Arithmetic of adjoint L-values.

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Abstract: We describe $R = T$ theorems and its implication to the adjoint L-values.

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§1. Set up.

- Fix a field identification $\mathbb{C} \cong \mathbb{C}_p$ which induces $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$.
- A p-adic analytic family $\mathcal{F}$ of modular forms is defined with respect to i_p.
- We write $|\alpha|_p$ for the p-adic absolute value (with $|p|_p = 1/p$) induced by i_p.
- Take a Dirichlet character $\psi : (\mathbb{Z}/Np^{r+1}\mathbb{Z})^\times \to W^\times$ with ($p \nmid N, r \geq 0$), and consider the space of elliptic cusp forms $S_{k+1,\psi} := S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ of weight $k+1$ with character ψ.
- Let the ring $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values ψ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively.
- We assume that the $\psi_p = \psi|_{(\mathbb{Z}/p^{r+1}\mathbb{Z})^\times}$ has conductor p^{r+1} if non-trivial and $r = 0$ if trivial.
- Since we will consider only $U(p)$-eigenforms with p-adic unit eigenvalues (under $|\cdot|_p$), this does not pose any restriction.
- We assume that N is cube-free not to worry about nilpotence in the Hecke algebra.
- We write N_ψ for Np^{r+1} if confusion is unlikely.

§2. Finite level Hecke algebra.

The Hecke algebra over $\mathbb{Z}[\psi]$ is the subalgebra of the linear endomorphism algebra of $S_{k+1}(\Gamma_0(N_\psi),\psi)$ generated by Hecke operators $T(n)$:

$$\begin{aligned} h = h_{k,\psi} &= h_k(\Gamma_0(N_\psi),\psi;\mathbb{Z}[\psi]) \\ &= \mathbb{Z}[\psi][T(n)|n=1,2,\cdots] \\ &\qquad \subset \mathrm{End}(S_{k+1}(\Gamma_0(N_\psi),\psi)), \end{aligned}$$

where $T(n)$ is the Hecke operator for an integer $n > 0$. We put $h_{k,\psi/A} = h \otimes_{\mathbb{Z}[\psi]} A$ for any $\mathbb{Z}[\psi]$-algebra A. For a prime $l|N$, $T(l)$ is often written as $U(l)$.

Let $S_{k+1,\psi/A}$ be the space of A-integral cusp forms of weight $k+1$, of character ψ and of level N_ψ. If $A \subset \mathbb{C}$, we have

$$S_{k+1,\psi/A} = \{f = \sum_{n=1}^{\infty} a(n,f)q^n \in S_{k+1,\psi}|a(n,f) \in A\}$$

for $\sum_n a(n,f)q^n$ is the q-expansion at the ∞ cusp. For any ring A (not necessarily in $\mathbb{C}$), we have $S_{k+1,\psi/A} = S_{k+1,\psi/\mathbb{Z}[\psi]} \otimes_{\mathbb{Z}[\psi]} A$.

§3. Ordinary part.

- For p-profinite ring A, the ordinary part $\mathbf{h}_{k,\psi/A} \subset h_{k,\psi/A}$ is the maximal ring direct summand on which $U(p)$ is invertible.
- Writing e for the idempotent of $\mathbf{h}_{k,\psi/A}$, and hence $e = \lim_{n\to\infty} U(p)^{n!}$ under the p-profinite topology of $h_{k,\psi/A}$.
- We write the image of the idempotent as $S^{ord}_{k+1,\psi/A}$ (as long as e is defined over A).
- Note here if $r = 0$ (i.e., $\psi_p = 1$), the projector e (actually defined over $\overline{\mathbb{Q}}$) induces a surjection $e : S_{k+1}(\Gamma_0(N), \psi; A) \to S^{ord}_{k+1}(\Gamma_0(Np), \psi; A)$ if $k > 1$.

§4. Duality.

