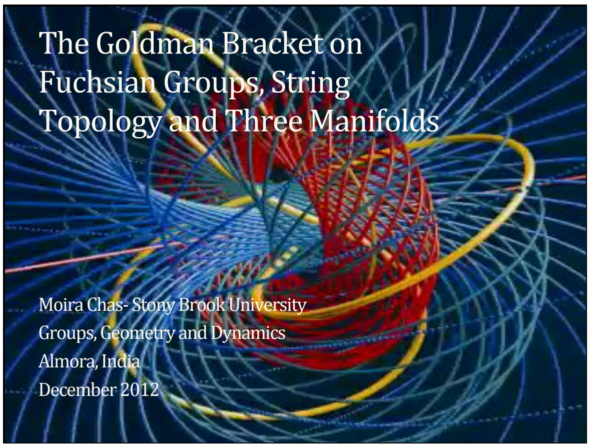


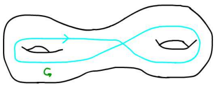
The Goldman Bracket on Fuchsian Groups, String Topology and Three Manifolds

Moira Chas- Stony Brook University
Groups, Geometry and Dynamics
Almora, India
December 2012

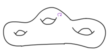
The plan

1. The Goldman Lie bracket and and its relation with the number of intersection points of curves on a surface.
   - Statement: Two surfaces have isomorphic Lie algebras (and a bit more) if and only if they are isomorphic.
   - Proof: Uses that the bracket "knows" about intersection of curves, and the curves "know" in which surface they live.
2. Orbifolds, closed curves, homotopy and intersection number of curves.
3. The Goldman Lie bracket on orbifolds and its relation with intersection number of curves.
4. The String bracket on Three Manifolds.
5. The String bracket on Three Manifolds and relationwith intersections of tori and curves
   - Statement: (Certain) pair of three manifolds have isomorphic Lie algebras if and only if they have common topological features.
   - Proof: Uses that the String bracket "knows" about intersection of tori and curves, and the tori and curve "know" in which manifold they live

Today's Goal: Study the Goldman Lie bracket of curves on surfaces



- Description of the vector space where the bracket is defined
- Definition of the bracket.
- Examples
- Relation between the bracket of two curves and number of intersection points.

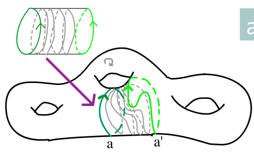


S is an oriented surface (with or without boundary)

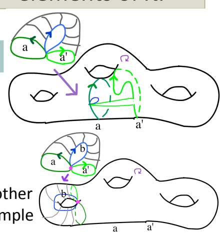
π = the fundamental group of S

π_0 = the set of free homotopy classes of directed closed curves on S.

π_0 ⟵ one to one ⟶ conjugacy classes of elements of π.



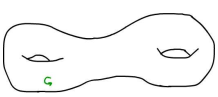
a' = bab^{-1}



Another example

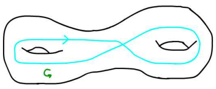
a is freely homotopic to a'

S is an oriented surface (with or without boundary)



π = fundamental group of S

π_0 = set of free homotopy classes of directed closed curves on S.



Denote by $V(\pi_0)$ the free module with basis π_0 (coefficients in Z or R, your choice)

Remark: If LS={f / f: oriented circle ➔S},

then π_0 ≈ path components of LS = $\pi_0(LS)$

Example:
An element of $V(\pi_0)$ is
0^{-1} - 2 0 + 3 <class of trivial loop>

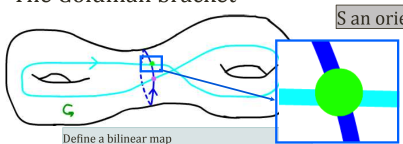
Remark: An element of $V(\pi_0)$ is a *formal* linear combination of elements in π_0.

Thus, if A in π_0, there is not a relation of the type $A^{-1} = -A$.

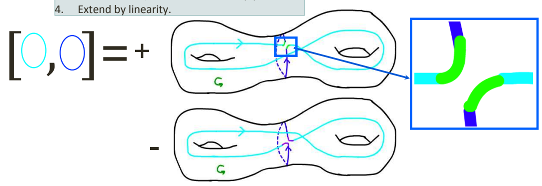
The Goldman bracket

S an oriented surface

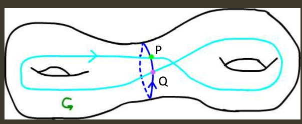
π_0 = set of free homotopy classes of closed oriented curves on S.



Define a bilinear map
[,]: $Z[\pi_0] \otimes Z[\pi_0] \rightarrow Z[\pi_0]$ as follows:
1. Take 0 and 0 in the basis π_0
2. Choose representatives in general position.
3. Define [0, 0] (intersection and loop product.
4. Extend by linearity.

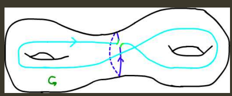
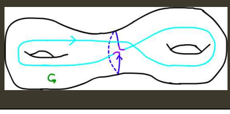


A, B in π_0, chose representatives in general position.



$[A,B] = A \cdot_P B - A \cdot_Q B$
where $A \cdot_R B$ means the free homotopy class of the curve that starts at P, goes around A, and then around B.

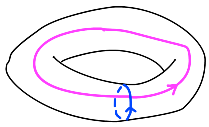
Theorem: (Goldman, 1986)
The bracket is
- well defined,
- skew symmetric and
- satisfies the Jacobi identity.

[A,B]= +  - 

Homework 1: Compute the Goldman Lie algebra of the torus.

Hint: Recall the the fundamental group of the torus is the free group on two generators, a and b.

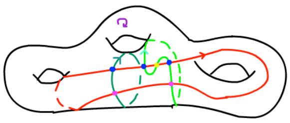
These generators can be taken as the based homotopy classes of the pink and blue loops depicted below.



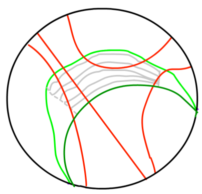
1. Use that there is a bijection between free homotopy classes of closed oriented curves on the torus and the set $\{a^i b^j$, i and j in Z.$\}$
2. Compute
   - [a,b],
   - $[a^i,b]$,
   - $[a^i,b^j]$,
   - $[a^i b, b^j]$,
   - $[a^i b^k, a^i b^j]$

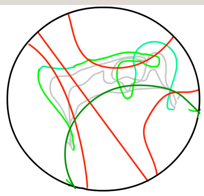
a ≈ a' then "[a,b]"="[a',b]"

- The light green a and dark green curve a' are freely homotopic.
- The two terms corresponding to the two pink points are equal.
- The three terms corresponding to the three blue points are equal.
- The term corresponding to the yellow point is to minus a term determined by a blue point.
- Thus, "[a,b]"
  = 0 + 0 + 0 + 0
  = 0 + 0 + 0 - 0
  = 0 + 0
  = " [a',b] "

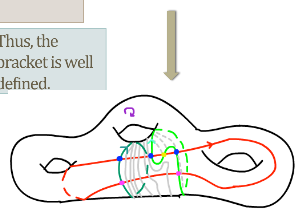


- Consider two homotopic curves a and a' in general position.
- Consider a homotopy between them.
- Lift the homotopy to the with universal cover.
- Consider all lifts of b that intersect the image of the homotopy.
- The intersection points of the lift of a and b (resp. a and b'), can be divided in two sets, S and G (resp. S' and G') so that all the terms corresponding to the projection of points in S (resp. S') cancel in pairs and there is a one to one correspondence between the terms corresponding to the projection of the point of G and the projeciton of the points in G'.



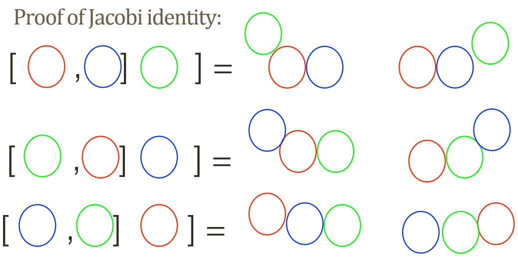


Thus, the bracket is well defined.



Jacobi identity: for a,b, c in $V(\pi_0)$,
$[[A,B], C]+[[B,C],A]+[[C,A],B]=0$

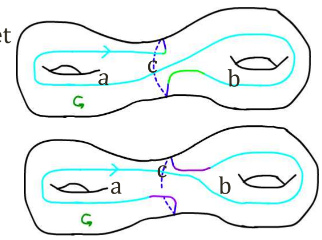
Proof of Jacobi identity:

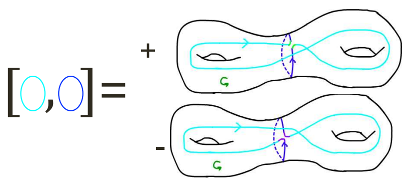


The Goldman bracket

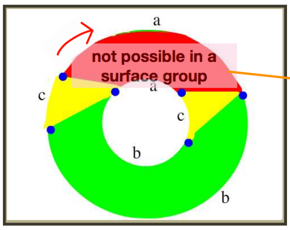
Claim: that these two terms are different.

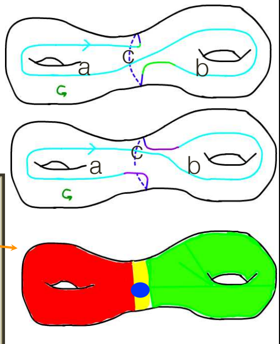
<abc> ≠ <acb>



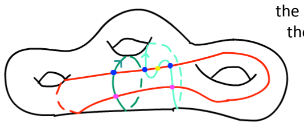


Observe that the two terms are different.
<abc> ≠ <acb>





The minimal intersection between the class of the green curves and the class of the red curve is 2.



