Arithmetic of adjoint L-values.

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Abstract: We now relate $|C_0(\lambda; W)|$ to the corresponding adjoint L-value.

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§1. Set up.

• Fix an an algebra homomorphism $\lambda : h_{k,\chi/\mathbb{Z}[\chi]} \to \overline{\mathbb{Q}}_p$ and write the associated Galois representation ρ_λ : Gal$(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(W)$ and a new form $f = f_\lambda \in S_{k+1,\chi}$ of conductor C; so, $N = C$.

• Writing the Hecke polynomial at a prime ℓ as

$$1 - \lambda(T(\ell))X + \chi(\ell)\ell^k X^2 = (1 - \alpha_\ell X)(1 - \beta_\ell X),$$

we have the following Euler product convergent absolutely if Re$(s) > 1$:

$$L(s, Ad(\lambda)) = \prod_\ell \left\{(1 - \frac{\alpha_\ell}{\beta_\ell}\ell^{-s})(1 - \ell^{-s})(1 - \frac{\beta_\ell}{\alpha_\ell}\ell^{-s})\right\}^{-1}.$$

• It has an analytic continuation to the whole complex s-plane (due to Shimura 1975 and Gelbart–Jacquet in 1978) and satisfies a functional equation of the form $1 \leftrightarrow 1 - s$ whose Γ-factor is

$$\Gamma(s, Ad(\lambda)) = \Gamma_{\mathbb{C}}(s + k)\Gamma_{\mathbb{R}}(s + 1),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$.

§2. Inner product formula.

Theorem 1 (Shimura). *Let χ be a primitive character modulo C. Let $\lambda : h_k(\Gamma_0(C), \chi; \mathbb{Z}[\chi]) \to \mathbb{C}$ be a $\mathbb{Z}[\chi]$-algebra homomorphism for $k \geq 1$. Then*

$$\Gamma(s, Ad(\lambda))L(s, Ad(\lambda))$$

has an analytic continuation to the whole complex s-plane and

$$\Gamma(1, Ad(\lambda))L(1, Ad(\lambda)) = 2^{k+1}C^{-1}\int_{\Gamma_0(C)\backslash\mathfrak{H}} |f|^2 y^{k-1} dxdy,$$

where $f = \sum_{n=1}^{\infty} \lambda(T(n))q^n \in S_{k+1}(\Gamma_0(C), \chi)$ and $z = x + iy \in \mathfrak{H}$. If $C = 1$, we have the following functional equation:

$$\Gamma(s, Ad(\lambda))L(s, Ad(\lambda)) = \Gamma(1-s, Ad(\lambda))L(1-s, Ad(\lambda)).$$

§3. Relation to a Rankin product.

We consider $L(s-k,\rho_\lambda\otimes\widetilde{\rho}_\lambda)$ for the Galois representation ρ_λ associated to λ and its contragredient $\widetilde{\rho}_]lm = {}^t\rho_\lambda$. Since $\rho_\lambda\otimes\widetilde{\rho}_\lambda = \mathbf{1}\oplus Ad(\rho_\lambda)$, we have

$$L(s,\rho_\lambda\otimes\widetilde{\rho}_\lambda) = L(s,Ad(\lambda))\zeta(s) \qquad (1)$$

for the Riemann zeta function $\zeta(s)$. Then, the Rankin-convolution method tells us [LFE, Theorem 9.4.1] that

$$\left(2^{2-s}\prod_{p|C}(1-\frac{1}{p^{s-k}})\right)\Gamma_{\mathbb{C}}(s)L(s-k,\rho_\lambda\otimes\widetilde{\rho}_\lambda) = \int_{\Gamma_0(C)\backslash\mathfrak{H}}|f|^2E'_{0,C}(s-k,\mathbf{1})y^{-2}dxdy,$$

where $E'_0(s,\mathbf{1})$ is an Eisenstein series of level C for the trivial character $\mathbf{1}$ defined in [LFE, page 297].

This expression gives a meromorphic continuation of $L(s,\rho_\lambda\otimes\widetilde{\rho}_\lambda)$ and $L(s,Ad(\lambda))$, as $\zeta(s)$ is shown meromophic from Riemann's time.

§4. Proof of Inner product formula.