Define a pairing $(\cdot,\cdot) : h_{k,\psi/A} \times S_{k+1,\psi/A} \to A$ by $(h,f) = a(1,f|h)$. Write $M^* = \mathrm{Hom}_A(M,A)$ for an A-module. By the celebrated formula of Hecke:

$$a(m,f|T(n)) = \sum_{0<d|(m,n),(d,N_\psi)=1} \psi(d)d^k a(\frac{mn}{d^2},f),$$

it is an easy exercise to show that, as long as A is a $\mathbb{Z}[\psi]$-algebra,

$$S^*_{k+1,\psi/A} \cong h_{k,\psi/A},\ S^{ord,*}_{k+1,\psi/A}, A) \cong \mathbf{h}_{k,\psi/A}$$
$$h^*_{k,\psi/A} \cong S_{k+1,\psi/A},\ \mathbf{h}^*_{k,\psi/A} \cong S^{ord}_{k+1,\psi/A}$$
$$\mathrm{Hom}_{A\text{-alg}}(h_{k,\psi/A}, A)$$
$$\cong \{f \in S_{k+1,\psi/A} : f|T(n) = \lambda(n)f,\ a(1,f) = 1\}.$$

The second isomorphism can be written as $\phi \mapsto \sum_{n=1}^{\infty} \phi(T(n))q^n$ for $\phi : h_{k,\psi/A} \to A$ and the last is just this one restricted to A-algebra homomorphisms.

§5. Big Hecke algebra.

Now we move Neben characters, puting $\psi_k := \psi\omega^{-k}$ for a character ψ modulo Np. Recall $\Lambda = W[[T]]$. Defining the Hecke algebra as $\Lambda[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}_\Lambda(\mathcal{S}_\Lambda)$ for the space $\mathcal{S}_\Lambda$ of ordinary Λ-adic forms interpolating cusp forms in $S^{ord}_{k+1,\psi_k\epsilon/W}$ for $k > 1$ and $\epsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}(W)$, we have the big Hecke algebra $\mathbf{h}$. Here is its characterizing properties:

(C1) $\mathbf{h}$ is free of finite rank over Λ equipped with $T(n) \in \mathbf{h}$ for all $1 \leq n \in \mathbb{Z}$ (so $U(l)$ for $l|Np$),

(C2) if $k \geq 1$ and $\epsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}$ is a character,

$$\mathbf{h} \otimes_{\Lambda, t \mapsto \epsilon(\gamma)\gamma^k} W[\epsilon] \cong \mathbf{h}_{k,\epsilon\psi_k} \ (\gamma = 1 + p),$$

sending $T(n)$ to $T(n)$ (and $U(l)$ to $U(l)$ for $l|Np$), where $W[\epsilon] \subset \mathbb{C}_p$ is the W-subalgebra generated by the values of ϵ.

§6. Irreducible components and families.

Let Spec($\mathbb{I}$) be a reduced irreducible component Spec($\mathbb{I}$) $\subset$ Spec($\mathbf{h}$). Write $a(n)$ for the image of $T(n)$ in $\mathbb{I}$ (so, $a(p)$ is the image of $U(p)$).

If a point P of Spec($\mathbb{I}$)($\overline{\mathbb{Q}}_p$) kills $(t-\epsilon(\gamma)\gamma^k)$ with $1 \leq k \in \mathbb{Z}$ (i.e., $P((t-\epsilon(\gamma)\gamma^k)) = 0$), we call it an *arithmetic* point and we write $\epsilon_P = \epsilon$, $\psi_P = \epsilon_P \psi \omega^{-k}$, $k(P) = k \geq 1$ and $p^{r(P)}$ for the order of ϵ_P.