For each A and B in π_0, the smallest number of points (counted with multplicity) in which a curve in A intersects a curve in B is *minimal intersection of A and B*, denoted by ***i(A,B)***.

---

w in $Z[\pi_0]$

$w = \mu_1 A_1 + \mu_2 A_2 + \ldots + \mu_n A_n$, A_i in π_0, $A_i \neq A_j$ if $i \neq j$.

The *Manhattan norm of w (or l_1 norm)*, is

M(w)$= |\mu_1| + |\mu_2| + \ldots + |\mu_n|$

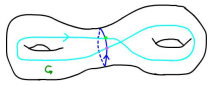
Goal: Study relation between M([A,B]) and i(A,B).

A and B in π_0, i(A,B) is the minimal intersection number between representatives of A and B.

M [X,Y] ≤ i(X,Y)

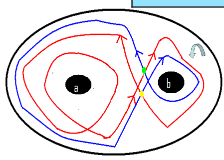
M[X,Y] = i(X,Y)?

In this example, yes.



But not always...

Combinatorial presentation of the Goldman bracket



aab ba = aabba

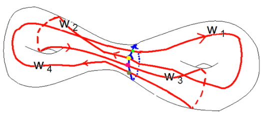
baa ab = baaab

[aab,ab]=0
i(aab,ab)=2

Consider X and Y are free homotopy classes of closed curves, such that X has a representative which does not intersect itself.

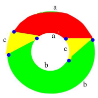
Theorem (Goldman, 1986) If [X,Y]=0 (that is, M[X,Y]=0) then i(X,Y)=0.

Theorem (C, 2009) M[X,Y] = i(X,Y).



Proof: Free product with amalgamation (or HNN structure)

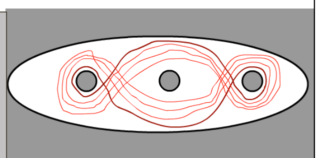
$\pm[w_1 w_2 w_3 w_4, X] =$
$w_1 X w_2 w_3 w_4 - w_1 w_2 X w_3 w_4 + w_1 w_2 w_3 X w_4 - w_1 w_2 w_3 w_4 X$

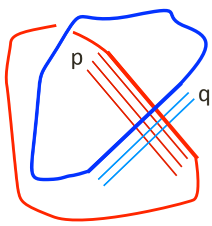


Combinatorial presentation of the Goldman bracket
Counting intersections theorem.

Counting Self-Intersections Theorem (Krongold – C.) In a surface with boundary, if X is the free homotopy class of a closed curve which is not the proper power of another class then

$M[X,X^3] = 3.\ 2.$ Self-intersection(X)





Theorem (Gadgil,C.) If p or q are large enough integers,

then $M\ [X^p,Y^q] = p.q\ i(X,Y)$

Holds for X=Y, (with p ≠ q).

Recall: The center of a Lie algebra L is the set of elements a of b such that [a,b] = 0 for all b in L.

0 is in the center.

Center = < 0 >

Theorem: (Etingof) The center of the Goldman Lie algebra of curves on surfaces consist in 0 if the surface has empty boundary

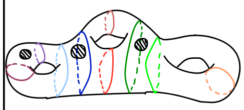
Conjecture: The center of the Goldman Lie algebra is a linear combination of classes of curves parallel to the boundary if surface has non-empty boundary.

0, 0, 0 are in the center.

Conjecture Center = < 0, 0, 0 >

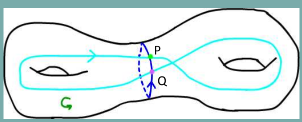
Homework: 2
Given the Goldman Lie algebra on $V(\pi_0)$ defined on the basis π_0, can the genus and number of boundary components of the surface be determined?

Hint 1: Assume the conjecture that the center is generated by the classes of curves parallel to boundary componens
Conjecture Center = < 0, 0, 0 >
Then dim center =3



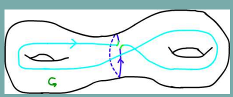
Recall: the maximal number of simple closed curves, not pairwise homotopic nor parallel to boundary components, that do not intersect is **3.genus -3+boundary comp.**

AA, B in π_0, chose representatives in general position.



$[A,B] = A \cdot_P B - A \cdot_Q B$
where $A \cdot_R B$ means the free homotopy class of the curve that starts at P, goes around A, and then around B.

[A,B]=



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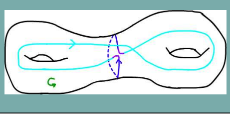

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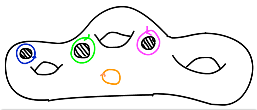
Given the Goldman Lie algebra, can one recover the surface?

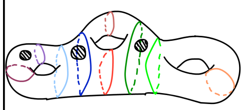
Number of boundary components?

Equals the dimension of the center, if the conjecture holds.



Conjecture Center = < 0, 0, 0 >
Then dim center =3

Then # boundary components = 3.



Compute the genus of the surface?

$[x,x^3]$ -theorem finds all simple closed curves.

[[x, simple]- theorem to determine disjointness.

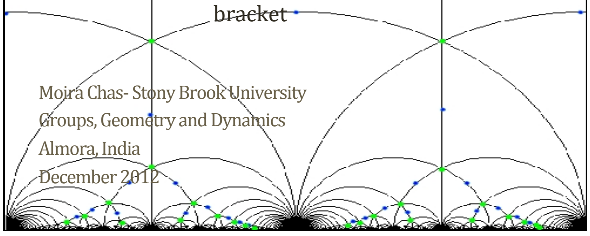
Needs geometric basis

3g+b-3 = 9
Since b=3, g=3

Recall; the maximal number of simple closed curves, not pairwise homotopic nor parallel to boundary components, that do not intersect is **3.genus -3 + boundary comp.**

The Goldman Bracket on Fuchsian Groups, String Topology and Three Manifolds

Orbifolds and the Goldman bracket



Moira Chas- Stony Brook University
Groups, Geometry and Dynamics
Almora, India
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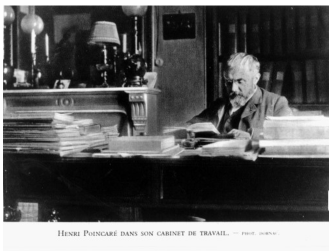
At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations that I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry.

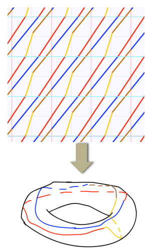
Poincare on his discovery of Fuchsian Groups



Outline

- Groups acting on spaces.
- Orbifolds.
- Two dimensional orbifolds.
- Examples
- “Closed curves” in orbifolds.
- “Homotopy” in orbifolds.
- Combinatorics of intersection of “closed curves” in orbifolds.

An example of a group G acting on a space X and the quotient X/G



Z+Z acts on R^2 by translations:

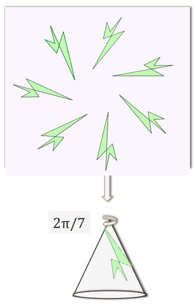
(m,n) in Z+Z ,(a,b) in R^2, (m,n).(a,b)=(a+m,b+n).

- This action induces a covering map, p:R^2→R^2/(Z+Z)
- Given the quotient R^2/(Z+Z) the action of the group and the cover space can be reconstructed.
- This action determines a tiling of R^2

An example of a group G acting on a space X and the quotient X/G

Z/7Z acts on R^2 by rotation of 2π/7.

- This action induces a *branched* covering map, p:R^2->R^2/(Z/7Z)
- Given the quotient R^2/(Z/7Z), can the action of the group and the cover space be reconstructed?
- This action determines a tiling of R^2.

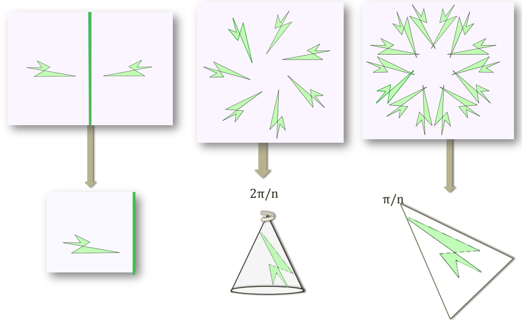


G=finite group of isometries of R^2 , X=R^2

G=Reflection along a line

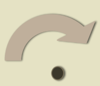
G=Rotation 2π/n around a point.

G=Dihedral group (Generated by a rotation 2π/n and a reflection)



An example of a group G acting on a space X and the quotient X/G

All angles



G is the group of all rotations about the origin of R^2 .

What is R^2/G ?

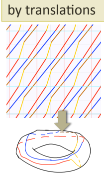
Given R^2/G , can the action and the cover space be reconstructed?

Consider a metric space X acted on by a group of isometries G.

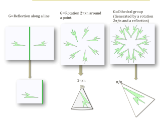
Suppose that the action of G on X is properly continuous and free.

If X is a manifold, the quotient X/G is a manifold of the same dimension.

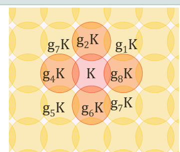
Example 1:
Z+Z acts on R^2
by translations



Example 2:
Finite groups of
isometries of R^2



Recall that a group G acts *properly discontinuously* on a metric space X if for every compact subset K of X, the set
{g in G : K intersects gK}
is finite.



In particular, if a group G acts properly discontinuously on X then the stabilizer of a point P that is, the set
{g in G : gP=P}
is finite.

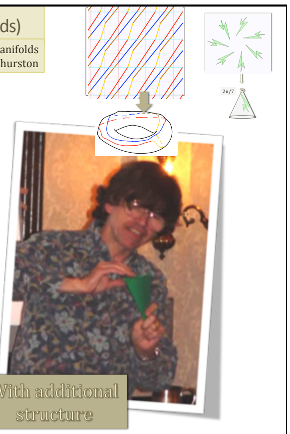
The action of G is *free* if the stabilzer of every point is trivial.