The Eisenstein series has a simple pole at $s = 1$ with constant residue: $\pi \prod_{p|C}(1 - \frac{1}{p})$, which yields

$$\left(\left(2^{1-k}\prod_{p|C}(1-\frac{1}{p})\right)\right) \times \mathrm{Res}_{s=k+1}\Gamma_{\mathbb{C}}(s)L(s-k,\rho_\lambda\otimes\widetilde{\rho}_\lambda) = \pi\prod_{p|C}(1-\frac{1}{p})\int_{\Gamma_0(C)\backslash\mathfrak{H}}|f|^2y^{-2}dxdy.$$

This combined with (1) yields the residue formula and analytic continuation of $L(s, Ad(\lambda))$ over the region of $Re(s) \geq 1$.

Since $\Gamma_{\mathbb{C}}(s)E'_{0,C}(s,\mathbf{1})$ satisfies a functional equation of the form $s \mapsto 1 - s$ (see [LFE, Theorem 9.3.1])), we have the continuation of

$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k)L(s-k,\rho_\lambda\otimes\widetilde{\rho}_\lambda)$$

and functional equation. In additin, we get holomorphy of $L(s, Ad(\rho_\lambda))$ around $s = 1$.

§5. Eichler-Shimura isomorphism.

Consider the inclusion $I : SL_2(\mathbb{Z}) \hookrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^2) = GL_2(\mathbb{C})$. Let us take the nth symmetric tensor representation $I^{sym\otimes n}$ whose module twisted by the action of χ, we write as $L(n, \chi; \mathbb{C})$. Recall the Eichler-Shimura isomorphism,

$$\delta : S_{k+1}(\Gamma_0(C), \chi) \oplus \overline{S}_{k+1}(\Gamma_0(C), \chi) \cong H^1_{cusp}(\Gamma_0(C), L(n, \chi; \mathbb{C})),$$

where $k = n + 1$, $\overline{S}_{k+1}(\Gamma_0(C), \chi)$ is the space of anti-holomorphic cusp forms of weight $k + 1$ of "Neben" type character χ, and

$$H^1_{cusp}(\Gamma_0(C), L(n, \chi; \mathbb{C})) \subset H^1(\Gamma_0(C), L(n, \chi; \mathbb{C}))$$

is the image of compactly supported cohomology in the ordinary cohomology group.

Identify $L(n, \chi; A) = AX^n + AX^{n-1}Y + \cdots + AY^j$ (the space of homogeneous polynomials) and let $\alpha \in SL_2(\mathbb{Z})$ act by $(X, Y) \mapsto \chi(d)(X, Y)^t\alpha^\iota$ with $\alpha^\iota\alpha = \det(\alpha)$ for the lower right cornar entry d of α.

§6. Eichler-Shimura as de Rham map.

The Eichler-Shimura map δ is specified in [LFE] as follows: We put

$$\omega(f) = \begin{cases} f(z)(X - zY)^n dz & \text{if } f \in S_k(\Gamma_0(C), \chi), \\ f(z)(X - \overline{z}Y)^n d\overline{z} & \text{if } f \in \overline{S}_k(\Gamma_0(C), \chi). \end{cases}$$

Then we associate to f the de Rham cohomology class of $[\omega(f)]$ which gives

$$S_{k+1}(\Gamma_0(C), \chi) \oplus \overline{S}_{k+1}(\Gamma_0(C), \chi) \\ \cong H^1_{cusp,dR}(X_0(C), \mathcal{L}(n, \chi; \mathbb{C})),$$

where $\mathcal{L}(n, \chi; \mathbb{C})$ is the C^∞-sheaf associated o $L(n, \chi; \mathbb{C})$. Then we associate to f the cohomology class of the 1-cocycle $\gamma \mapsto \int_z^{\gamma(z)} \omega(f)$ of $\Gamma_0(C)$ for a fixed point z on the upper half complex plane. The map δ does not depend on the choice of z.

§7. Poincaré duality.

Define $[\ ,\ ] : L(n,\chi;A) \times L(n,\overline{\chi};A) \to A$ by

$$[\sum_j a_j X^{n-j}Y^j, \sum_j b_j X^{n-j}Y^j] = \sum_{j=0}^{n} (-1)^j \binom{n}{j}^{-1} a_j b_{n-j}.$$

