Arithmetic of adjoint L-values.

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June 14, 2014

Abstract: We relate the differential module to Selmer group and its characteristic ideal to the annihilator of the congruence module.

*Pune (India) lecture No.2 on June 14, 2014. The authors are partially supported by the NSF grants: DMS 0753991.

§1. Group algebra and complete intersection.

Decompose $C_F = \bigoplus_{i=1}^{m} C_i$ for cyclic group C_i. Then sending $X_i \mapsto \gamma_i - 1$ for a generators γ_i of C_i, we have

$W[C_F] \cong W[[X_1, \ldots, X_m]]/(x_1^{|C_1|} - 1, \ldots, x_m^{|C_m|} - 1)$

for $x_i = 1 + X_i$.

Let A be a complete normal local domain (for example, a complete regular local rings like $A = W$ or $A = W[[T]]$ or $A = W[[T_1, \ldots, T_r]]$ (power series ring)). Any local A-algebra R free of finite rank over A has a presentation

$$R \cong A[[X_1, \ldots, X_n]]/(f_1, \ldots, f_m)$$

for $f_i \in A[[X_1, \ldots, X_n]]$ with $m \geq n$. If $m = n$, then R is called a local complete intersection over A.

§2. Pseudo module theory.

For two A-modules M, N of finite type, a morphism $\phi : M \to N$ is called a pseudo isomorphism if for the annihilator ideals $\mathfrak{a}$ and $\mathfrak{b}$ of Ker(ϕ) and Coker(ϕ), respectively the closed subschemes Spec($A/\mathfrak{a}$) and Spec($A/\mathfrak{b}$) of Spec(A) have co-dimension at least 2.

For a torsion A-modules M of finite type, we have a pseudo isomorphism $M \to \bigoplus_i A/\mathfrak{f}_i$ for finitely many reflexive ideal $0 \neq \mathfrak{f}_i \in A$. An ideal $\mathfrak{f}$ is *reflexive* if $\mathrm{Hom}_A(\mathrm{Hom}_A(\mathfrak{f}, A), A) \cong \mathfrak{f}$ canonically as A-modules (and equivalently $\mathfrak{f} = \bigcap_{\lambda \in A, (\lambda) \supset \mathfrak{f}} (\lambda)$; i.e., close to be principal).

The characteristic ideal char(M) of M is defined by $\mathrm{char}(M) := \prod_i \mathfrak{f}_i \subset A$. If A is a unique factorization domain, any reflexive ideal is principal.

If $A = W$, a pseudo-isomorphism is an isomorphism, and if $A = W[[T]]$, it is an isogeny.

§3. Tate's theorem.

Theorem 1 (J. Tate). *Assume that R is a local complete intersection over a complete normal noetherian local domain A with an algebra homomorphism $\lambda : R \to A$. If after tensoring the quotient field Q or A, $R \otimes_A Q = (\mathrm{Im}(\lambda) \otimes_A Q) \oplus S$ as algebra direct sum for some Q-algebra S, then $C_j(\lambda; A)$ is a torsion A-module of finite type, and we have*

$$\mathrm{Ann}(C_0(\lambda; A)) = \mathrm{char}(C_0(\lambda; A)) = \mathrm{char}(C_1(\lambda; A)).$$

See [MFG, §5.3.4] for a proof, which uses Koszul complexes.

If $A = \mathbb{Z}_p$, $\mathrm{char}(M) = (|M|)$, and hence, the above theorem is a proper generalization of our result for group algebras.

§4. Two dimensional deformations.

Fix a positive integer N prime to p. Let $\overline{\rho}$: Gal$(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F})$ be the Galois representation unramified outside Np. Let ψ be the Teichmüller lift of det$(\overline{\rho})$. For simplicity, assume that ψ and $\overline{\psi}$:= det$(\overline{\rho})$ has conductor divisible by N and $p \nmid \varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$. We consider deformations (over W) satisfying the following three properties:

(D1) ρ is unramified outside Np,

(D2) $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ with δ unramified while ϵ ramified,

(D3) For each prime $l|N$, writing I_l for the inertia group $\rho|_{I_l} \cong \left(\begin{smallmatrix} \psi_l & * \\ 0 & 1 \end{smallmatrix}\right)$ regarding $\psi_l = \psi|_{\mathbb{Z}_l^\times}$ as the character of I_l by local class field theory,

In particular, we assume (D1–3) for $\overline{\rho}$. Writing $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \overline{\epsilon} & * \\ 0 & \overline{\delta} \end{pmatrix}$, we always assume that $\overline{\epsilon}$ is ramified while $\overline{\delta}$ is unramified.