Symmetry and Spaces of Orbits (Orbifolds)

From "Notes of Geometry and Topology of Three Manifolds
By Bill Thurston

When a group G acts properly discontinuously on a (*nice*) space X,

- the information of the action is recorded by the covering map X ➔X/G.
- A symmetric pattern on X is determined by drawing the pattern on X/G and lifting.
- If X is simply connected, then all we need to know to reconstruct the action is the quotient X/G.
- But even though these objects are mathematically equivalent, they give images that are quite different in appearance.
- Because of this contrast, a lot is to be gained by going back and forth comparing them.
- We will enlarge our vocabulary, so we can discuss groups that act properly discontinuously but not necessarily act freely.
- The quotient spaces of these groups (equipped with enough additional structure to describe the way they act) we will call



Action of G on X | Covering map X ➔X/G | With additional structure

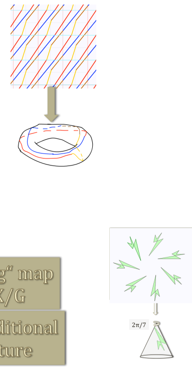
Symmetry and Spaces of Orbits (Orbifolds)

- G acts freely, properly discontinuously on X.

Action of G on X | Covering map X ➔X/G

- G acts ~~freely~~, properly discontinuously on X.

Action of G on X | "Covering" map X ➔X/G | With additional structure



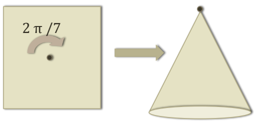
Note: If the action is not free, the quotient map is not an "honest" covering map. A precise description of this map will be given later.

Symmetry and Spaces of Orbits (Orbifolds)
The additional structure: Example

"Covering" map X ➔X/G with additional structure

G=Z/7Z
X = R^2

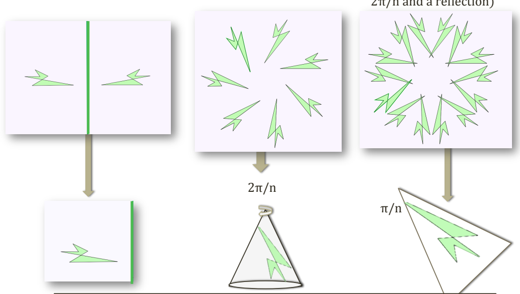
X/G
Order = 7
Is the additional structure



- The quotient spaces of these groups (equipped with enough additional structure to describe the way they act) we will call **orbifolds**.

G=finite group of isometries of R^2 , X=R^2

G=Reflection along a line | G=Rotation 2π/n around a point. | G=Dihedral group (Generated by a rotation 2π/n and a reflection)



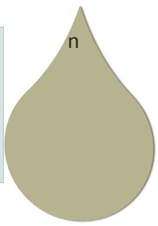
Every point in the quotient spaces has a (small) neighborhood U ≈ R^2/G, where G is trivial, reflection, rotation or dihedral.

The Teardrop

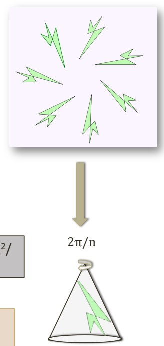
Every point except one, P has a neighborhood ≈ to R^2.

P has a neighborhood ≈ to a cone of the form $R^2/(Z/nZ)$, for some n.

Thus, P is labeled by the integer n.



Rotation 2π/n around a point.



Every point in the teardrop has a (small) neighborhood U ≈ R^2/G, where G is trivial, reflection, rotation or dihedral.

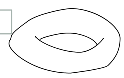
We are going to expand the concept of orbifold, to include examples like the teardrop that are not necessarily quotients.

An ***n-dimensional orbifold*** O is a (Hausdorff, paracompact) topological space X_0 which is locally homeomorphic to the quotient space of R^n by a finite group action.

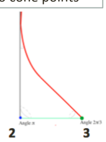
The set of points where the finite group it is not trivial (roughly speaking, where there is extra structure) is called the singular locus.

Some examples of two dimensional orbifolds

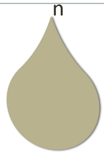
Singular locus is ø



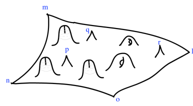
Singular locus : the two cone points

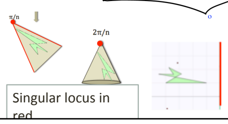


Singular locus : the cone point.



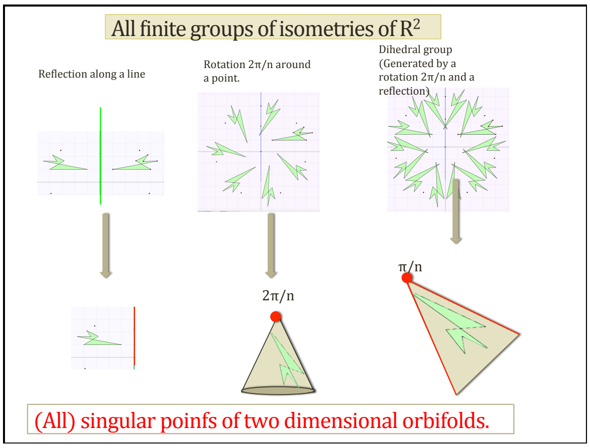
Singular locus : all the cone points and the boundary

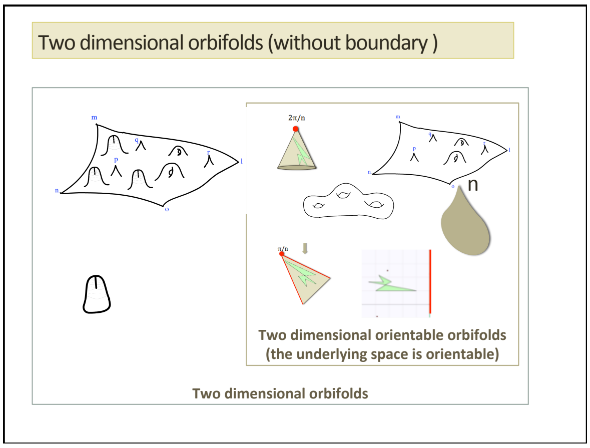


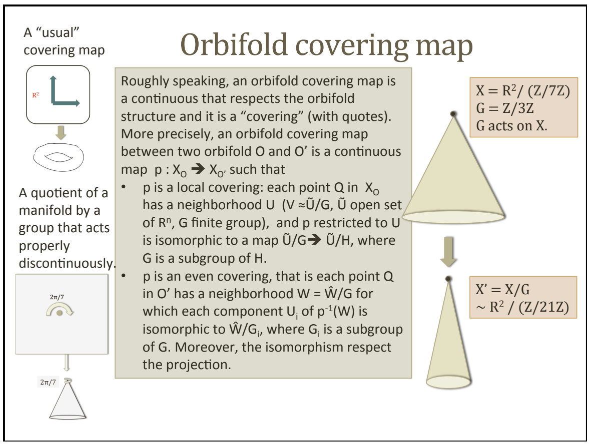


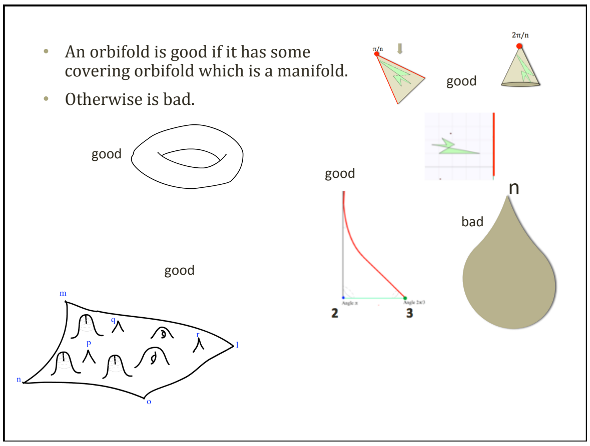
Singular locus in red

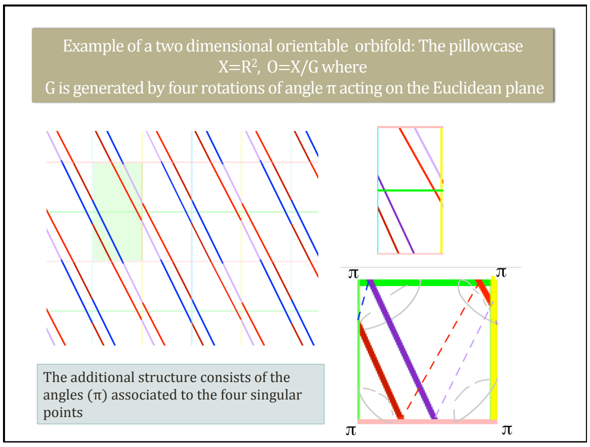
Note that the orbifold O and the topological space X_0 are different objects.

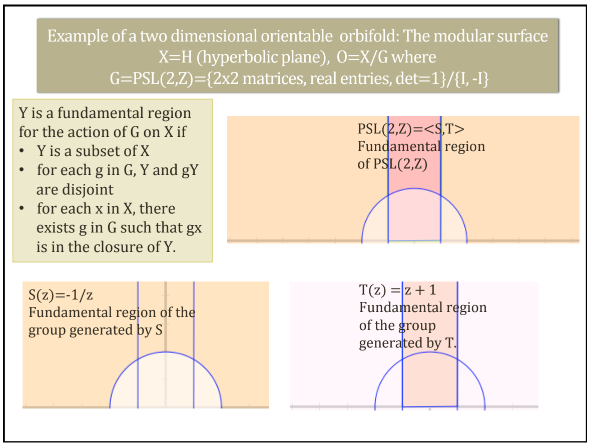

All finite groups of isometries of R^2
Reflection along a line
Rotation $2\pi/n$ around a point.
Dihedral group (Generated by a rotation $2\pi/n$ and a reflection)
$2\pi/n$
π/n
(All) singular poinfs of two dimensional orbifolds.


