. Stable solutions are denoted by solid curves and unstable by dashed curves.

Steady-state configurations

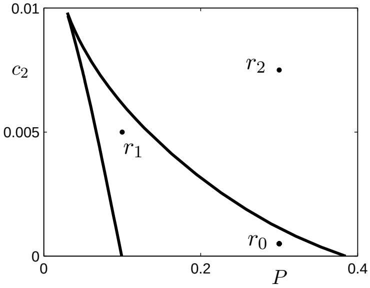


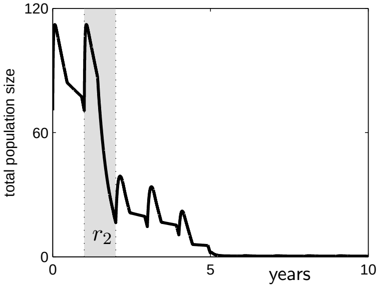
A two-parameter bifurcation diagram with the predator population size and wintering mortality rate as bifurcation parameters.

Inter-annual variations



Inter-annual variations





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