§5. Universal couple exists.

We admit the following fact and study its consequences:

Theorem 2 (B. Mazur)**.** *We have an universal couple $(R, \boldsymbol{\rho})$ of a W-algebra R and a Galois representation $\boldsymbol{\rho}$:* Gal$(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(R)$ *such that $\boldsymbol{\rho}$ is universal among deformations satisfying (D1–3). The algebra R is a $W[[C_{\mathbb{Q}}(Np^{\infty})]]$-algebra canonically by the universality of $W[[C_{\mathbb{Q}}(Np^{\infty})]]$ applied to* det$(\boldsymbol{\rho})$*. In particular, R is an algebra over $\Lambda = W[[T]] \subset W[[C_{\mathbb{Q}}(Np^{\infty})]]$. If we add one more property to* (D1–3)*:*
(det) det$(\rho) = \psi\epsilon\omega^{-k}\nu^k$ *for the p-adic cyclotomic character ν and $k = k(P)$*
for a character ϵ : Gal$(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \to \mu_{p^\infty}(W)$*, then the residual couple $(R/PR, \boldsymbol{\rho}_P)$ with $\boldsymbol{\rho}_P = \boldsymbol{\rho}$* mod P *for $P = (t - \epsilon([\gamma, \mathbb{Q}_p])\gamma^k) \subset \Lambda$ is universal among deformations satisfying* (D1–3) *and* (det)*.*

See [MFG,§3.2.4] and [HMI,§3.2] for a proof.

§6. Adjoint Galois representation.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(A)$ be a deformation satisfying (D1–3). Write $V(\rho) = A^2$ on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via ρ. Since $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ for an unramified δ, we have a filtration

$$V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}) \hookrightarrow V(\rho) \twoheadrightarrow V_{\mathfrak{p}}(\delta_{\mathfrak{p}})$$

stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})$ for a prime $\mathfrak{p}|p$ of F.

We now let $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ act on $M_2(A) = \mathrm{End}_A(V(\rho))$ by conjugation: $x \mapsto \rho(\sigma)x\rho(\sigma)^{-1}$. The trace zero subspace

$$\mathfrak{sl}(A) = \{x \in M_2(A) | \mathrm{Tr}(x) = 0\}$$

is stable under this action. This new Galois module of dimension 3 is called the adjoint representation of ρ and written as $Ad(\rho)$. Thus

$$V = V(Ad(\rho)) = \left\{T \in \mathrm{End}_A(V(\rho)) \middle| \mathrm{Tr}(T) = 0\right\}.$$

§7. Adjoint Selmer groups.

This space $V(Ad(\rho))$ has a three step filtration: $0 \subset V_{\mathfrak{p}}^{+} \subset V_{\mathfrak{p}}^{-} \subset V$ given by

$$V_{\mathfrak{p}}^{+} = \left\{T \in V_{\mathfrak{p}}^{-}(Ad(\rho)) \middle| T(V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}))) = 0\right\} \sim \left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\},$$
$$V_{\mathfrak{p}}^{-} = \left\{T \in V(Ad(\rho)) \middle| T(V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}})) \subset V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}})\right\} \sim \left\{\begin{pmatrix} * & * \\ 0 & -* \end{pmatrix}\right\}.$$

Writing $A^{\vee}$ for the Pontryagin dual module

$$\mathrm{Hom}_W(A, K/W) \cong \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

for the quotient field K of W. Then for any A-modules M, we put $M^* = M \otimes_A A^{\vee}$. In particular, $V(Ad(\rho))^*$ and $V_{\mathfrak{p}}(?)^*$ are divisible Galois modules. We define

$$\mathrm{Sel}_F(Ad(\rho)) = \mathrm{Ker}(H^1(F, V(Ad(\rho))^*)$$
$$\to \prod_{\mathfrak{p}|p} H^1(I_{\mathfrak{p}}, \frac{V(Ad(\rho)^*}{V_{\mathfrak{p}}^{+}(Ad(\rho))^*}) \times \prod_{\mathfrak{l}\nmid p} H^1(I_{\mathfrak{l}}, V(Ad(\rho)^*)),$$

where $\mathfrak{p}|p$ and $\mathfrak{l} \nmid p$ are primes of F and $I_{\mathfrak{l}}$ is the inertia subgroup at $\mathfrak{l}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

§8. Mazur's theorems.