Two dimensional orbifolds (without boundary)
$2\pi/n$
n
Two dimensional orientable orbifolds (the underlying space is orientable)
Two dimensional orbifolds

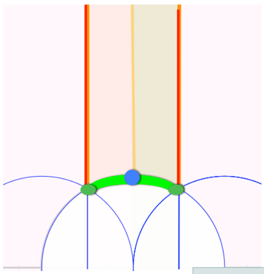

A "usual" covering map
R^2
A quotient of a manifold by a group that acts properly discontinuously.
$2\pi/7$
$2\pi/7$
Orbifold covering map
Roughly speaking, an orbifold covering map is a continuous that respects the orbifold structure and it is a "covering" (with quotes). More precisely, an orbifold covering map between two orbifold O and O' is a continuous map p : X_O ➔ $X_{O'}$ such that
• p is a local covering: each point Q in X_O has a neighborhood U ($V \approx \tilde{U}/G$, $\tilde{U}$ open set of R^n, G finite group), and p restricted to U is isomorphic to a map $\tilde{U}/G$➔ $\tilde{U}/H$, where G is a subgroup of H.
• p is an even covering, that is each point Q in O' has a neighborhood $W = \hat{W}/G$ for which each component U_i of $p^{-1}(W)$ is isomorphic to $\hat{W}/G_i$, where G_i is a subgroup of G. Moreover, the isomorphism respect the projection.
$X = R^2/ (Z/7Z)$
$G = Z/3Z$
G acts on X.
$X' = X/G$
$\sim R^2 / (Z/21Z)$

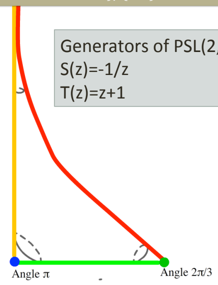

• An orbifold is good if it has some covering orbifold which is a manifold.
• Otherwise is bad.
$2\pi/n$
good
good
good
n
bad
good
2
3


Example of a two dimensional orientable orbifold: The pillowcase
$X=R^2$, O=X/G where
G is generated by four rotations of angle π acting on the Euclidean plane
π
π
π
π
The additional structure consists of the angles (π) associated to the four singular points


Example of a two dimensional orientable orbifold: The modular surface
X=H (hyperbolic plane), O=X/G where
G=PSL(2,Z)={2x2 matrices, real entries, det=1}/{I, -I}
Y is a fundamental region for the action of G on X if
• Y is a subset of X
• for each g in G, Y and gY are disjoint
• for each x in X, there exists g in G such that gx is in the closure of Y.
PSL(2,Z)=<S,T>
Fundamental region of PSL(2,Z)
S(z)=-1/z
Fundamental region of the group generated by S
T(z) = z + 1
Fundamental region of the group generated by T.

Example of a two dimensional orientable orbifold: The modular surface X=H (hyperbolic plane), O=X/G where G=PSL(2,Z)={2x2 matrices, real entries, det=1}/{I, -I}

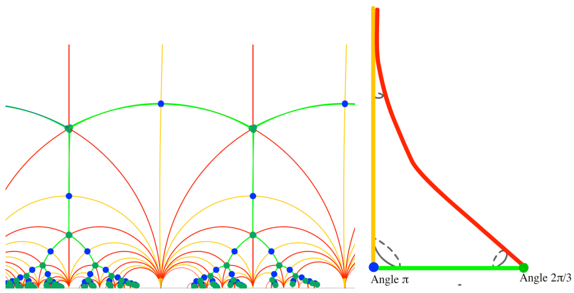




Generators of PSL(2,Z)
S(z)=-1/z
T(z)=z+1

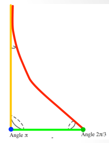
The additional structure consists of the intergers, (2,3) or equivalently, the angles (π, 2π/3) associated to the two singular points

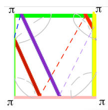
Example of a two dimensional orientable orbifold: The modular group G=PSL(2,Z) acting on the hyperbolic plane H.



Example of an orbifold: The billiard table
G is generated by four reflections in the four sides of a square

- The modular surface, the billiard and the pillowcase are examples quotient spaces of groups acting by isometries.
- In these cases, the orbifold has a geometric structure along with the topological structure.
- The geometric structure in enough to capture the essence of the cases.
- For a topological orbifold, not equipped with a geometric structure, an additional piece of information is necessary (for instance, the order at the cone points).

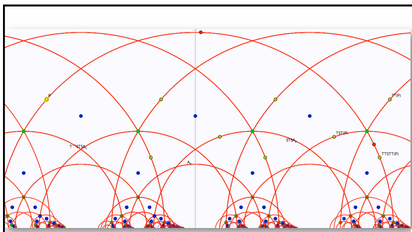




Homework

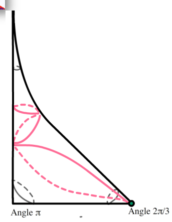
Problem 5.1.8 from Thurston's Notes:

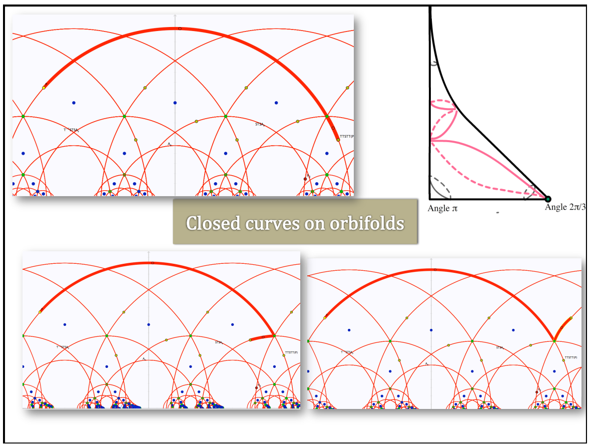
Consider a tetrahedron T on the Euclidean space which is made of four congruent triangles. Show that T is isometric to the quotient space of the Euclidean plane by a discrete group. Construct a physical model and watch what happens when you roll it carefully on a table. Wrap a ribbon or string around the tetrahedron so that it follows a geodesic. It should never cross itself. Why?

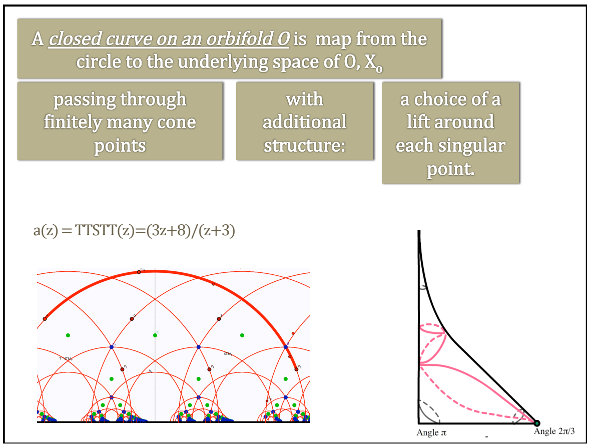


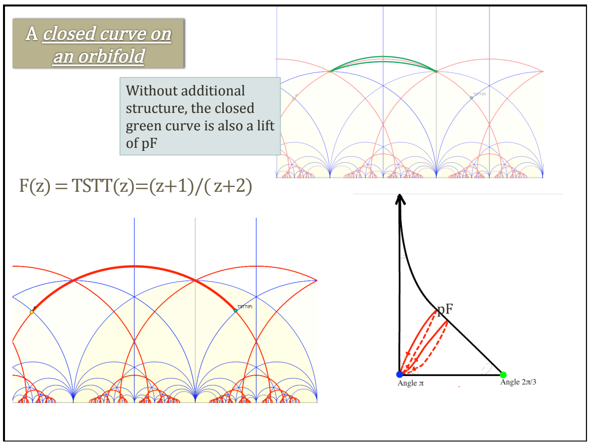
In red we have the axis of a(z) =TTTSTTT(z)=(3z+8)/(z+3) and its conjugates.