By definition, $[(X-zY)^n, (X-\overline{z}Y)^n] = (z-\overline{z})^n$. It is an easy exercise to check that $[\gamma P, \gamma Q] = \det \gamma^n [P,Q]$ for $\gamma \in GL_2(A)$. Thus we have a Γ-homomorphism $L(n,\chi;A) \otimes_A L(n,\chi^{-1};A) \to A$, and we get the cup product pairing for $Y = \Gamma_0(N)\backslash\mathfrak{H}$

$$[\ ,\ ] : H^1_c(Y, \mathcal{L}(n,\chi;A)) \times H^1(Y, \mathcal{L}(n,\chi^{-1};A)) \longrightarrow H^2_c(Y,A) \cong A.$$

This pairing induces the cuspidal pairing

$$[\ ,\ ] : H^1_{cusp}(Y, \mathcal{L}(n,\chi;A)) \times H^1_{cusp}(Y, \mathcal{L}(n,\chi^{-1};A)) \longrightarrow A.$$

This pairing is perfect on the ordinary part (or if $n! \in A^\times$).

§8. Hecke equivariance of the pairing.

The action of $\tau = \begin{pmatrix} 0 & -1 \\ C & 0 \end{pmatrix}$ defines a quasi-involution on the cohomology

$$\tau : H^1_{cusp}(\Gamma_0(C), L(n,\chi;A)) \\ \to H^1_{cusp}(\Gamma_0(C), L(n,\chi^{-1};A)),$$

which is given by $u \mapsto \{\gamma \mapsto \tau u(\tau\gamma\tau^{-1})\}$ for each homogeneous 1-cocycle u. The cocycle $u|\tau$ has values in $L(n,\chi^{-1};A)$ because conjugation by τ interchanges the diagonal entries of γ. We have $\tau^2 = (-C)^n$ and $[x|\tau, y] = [x, y|\tau]$. Then we modify the above pairing [,] by τ and define $\langle x, y\rangle = [x, y|\tau]$.

The operator $T(n)$ is symmetric with respect to this pairing:

$$\langle x|T(n), y\rangle = \langle x, y|T(n)\rangle. \qquad (2)$$

§9. Periods.

We now regard $\lambda : h_{k,\chi/\mathbb{Z}[\chi]} \to \mathbb{C}$ as actually having values in $W \cap \overline{\mathbb{Q}}$ (via the fixed embedding: $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). Put $A = W \cap \mathbb{Q}(\lambda)$, and A is a valuation ring of $\mathbb{Q}(\lambda)$ of residual characteristic p. The Frobenius c at ∞ acts on the cohomology, and the corresponding $\pm$-eigenspaces are indicated by adding $[\pm]$.

For the image $L_\pm$ of $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))[\pm]$ in $H^1_{cusp}(\Gamma_0(C), L(n,\chi;\mathbb{Q}(\lambda)))[\pm]$,

$$H^1_{cusp}(\Gamma_0(C), L(n,\chi;\mathbb{Q}(\lambda)))[\lambda,\pm] \cap L_\pm = A\xi_\pm$$

for a generator $\xi_\pm$. Hereafter, we write L for $L[+]$.

For the normalized eigenform $f \in S_\kappa(\Gamma_0(C),\chi)$ with $T(n)f = \lambda(T(n))f$, we define $\Omega(\pm,\lambda;A) \in \mathbb{C}^\times$ by

$$\delta(f) \pm c(\delta(f)) = \Omega(\pm,\lambda;A)\xi_\pm.$$

§10. Rationality.

Theorem 2. *Let χ be a character of conductor C. Let $\lambda : h_k(\Gamma_0(C),\chi;\mathbb{Z}[\chi]) \to \overline{\mathbb{Q}}$ ($k \geq 1$) be a $\mathbb{Z}[\chi]$-algebra homomorphism. Then for a valuation ring A of $\mathbb{Q}(\lambda)$, we have, up to sign,*

$$\frac{i^k W(\lambda) C^{(k+1)/2}\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)} = \langle \xi_+, \xi_- \rangle \in \mathbb{Q}(\lambda).$$

If the residual characteristic of A is prime to $n!$($\Leftrightarrow$ $p > n$) or f is ordinary, then $\langle \xi_+, \xi_- \rangle \in A$.

We note

$$\langle \Omega(+,\lambda;A)\xi_+, \Omega(-,\lambda;A)\xi_- \rangle = \Omega(+,\lambda;A)\Omega(-,\lambda;A)\langle \xi_+, \xi_- \rangle$$

and $\delta(f)|\tau = W(\lambda)(-1)^n C^{(n/2)}\delta(f_c)$, where $f_c = \sum_{m=1}^{\infty} \overline{\lambda(T(m))} q^m$ and $f|\tau = W(\lambda) f_c$ for and $W(\lambda) \in \mathbb{C}$ with $|W(\lambda)| = 1$.