If P is arithmetic, by (C2), we have a Hecke eigenform $f_P \in S_{k+1}(\Gamma_0(Np^{r(P)+1}), \epsilon\psi_k)$ such that its eigenvalue for $T(n)$ is given by

$$a_P(n) := P(a(n)) \in \overline{\mathbb{Q}}_p$$

for all n. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \{f_P | \text{arithemtic } P \in \text{Spec}(\mathbb{I})\}$ of Hecke eigenforms, and an $\mathbb{I}$-adic forms $\mathcal{F} := \sum_n a(n)q^n$.

§7. Modular Galois representation.

Each connected component Spec($\mathbb{T}$) $\subset$ Spec($\mathbf{h}$) has a 2-dimensional absolutely irreducible continuous representation $\rho_{\mathbb{T}}$ of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) with coefficients in the quotient field of $\mathbb{T}$. The representation $\rho_{\mathbb{T}}$ restricted to the p-decomposition group D_p is reducible with unramified *quotient* character.

As is well known now, $\rho_{\mathbb{T}}$ is unramified outside Np and satisfies

$$\mathrm{Tr}(\rho_{\mathbb{T}}(Frob_l)) = a(l) \quad (l \nmid Np),$$
$$\rho_{\mathbb{T}}([\gamma^s, \mathbb{Q}_p]) \sim \begin{pmatrix} t^s & * \\ 0 & 1 \end{pmatrix} \text{ and } \rho_{\mathbb{T}}([p, \mathbb{Q}_p]) \sim \begin{pmatrix} * & * \\ 0 & a(p) \end{pmatrix},$$

where $[x, \mathbb{Q}_p]$ is the local Artin symbol.

For each $P \in$ Spec($\mathbb{T}$), writing $\kappa(P)$ for the residue field of P, we also have a semi-simple Galois representation ρ_P : Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) $\to GL_2(\kappa(P))$ unramified outside Np such that $\mathrm{Tr}(\rho_P(Frob_l)) = a(l)$ mod P for all primes $l \nmid Np$.

§8. $\rho_{\mathbb{T}}$ as deformation of $\overline{\rho}$.

Start with a connected component Spec($\mathbb{T}$) of Spec($\mathbf{h}$) of level Np^∞ with character ψ.

Recall the deformation properties (D1–3):
(D1) ρ is unramified outside Np,
(D2) $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \epsilon & * \\ 0 & \delta \end{pmatrix}$ with δ unramified while ϵ ramified,
(D3) For each prime $l|N$, writing I_l for the inertia group $\rho|_{I_l} \cong \begin{pmatrix} \psi_l & * \\ 0 & 1 \end{pmatrix}$ regarding $\psi_l = \psi|_{\mathbb{Z}_l^\times}$ as the character of I_l by local class field theory.

We assume that $\overline{\rho}$ is **absolutely irreducible**. Under this condition, $\rho_{\mathbb{T}}$ has values in $GL_2(\mathbb{T})$ and its isomorphism class is unique (Caraylol, Serre). Thus ρ_P has values in $GL_2(\mathbb{T}/P)$. In particular, ρ_P is a deformation of $\overline{\rho}$ satisfying (D1–3) and (det).

§9. Hecke algebras are universal.

For simplicity, assume that $N = C$ for the prime-to-p conductor C of $\overline{\rho}$. Here is the "$R = \mathbb{T}$" theorem:

Theorem 1 (Wiles et al). *If $\overline{\rho}$ is absolutely irreducible over* Gal$(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_p])$, *$(\mathbb{T}, \rho_{\mathbb{T}})$ is a local complete intersection over Λ and is universal among deformations satisfying* (D1–3).