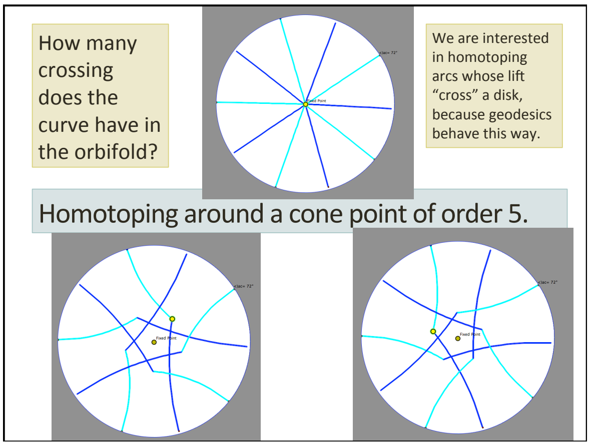
Closed curves on orbifolds

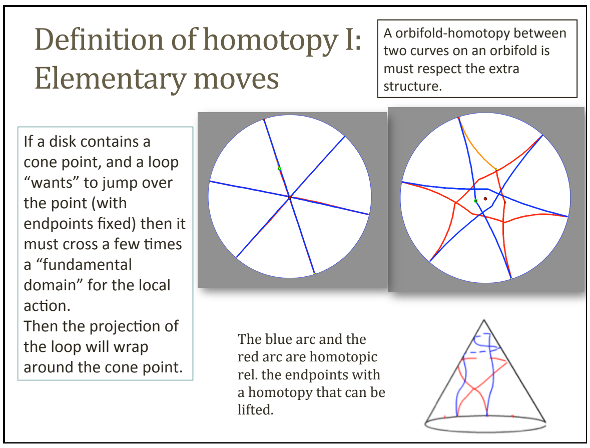


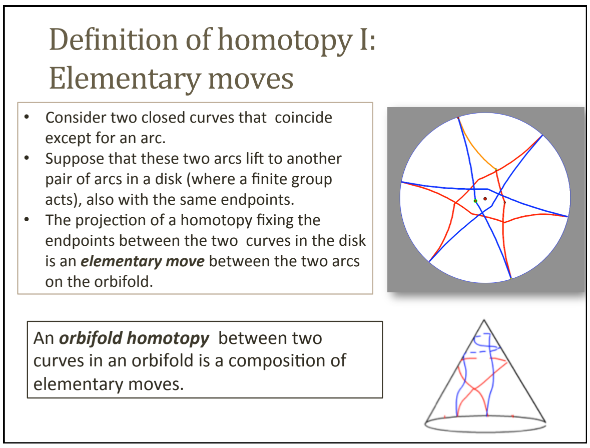

Closed curves on orbifolds
Angle π
Angle 2π/3


A closed curve on an orbifold O is map from the circle to the underlying space of O, X₀
passing through finitely many cone points
with additional structure:
a choice of a lift around each singular point.
a(z) = TTSTT(z)=(3z+8)/(z+3)
Angle π
Angle 2π/3


A closed curve on an orbifold
Without additional structure, the closed green curve is also a lift of pF
F(z) = TSTT(z)=(z+1)/(z+2)
pF
Angle π
Angle 2π/3

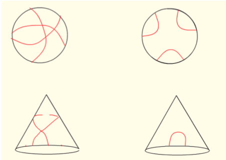

How many crossing does the curve have in the orbifold?
We are interested in homotoping arcs whose lift "cross" a disk, because geodesics behave this way.
Homotoping around a cone point of order 5.


Definition of homotopy I: Elementary moves
A orbifold-homotopy between two curves on an orbifold is must respect the extra structure.
If a disk contains a cone point, and a loop "wants" to jump over the point (with endpoints fixed) then it must cross a few times a "fundamental domain" for the local action.
Then the projection of the loop will wrap around the cone point.
The blue arc and the red arc are homotopic rel. the endpoints with a homotopy that can be lifted.


Definition of homotopy I: Elementary moves
Consider two closed curves that coincide except for an arc.
Suppose that these two arcs lift to another pair of arcs in a disk (where a finite group acts), also with the same endpoints.
The projection of a homotopy fixing the endpoints between the two curves in the disk is an elementary move between the two arcs on the orbifold.
An orbifold homotopy between two curves in an orbifold is a composition of elementary moves.

Definition of homotopy I: Elementary moves

There is not a homotopy rel. to endpoints between left and right arcs in the orbifold.

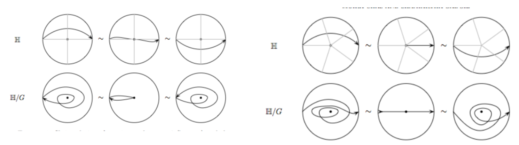


- Consider two closed curves that coincide except for an arc.
- Suppose that these two arcs lift to another pair of arcs in a disk (where a finite group acts), also with the same endpoints.
- The projection of a homotopy fixing the endpoints between the two curves in the disk is an ***elementary move*** between the two arcs on the orbifold.

An ***orbifold homotopy*** between two curves in an orbifold is a composition of elementary moves.

Definition of homotopy II: Combinatorial elementary moves

Cone point of order 4 | Cone point of order 5

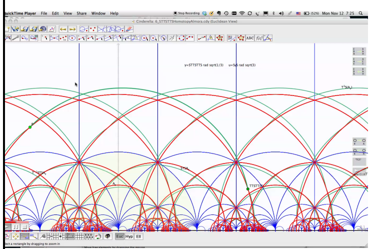


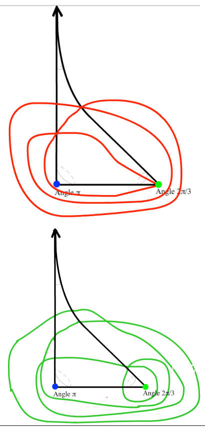
A loop with no crossings passing through a cone point of order n even can be homotoped to a loop going n/2 times around the cone point.

A path with no crossings joining opposite ends of a disk and pasing through a cone point of order odd can be homotoped to a loop going (n-1)/2 times around the cone point.

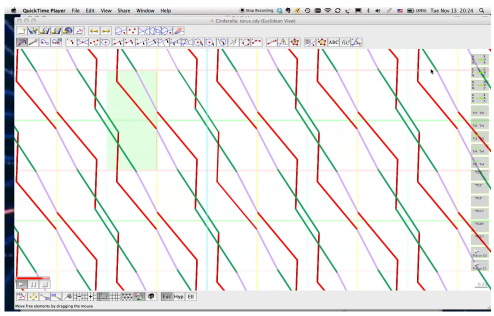
A H/G-homotopy between two … composition of finitely many elementary moves and homotopies in the complement of the cone points.

Lifting of a homotopy of a close curve … surface with a cone point of order 3 o… the modular





Homotoping a closed curve on the pillowcase.



Recall that in a surface

π_0 = free homotopy classes of curves on a surface ⟷ one to one ⟷ conjugacy classes of elements of π_1(surface).

The orbifold-fundamental group is the group of deck transformations of the universal cover of the orbifold.
It is also the set of based loops up to based orbifold-homotopy.

Orbifold-homotopy classes of closed directed curves on an orbifold. ⟷ one to one ⟷ Conjugacy classes of the orbifold fundamental group.

In particular, if G is a discrete group of orientation preserving isometries of the hyperbolic plane H,

H/G homotopy classes of closed curves in H/G ⟷ one to one ⟷ Conjugacy classes of G

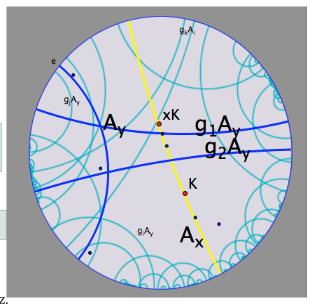
Scott's combinatorial description of the intersection of geodesics in an orbifold

- g : [K,xK) ∩ gAy ≠ ø
- # {g<y> : [K,xK) ∩ gAy } ≠ ø
- # {<x>g<y> : Ax ∩ gAy } ≠ ø

Intersection points of p(x) and p(y)

One to one

{<x> g <y> : Ax ∩ gAy ≠ ø }



[K, xK) interval of Ax
<z>=cyclic group generated by z.

The Goldman Bracket on Fuchsian Groups, String Topology and Three Manifolds

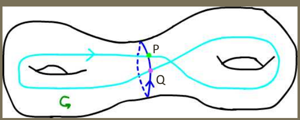
Orbifolds and the Goldman bracket

Moira Chas- Stony Brook University
Groups, Geometry and Dynamics
Almora, India
December 2012

Outline

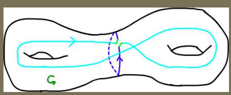
- Definition of the bracket.
- Combinatorial definition of the bracket on an orbifold H/G, where G is a discrete subgroup of orientation preserving isometries of the hyperbolic plane.
- The bracket counts intersection.

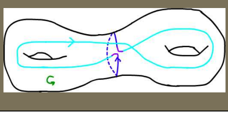
A, B in π_0 of an orbifold. Chose representatives in general position.



[[A,B] = A $._P$ B - A $._Q$ B where A$._R$ B means the orbifold-homotopy class of the curve that starts at P, goes around A, and then around B.

[A,B]=

+ 

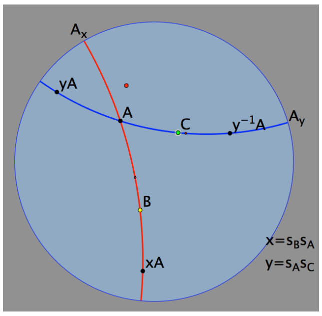
- 

The product of two hyperbolic elements whose axes intersect is hyperbolic

C is the midpoint of A and y^{-1} A

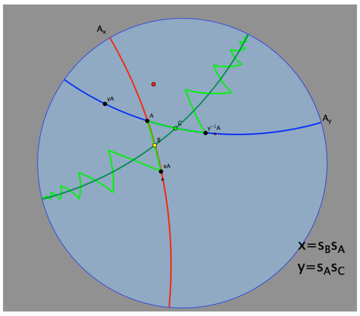
B is the midpoint of A and xA

s_P rotation of angle π around P.



x y = s_B s_C

The zigzag of a product



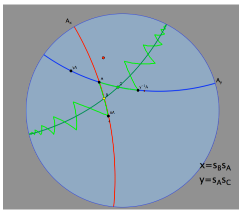
Combinatorial definition of the bracket

$$[x,y] = \sum_{<x>g<y>, A_x \cap gA_y \neq \emptyset} sign(x, gyg^{-1}) < xgyg^{-1} >$$

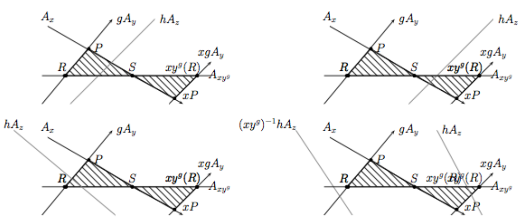
Intersection points of p(x) and p(y)

One to one

{<x> g <y> : Ax ∩ gAy ≠ ø }



The bracket satisfies Jacobi



Two terms of the bracket $[x^p,y^q]$ with different sign do not cancel if p or q are large enough.