§**11. Proof.** By definition, we have

$$2\Omega(+,\lambda;A)\Omega(-,\lambda;A)\langle\xi_+,\xi_-\rangle = [\delta(f)+c\delta(f),(\delta(f)-c\delta(f))|\tau],$$

which is equal to, up to sign,

$$\begin{aligned}4i\int_{Y_0(C)}&[\delta(f)|\tau,c\delta(f)]dx\wedge dy\\ &= 2^{k+1}i^kW(\lambda)C^{((k-1)/2}\int_{Y_0(C)}|f_c|^2y^{k-1}dxdy\\ &= 2^{k+1}i^kW(\lambda)C^{(k-1)/2}\int_{Y_0(C)}|f|^2y^{k-1}dxdy\\ &= i^kW(\lambda)C^{(k+1)/2}\Gamma(1,Ad(\lambda))L(1,Ad(\lambda)),\end{aligned}$$

where $Y_0(C)=\Gamma_0(C)\backslash\mathfrak{H}$. $\square$

Let $\mathbb{T}=\mathbb{T}_k$ be the local ring of $h_{k,\chi/W}$ through which λ factor through. Let $1_{\mathbb{T}}$ be the idempotent of $\mathbb{T}$. Since the conductor of χ coincides with C, $h_{k,\chi/W}$ is reduced. Thus for the quotient field K of W, the unique local ring $\mathbb{I}_K$ of $h_{k,\chi/K}$ through which λ factors is isomorphic to K. Let 1_λ be the idempotent of $\mathbb{I}_K$ in $h_{k,\chi/K}$.

§12. The adjoint L-value as an index.

Corollary 1. *Let the assumption be as in* Theorem 2. *Let A be a valuation ring of residual characteristic $p > 3$. Suppose that $\langle\ ,\ \rangle$ induces a perfect duality on $L := 1_{\mathbb{T}} H^1_{cusp}(\Gamma_0(C), L(n,\chi;W))$ (for example if f is ordinary). Then*

$$\left|\frac{i^{k+1}W(\lambda)C^{k/2}\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)}\right|_p^{-r(W)} = |L^{\lambda}/L_{\lambda}|,$$

where $r(W) = \mathrm{rank}_{\mathbb{Z}_p} W$, $L^{\lambda} = 1_{\lambda} L$, and L_{λ} is given by the intersection

$$L \cap H^1_{cusp}(\Gamma_0(C), L(n,\chi;K))[\lambda,+].$$

§13. Proof of the corollary.

By our choice, ξ_+ is the generator of L_λ.

Similarly we define $M^\lambda = 1_\lambda M$ for

$$M := 1_{\mathbb{T}} H^1_{cusp}(\Gamma_0(C), L(n,\chi;W))[-],$$

and $M_\lambda = M^\lambda \cap M$ in $H^1_{cusp}(\Gamma_0(C), L(n,\chi;K))[-]$. Then ξ_- is a generator of M_λ.

Since the pairing is perfect, $L_\lambda \cong \mathrm{Hom}_W(M^\lambda, W)$ and $L^\lambda \cong \mathrm{Hom}_W(M_\lambda, W)$ under $\langle\ ,\ \rangle$. Then it is an easy exercise to see that

$$|\langle \xi_+, \xi_- \rangle|_p^{-1} = |L^\lambda / L_\lambda|.$$

$\square$

§14. Conclusion.

Decompose $\mathbb{T} \otimes_W K = K \oplus X$ as an algebra direct sum so that the projection to K is induced by λ.

If $L \cong \mathbb{T}$ as $\mathbb{T}$-modules, by definition, we have

$$\mathrm{Im}(\lambda) \cong L^{\lambda} \quad \text{and} \quad L_{\lambda} \cong (K \oplus 0) \cap \mathbb{T}.$$

Thus $L^{\lambda}/L_{\lambda} \cong C_0(\lambda; W)$, which implies the non-abelian class number formula.

The freeness of L over $\mathbb{T}$ is a by-product of the Taylor-Wiles argument (proving Taylor–Wiles theorem: $R = \mathbb{T}$ theorem) applied to cohomology groups

$$H^1_{cusp}(\Gamma_0(C \prod_{q \in Q} q), L(n, \chi; K))[+]$$

for a suitably chosen sets of finitely many promes Q outside Cp.