Let $\widetilde{\mathbb{I}}$ be the integral closure of $\mathbb{I}$ in its quotient field. Replace $\mathbb{T}$ by $\mathbb{T}_{\mathbb{I}} := \mathbb{T} \otimes_{\Lambda} \widetilde{\mathbb{I}}$ and π by the composite $\boldsymbol{\lambda} : \mathbb{T}_{\mathbb{I}} \xrightarrow{\pi \otimes 1} \mathbb{I} \otimes_{\Lambda} \widetilde{\mathbb{I}} \xrightarrow{a \otimes b \mapsto ab} \widetilde{\mathbb{I}}$. The local complete intersection property of $\mathbb{T}$ implies the following facts for $\boldsymbol{\lambda} : \mathbb{T}_{\mathbb{I}} \to \widetilde{\mathbb{I}}$:

Corollary 1. *We have (1) $C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}) = \widetilde{\mathbb{I}}/(L_p)$ for $L_p \in \widetilde{\mathbb{I}}$.*
(2) char$(C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}))$ = char$(C_1(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}))$.

Since $\widetilde{\mathbb{I}}$ is free of finite rank over Λ, we get Gorensteinness: $\mathbb{T}^*_{\mathbb{I}} :=$ Hom$_{\widetilde{\mathbb{I}}}(\mathbb{T}_{\mathbb{I}}, \widetilde{\mathbb{I}}) \cong \mathbb{T}_{\mathbb{I}}$ as $\mathbb{T}_{\mathbb{I}}$-modules from Hom$_{\Lambda}(\mathbb{T}, \Lambda)$.

§10. Proof of Corollary.

The assertion (2) is the consequence of Tate's theorem.

The fact (1) can be shown as follows. Write $\mathfrak{b} = \mathrm{Ker}(\boldsymbol{\lambda})$; i.e., we have an exact sequence

$$0 \to \mathfrak{b} \to \mathbb{T}_{\mathbb{I}} \to \widetilde{\mathbb{I}} \to 0.$$

For dimension 2 regular ring, reflexive modules are free. Thus $\mathfrak{b}$ is the $\widetilde{\mathbb{I}}$-direct summand of $\mathbb{T}$. By taking $\widetilde{\mathbb{I}}$-dual, we have another exact sequence $0 \to \widetilde{\mathbb{I}}^* \to \mathbb{T}_{\mathbb{I}}^* \to \mathfrak{b}^* \to 0$. For $Q = \mathrm{Frac}(\Lambda)$, $\mathrm{Frac}(\mathbb{T}_{\mathbb{I}}) = \mathbb{T}_{\mathbb{I}} \otimes_{\Lambda} Q = \mathrm{Frac}(\mathbb{I}) \oplus X$ for $X := \mathfrak{b}^* \otimes_{\Lambda} Q$ which is the complementary ring summand of $\mathrm{Frac}(\widetilde{\mathbb{I}})$. Thus $\mathrm{Im}(\widetilde{\mathbb{I}}^* \hookrightarrow \mathbb{T}_{\mathbb{I}}^* = \mathbb{T}_{\mathbb{I}}) \subset \mathbb{T}_{\mathbb{I}}$ is the ideal $\mathfrak{a} = (\mathrm{Frac}(\mathbb{I}) \oplus 0) \cap \mathbb{T}$. Since $C_0(\boldsymbol{\lambda}; \mathbb{I}) = \widetilde{\mathbb{I}}/\mathfrak{a}$ and $\mathfrak{a} \cong \widetilde{\mathbb{I}}^* \cong \widetilde{\mathbb{I}}$ is principal.

By Mazur's theorem, note

$$C_1(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}) = \Omega_{\mathbb{T}_{\mathbb{I}}/\widetilde{\mathbb{I}}} \otimes_{\mathbb{T}_{\mathbb{I}}} \widetilde{\mathbb{I}} = \Omega_{\mathbb{T}/\Lambda} \otimes_{\mathbb{T}} \widetilde{\mathbb{I}} = C_1(\pi, \widetilde{\mathbb{I}}) \cong \mathrm{Sel}(\rho_{\widetilde{\mathbb{I}}})^{\vee}.$$

§11. Fixed determinant deformation.

For an arithmetic point P of Spec(Λ), recall $\psi_P = \epsilon_P\psi\omega^{-k(P)}$.