- Consider two terms $\pm <x^p y^q>$ and $\pm <x^p b\, y^q b^{-1}>$ where b is in G.
- If $<x^p y^q> = <x^p b\, y^q b^{-1}>$ then there exists a in G such that $x^p y^q = a\, x^p b\, y^q b^{-1} a^{-1}$.
- Thus $x^p y^q$, $a\, x^p b\, y^q b^{-1} a^{-1}$, the (x^p,y^q)-zigzag curve and the $(a\, x^p a^{-1}, aby^q b^{-1} a^{-1})$-zig-zag curve have the same endpoints.
- The axis of $x^p y^q$ is at bounded distance of the (x^p,y^q)-zigzag curve.
- The axis of $a\, x^p b\, y^q b^{-1} a^{-1}$ is at bounded distance of the $(a\, x^p a^{-1}, aby^q b^{-1} a^{-1})$-zigzag curve.
- Then the (x^p,y^q)-zigzag curve and the $(a\, x^p a^{-1}, aby^q b^{-1} a^{-1})$-zigzag are at bounded distance from each other.
- The bound of the distance depends only on the translation length of x, the translation length of y and G. It does not depend on p or q.

Example: [A,B]

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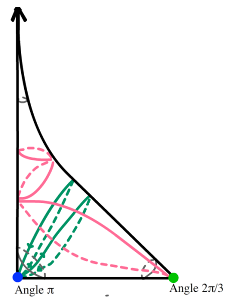
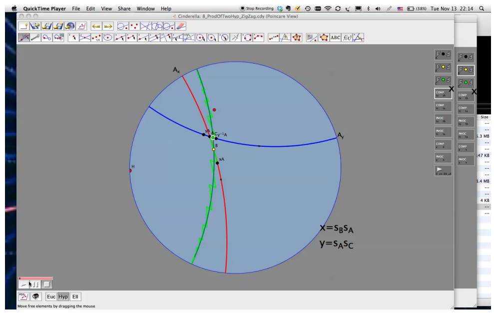
B=conjugay class of b(z)=(z+1)/(z+2)



Two terms of the bracket $[x^p,y^q]$ with different sign do not cancel if p or q are large enough.



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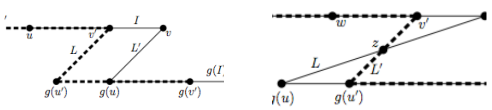

- Elementary hyperbolic geometry implies that these two pictures are not possible.
- Then the zigzag curves coincide.
- This implies that the two terms have the same sign.
- Therefore, they do not cancel.

The bracket counts intersection and self-intersection.

Mutual Intersection Theorem (Gadgil – C.) For x and y non-conjugate hyperbolic elements in a finitely generated discrete subgroup of isometries of the hyperbolic plane

$$M\ [\langle x^p\rangle, \langle y^q\rangle] = p.q\ i(<x>,<y>),$$

where we only consider p and q so that the ratio of the translation length of x to the translation length of y is not q/p.

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for all but finitely many values of p and q such that $p \neq q$.

The Goldman Bracket on Fuchsian Groups, String Topology and Three Manifolds

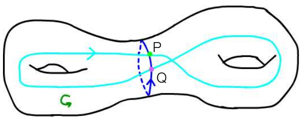
Orbifolds and the Goldman bracket

Moira Chas- Stony Brook University
Groups, Geometry and Dynamics
Almora, India
December 2012

Outline

- Definition of the bracket.
- Combinatorial definition of the bracket on an orbifold H/G, where G is a discrete subgroup of orientation preserving isometries of the hyperbolic plane.
- The bracket counts intersection.

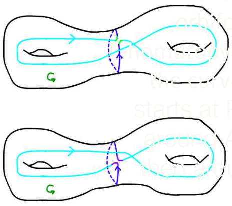
A, B in π_0 of an orbifold. Chose representatives in general position.



[[A,B] = A $\cdot_P$ B - A $\cdot_Q$ B where A$\cdot_R$ B means the orbifold-homotopy class of the curve that starts at P, goes around A, and then around B.

[A,B]=

+



-

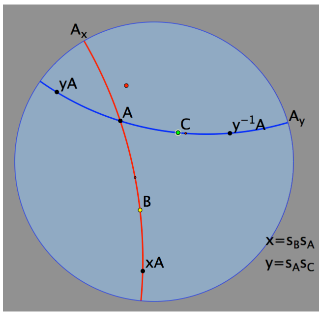
The product of two hyperbolic elements whose axes intersect is hyperbolic

C is the midpoint of A and y^{-1} A

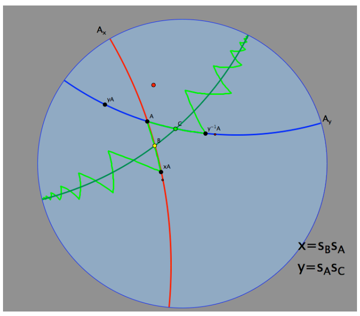
B is the midpoint of A and xA

s_P rotation of angle π around P.

x y = s_B s_C



The zigzag of a product



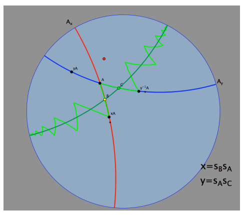
Combinatorial definition of the bracket

$$[x,y] = \sum_{<x>g<y>, A_x \cap gA_y \neq \emptyset} sign(x, gyg^{-1}) < xgyg^{-1} >$$

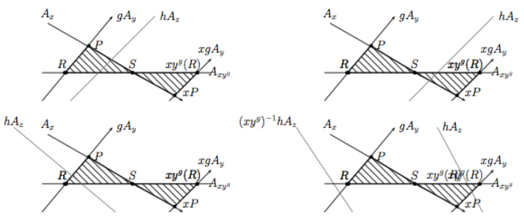
Intersection points of p(x) and p(y)

One to one

{<x> g <y> : Ax ∩ gAy ≠ ø }



The bracket satisfies Jacobi



Two terms of the bracket $[x^p,y^q]$ with different sign do not cancel if p or q are large enough.

- Consider two terms $\pm \langle x^py^q \rangle$ and $\pm \langle x^pb\, y^q b^{-1} \rangle$ where b is in G.
- If $\langle x^py^q \rangle = \langle x^pb\, y^q b^{-1} \rangle$ then there exists a in G such that $x^py^q = a\, x^pb\, y^q b^{-1} a^{-1}$.
- Thus x^py^q, $a\, x^pb\, y^q b^{-1} a^{-1}$, the (x^p,y^q)-zigzag curve and the $(a\, x^p a^{-1}, aby^q b^{-1} a^{-1})$-zig-zag curve have the same endpoints.
- The axis of x^py^q is at bounded distance of the (x^p,y^q)-zigzag curve.
- The axis of $a\, x^pb\, y^q b^{-1} a^{-1}$ is at bounded distance of the $(a\, x^p a^{-1}, aby^q b^{-1} a^{-1})$-zigzag curve.
- Then the (x^p,y^q)-zigzag curve and the $(a\, x^p a^{-1}, aby^q b^{-1} a^{-1})$-zigzag are at bounded distance from each other.
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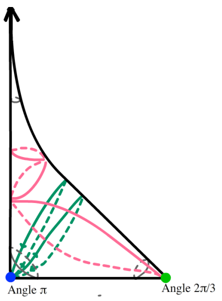

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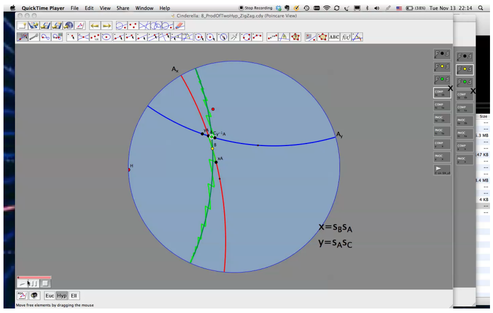

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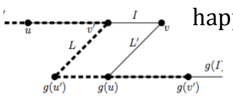

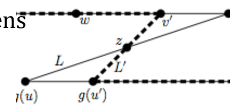

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Mutual Intersection Theorem (Gadgil – C.) For x and y non-conjugate hyperbolic elements in a finitely generated discrete subgroup of isometries of the hyperbolic plane

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The Goldman Bracket on Fuchsian Groups, String Topology and Three Manifold

Moira Chas- Stony Brook University
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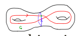
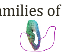
Almora, India
December 2012

The String Bracket and Three Manifolds

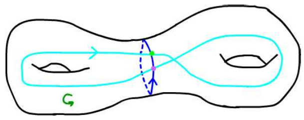
S = orientable surface (or orbifold)	M^3 = compact, orientable, irreducible, with contractible universal cover.
Goldman Bracket:  Lie Bracket on (linear combination of) closed, oriented free homotopy classes of curves.	String bracket: Lie bracket on (linear combination of) families of oriented closed curves. 
Combinatorial presentation	
The bracket encodes the intersection structure in terms of the Manhattan norm.	
Different surfaces have different Goldman Lie algebras	String bracket gives the H-S graph of the graph of groups in the celebrated torus decomposition

Outline

The String Bracket of on a Three Manifold and Geometrization

- The free loop space of a manifold M, LM and the equivariant homology groups of LM.
- The 0-th and first equivariant homology groups of LM, H_0 and H_1.
- The String bracket on H_0 and H_1
- The classification of three manifolds.
- The String bracket counts intersection between tori and curves.
- The string bracket detects data of irreducible three manifolds.

The Goldman bracket

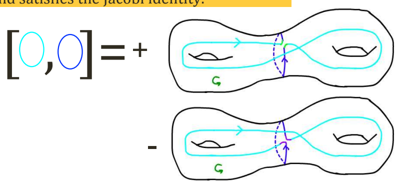


S an orientable surface

π_0 denotes the set of free homotopy classes of closed oriented curves on S.
NOTE: $\pi_0 = \pi_0$(free loop space of the surface)

Theorem: (Goldman, 1986)
The bracket is well defined, skew symmetric and satisfies the Jacobi identity.

$[\ ,\ ]: Z[\pi_0] \otimes Z[\pi_0] \to Z[\pi_0]$



S^1-equivariant homology of a space X with a S^1-action.

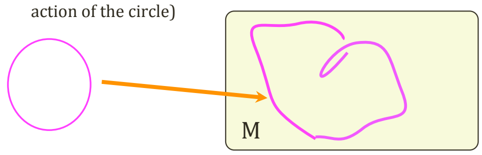
- Consider a space X with a circle action.
- If the action is free, **the equivariant homology of X (equivariant with respect to the circle action)** is the homology of X/circle.
- If the action is not free, we "make it" free by crossing X with a contractible space Y with a free circle action. The equivariant homology of X (equivariant with respect to the circle action is the homology of (X x Y) /(diagonal circle action).

LM = space of maps from the circle to M.

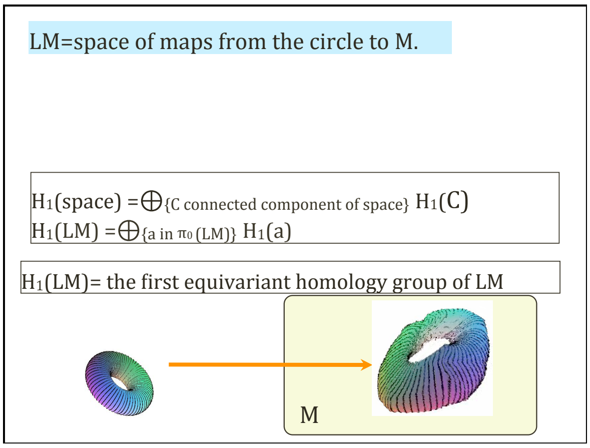
M is a 3 manifold

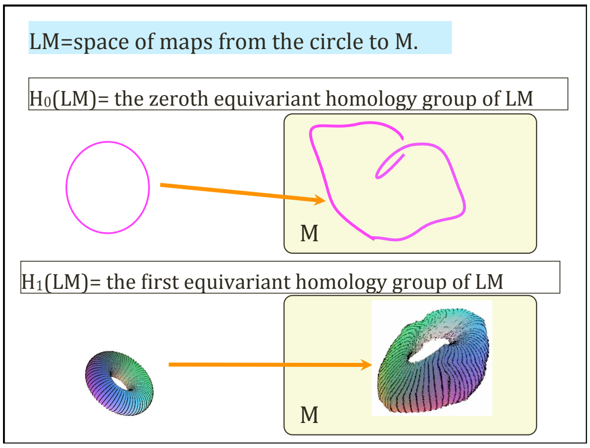
H_0(LM)= the zeroth equivariant homology group of LM

(equivariant with respect to the circle action of the circle)



Recall that $H_0(\text{space}) = \bigoplus_{\{C \text{ is a connected component of space}\}} Z.C$
Since the circle is connected,
$H_0(LM) = \bigoplus_{\{a \text{ in } \pi_0(M)\}} Z.a$


LM=space of maps from the circle to M.
H_1(space) =$\bigoplus_{\{C \text{ connected component of space}\}}$ $H_1(C)$
$H_1(LM)$ =$\bigoplus_{\{a \text{ in } \pi_0(LM)\}}$ $H_1(a)$
$H_1(LM)$= the first equivariant homology group of LM
M


LM=space of maps from the circle to M.
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M
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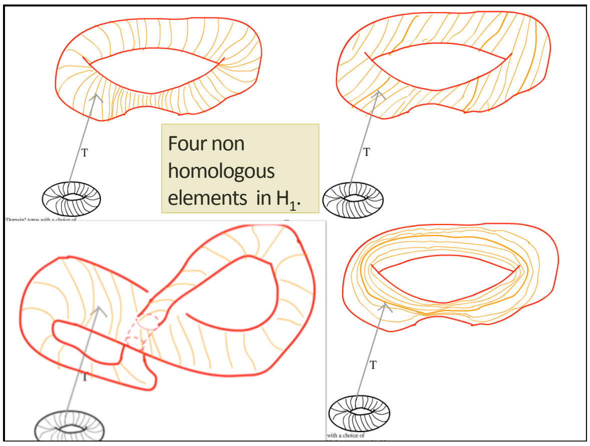
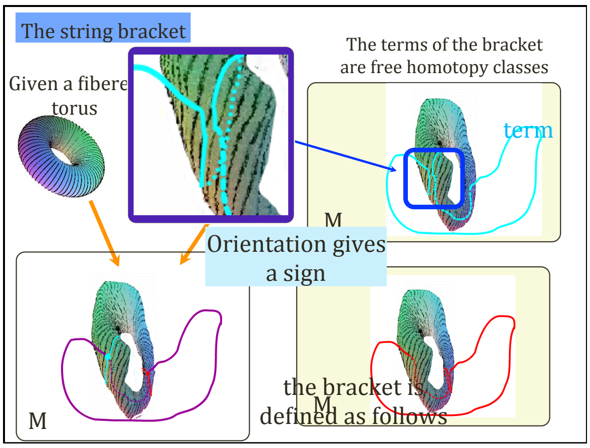
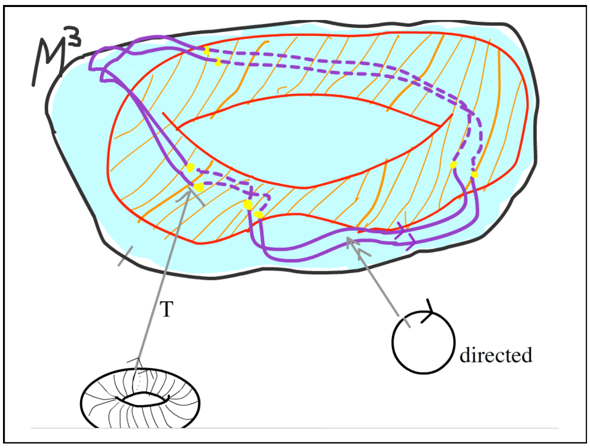
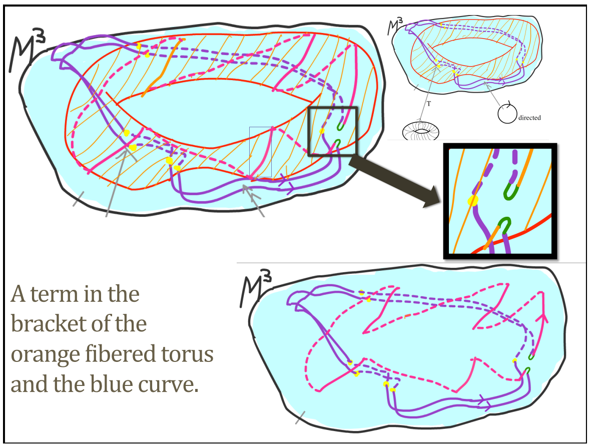
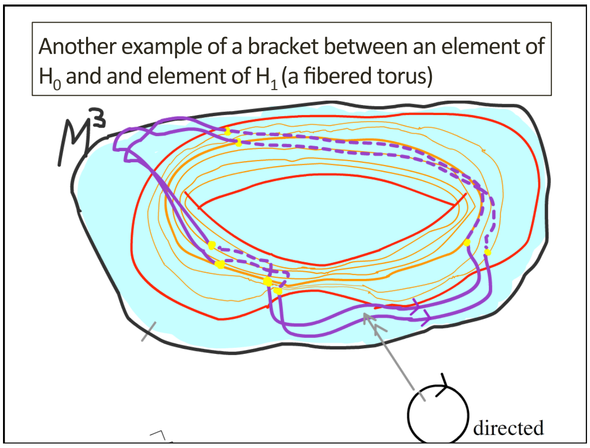
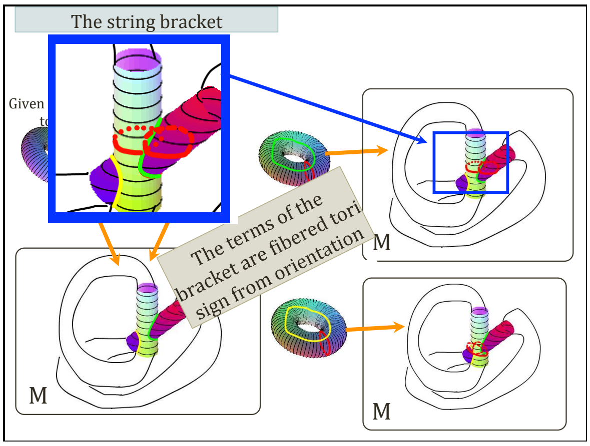