Theorem 2 (Wiles et al)**.** *If $\overline{\rho}$ is absolutely irreducible over* Gal($\overline{\mathbb{Q}}/\mathbb{Q}[\mu_p]$), $(\mathbb{T}/P\mathbb{T}, (\rho_{\mathbb{T}} \mod P))$ *is a local complete intersection over W and is universal among deformations ρ satisfying* (D1–3) *and*

(det) $\det(\rho) = \psi_P\nu^k$ *for the p-adic cyclotomic character ν and $k = k(P)$.*

Actually Wiles–Taylor proved the first cases of Theorem 2 which actually implies Theorem 1 (see [MFG, Theorem 5.29] for this implication). In the same way as Corollary 1, we can prove

Corollary 2. *Let $\lambda = (\boldsymbol{\lambda} \mod \mathfrak{P})$ for an arithmetic $\mathfrak{P} \in$* Spec($\widetilde{\mathbb{I}}$)($\overline{\mathbb{Q}}_p$)*. We have $C_0(\lambda; \widetilde{\mathbb{I}}/\mathfrak{P}) = \widetilde{\mathbb{I}}/(L_p(\mathfrak{P}))$ for $L_p \in \widetilde{\mathbb{I}}$ in* Corollary 1, *where $L_p(\mathfrak{P}) = \mathfrak{P}(L_p)$.*

§**12. One variable adjoint main conjecture.** Assume $p \nmid \varphi(C)$.

By Mazur's theorem, $C_1(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}) \cong \mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\widetilde{\mathbb{I}}}))^{\vee}$, and $\mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\widetilde{\mathbb{I}}})^{\vee}) = \mathrm{char}(C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}})) = (L_p)$ and $\mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\mathfrak{P}})^{\vee}) = \mathrm{char}(C_0(\lambda; \widetilde{\mathbb{I}}/\mathfrak{P})) = (L_p(\mathfrak{P}))$.

We will relate $L_p(\mathfrak{P})$ with $L(1, Ad(\rho_{\mathfrak{P}}))$ in the forthcoming last lecture, and we get the one variable adjoint main conjecture:

Corollary 3. *There exists a p-adic L-function $L_p \in \widetilde{\mathbb{I}}$ such that we have* $\mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\widetilde{\mathbb{I}}})^{\vee}) = (L_p)$ *and* $\mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\mathfrak{P}})^{\vee}) = (L_p(\mathfrak{P}))$ *for all arithmetic points $\mathfrak{P} \in \mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$.*

§13. Two variable adjoint main conjecture. In [H90], I made L_p to a two variable p-adic L-function $L \in \widetilde{\mathbb{I}}[[T]]$ interpolating $L(1+m, Ad(\rho_{\mathbb{I}}))$ (i.e., $\mathfrak{P}(L)(\gamma^m - 1) \doteq L(1+m, Ad(\rho_{\mathfrak{P}}) \otimes \omega^{-m})$ essentially).

Eric Urban [U06] proved the divisibility

$$L|\,\mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(\rho_{\widetilde{\mathbb{I}}} \otimes \kappa)^{\vee})$$

for the universal character $\kappa : \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \to W[[T]]^{\times}$ deforming the identity character unramified outside $p\infty$ in many cases (applying Eisenstein techniques to $GSp(4)$ of Ribet-Greenberg-Wiles).

The function has an exceptional zero at $T = 0$; that is, we expect that $(L/T)|_{T=0} \doteq TL_p$? up to an $\mathcal{L}$-invariant. The $\mathcal{L}$-invariant is up to a non-zero constant in $\mathbb{Z}_p$ given by $t\frac{da(p)}{dt}$ [H11]. The $\mathcal{L}$-invariant formula of Greenberg–Tilouine–Rosso tells us the equality if the family contains a cusp form giving rise to an elliptic curve with multiplicative reduction at p.