T
Four non homologous elements in H_1.
T
T

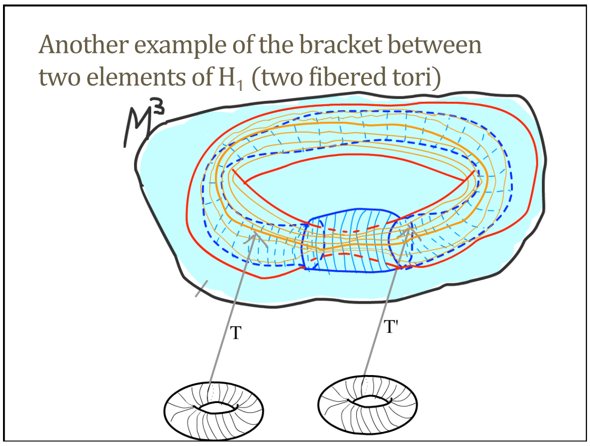

The string bracket
Given a fibered torus
The terms of the bracket are free homotopy classes
term
M
Orientation gives a sign
M
the bracket is defined as follows
M

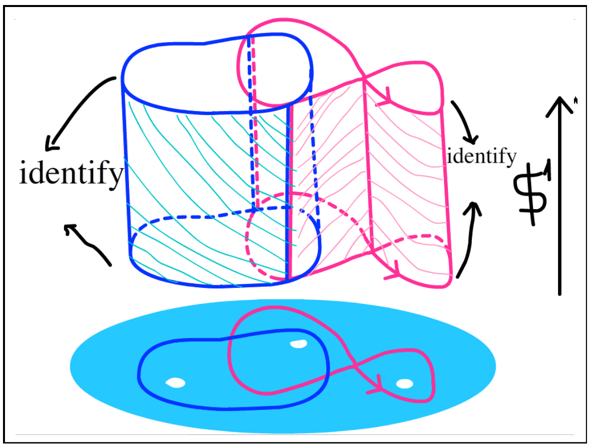

M^3
T
directed



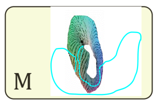




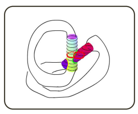




Theorem (Sullivan, C, 1999)



- $H_0 \otimes H_1 \to H_0$ is a Lie module.
- $H_1 \otimes H_1 \to H_1$ is a Lie algebra.



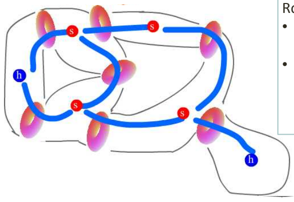
- The string bracket is well defined
- The string bracket satisfies Jacobi
- *(In general, $H \otimes H \to H$ is a Lie algebra of degree 2-d, d is the dimension of the manifold. When d=2, we get the Goldman bracket $H_0 \otimes H_0 \to H_0$).*

The classification of three manifolds
Step 1:

- ***Theorem (Alexander).*** *An embedded 2-sphere in a 3-sphere cuts the 3-sphere into two closed piecewise-linear 3-cells.*
- A closed 3-manifold is ***irreducible*** if every 2-sphere bounds a 3-ball.
- A closed 3-manifold is ***prime*** if every separating embedded 2-sphere bounds a 3-ball.
- ***Theorem (Kneser and Milnor)****. Any 3-manifold has a unique connected sum decomposition into prime 3-manifolds (here, uniqueness means that the list of summands is unique up to order).*
- From now on, we will consider irreducible manifolds (with infinite fundamental group and contractible universal cover)

The classification of three manifolds
Step 2:

JSJ Theorem: There exists a minimal collection of π_1 injective tori T (not pairwise isotopic) such that each component of M\T is either Seifert or hyperbolic.



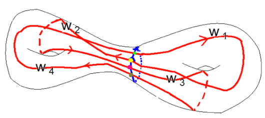
Roughly speaking,
- A ***Seifert manifold*** is "almost" a product Surface x Circle.
- A manifold is ***hyperbolic*** if every map from the torus to the manifold is not π_1 – injective or can be isotoped to the boundary.

M orientable, compact, irreducible 3-manifold.

Recall that in a surface, if X has an embedded representative then the M[X,Y]= i(X,Y) and

$\pm[\, w_1w_2w_3w_4, X\,]=$

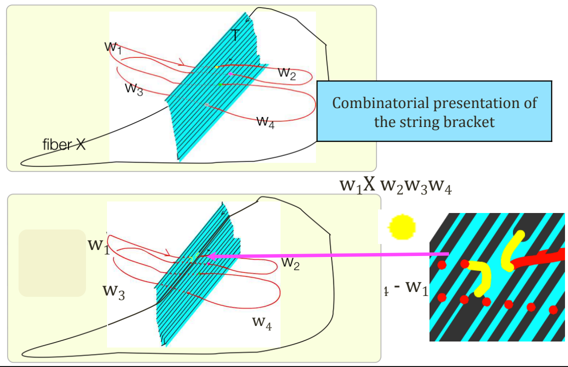
$w_1Xw_2w_3w_4 - w_1w_2Xw_3w_4 + w_1w_2w_3\, X\, w_4 - w_1w_2w_3w_4\, X$



If T is fibered torus and W is a free homotopy class W then M[T,W] ≤ i(T,W)

Does M[T, W]= i(T,W) hold, possibly assuming T embedded?

IT is a separating, embedded fibered torus with fiber X



If T is an embedded fibered torus and W is a free homotopy class, is M[T, W] =i(T,W)?

Not always

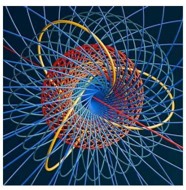


Image by Jos Leys

Recall: A Seifert fibered manifold is a manifold that is a disjoint union of circles organized in a particular way:

Except for finitely many, the circles are freely homotopic to a curve h.

The center of the fundamental group of a Seifert manifold is typically generated by h.

Suppose that T is a fibered torus in a Seifert manifold and the fiber of T is h.

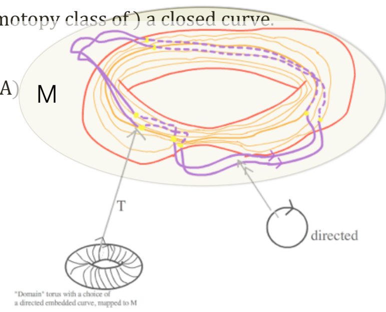
$[w_1w_2w_3w_4, <T,h>]$
$= w_2w_3w_4w_1h - w_3w_4w_1w_2h + w_4w_1w_2w_3h - w_1w_2w_3w_4h$
$= 0$

Theorem (Gadgil, C) Let T be (the homology class corresponding to) an embedded fibered torus in an irreducible manifold whose fiber is not the generic fiber of a Seifert piece.

Let A be (free homotopy class of) a closed curve.

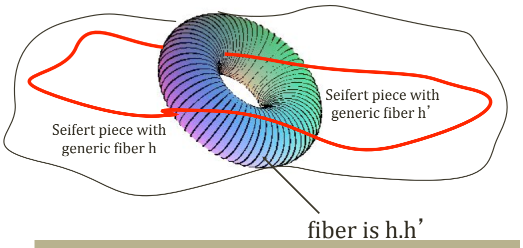
Then

$M[\,T, A^2] = 2\, i(T, A)$



Why A^2?

Why A^2?



$[w_1 w_2, \langle T, h.h' \rangle] = w_1 w_2\, h.h' - w_1 h.h'\, w_2 = 0$

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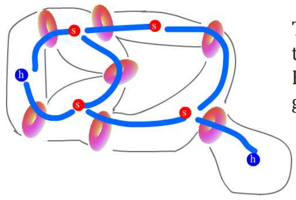
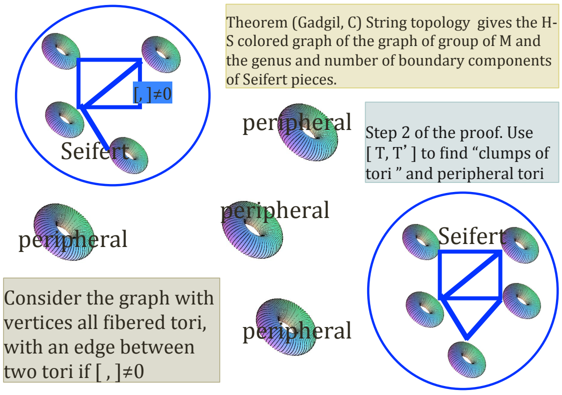


Thm (Gadgil, C) String topology determines the H-S colored graph of the graph of groups of M.

Theorem (Gadgil, C) String topology gives the H-S colored graph of the graph of group of M and the genus and number of boundary components of Seifert pieces.

Step 1 of the proof. Use the "p-q"-intersetion Theorem to determine whether a torus is embedded or not.

Theorem (Gadgil, C) String topology gives the H-S colored graph of the graph of group of M and the genus and number of boundary components of Seifert pieces.



Step 2 of the proof. Use [T, T'] to find "clumps of tori " and peripheral tori

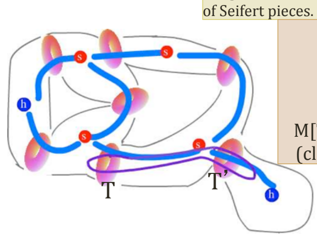
Consider the graph with vertices all fibered tori, with an edge between two tori if [,]≠0

Theorem (Gadgil, C) String topology gives the H-S colored graph of the graph of group of M and the genus and number of boundary components of Seifert pieces.

Step 3 of the proof. Say two fibered tori T, T' are equivalent if $M[T,A^2]=M[T',A^2]$ for all (classes of) curves A.

Thus (for most tori) T and T' are equivalent if and only if they are the same torus, with different fiber.

Theorem (Gadgil, C) String topology gives the H-S colored graph of the graph of group of M and the genus and number of boundary components of Seifert pieces.



$M[\,T, 0^2\,] \neq 0$

$M[\,T', 0^2\,] \neq 0$

$M[T'', 0^2\,] = 0$ for all other (classes of) peripheral tori

Step 3. Use $M[\,T, A^2] = 2\, i(T, A)$ to "reconstruct' the graph and Seifert pieces genus and number of boundary components.