Theorem 3 (B. Mazur). *Let* $(R, \boldsymbol{\rho})$ *be the universal couple among deformations satisfying* (D1–3) *and* (det). *If* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(W)$ *is a deformation, then we have a canonical isomorphism*

$$\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho))^{\vee} \cong \Omega_{R/W} \otimes_R W \cong C_1(\lambda; W)$$

as W*-modules for* $\lambda : R \to W$ *with* $\rho \cong \lambda \circ \boldsymbol{\rho}$*.*

Considering the universal ring for deformations satisfying only (D1–3), we get

Theorem 4 (B. Mazur). *Let* $(R, \boldsymbol{\rho})$ *be the universal couple among deformations satisfying* (D1–3). *Let* $\mathrm{Spec}(\mathbb{I})$ *be an irreducible component of* $\mathrm{Spec}(R)$. *Writing* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{I})$ *for the deformation corresponding to the projection* $R \twoheadrightarrow \mathbb{I}$*, we have a canonical isomorphism*

$$\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho))^{\vee} \cong \Omega_{R/\Lambda} \otimes_R \mathbb{I} \cong C_1(\lambda; \mathbb{I})$$

as $\mathbb{I}$*–modules for* $\lambda : R \to \mathbb{I}$ *with* $\rho \cong \lambda \circ \boldsymbol{\rho}$*.*

§9. Proof assuming (D1–3) and (det).

We repeat the "farfetched" proof in the $GL(2)$-case. Let $\Phi(A)$ be the set of deformations of $\overline{\rho}$ satisfying (D1–3) and (det) with values in $GL_2(A)$.

Let X be a profinite R-module. Then $R[X]$ is an object in CL_W. We consider the W-algebra homomorphism $\xi : R \to R[X]$ with $\xi \mod X = \mathrm{id}$. We have $\xi(r) = r \oplus d_\xi(r)$ and we get $d_\xi(rr') = rd_\xi(r') + r'd_\xi(r)$ and $d_\xi(W) = 0$. Thus $d_\xi \in Der_W(R, X)$. For any derivation $d : R \to X$ over W, $r \mapsto r \oplus d(r)$ is obviously an W-algebra homomorphism, and we get

$$\begin{aligned}
&\left\{\pi \in \Phi(R[X]) \middle| \pi \mod X = \boldsymbol{\rho}\right\} / \approx_X \\
&\qquad \cong \left\{\pi \in \Phi(R[X]) \middle| \pi \mod X \cong \boldsymbol{\rho}\right\} / \cong \\
&\qquad \cong \left\{\xi \in \mathrm{Hom}_{W\text{-alg}}(R, R[X]) \middle| \xi \mod X = \mathrm{id}\right\} \\
&\qquad \cong Der_W(R, X) \cong \mathrm{Hom}_R(\Omega_{R/W}, X),
\end{aligned}$$

where "$\approx_X$" is conjugation under $(1 \oplus M_n(X)) \cap GL_2(R[X])$.

§10. Making cocycle out of a deformation.

Let π be the deformation in the left-hand side. Write $\pi(\sigma) = \boldsymbol{\rho}(\sigma) \oplus u'_\pi(\sigma)$. Comparing

$$
\begin{aligned}
&(\boldsymbol{\rho}(\sigma) \oplus u'_\pi(\sigma))(\boldsymbol{\rho}(\tau) \oplus u'_\pi(\tau)) \\
&\qquad = \boldsymbol{\rho}(\sigma\tau) \oplus (\boldsymbol{\rho}(\sigma)u'_\pi(\tau) + u'_\pi(\sigma)\boldsymbol{\rho}(\tau)),
\end{aligned}
$$

we have

$$u'_\pi(\sigma\tau) = \boldsymbol{\rho}(\sigma)u'_\pi(\tau) + u'_\pi(\sigma)\boldsymbol{\rho}(\tau).$$

Define $u_\pi(\sigma) = u'_\pi(\sigma)\boldsymbol{\rho}(\sigma)^{-1}$. Then, $x(\sigma) = \pi(\sigma)\boldsymbol{\rho}(\sigma)^{-1}$ has values in $SL_2(R[X])$, and $x = 1 \oplus u \mapsto u = x - 1$ is an isomorphism from the multiplicative group of the kernel of the reduction map $SL_2(R[X]) \twoheadrightarrow SL_2(R)$ given by

$$\{x \in SL_2(R[X]) | x \equiv 1 \mod X\}$$

onto the additive group

$$Ad(X) = \{x \in M_2(X) | \mathrm{Tr}(x) = 0\} = V(Ad(\boldsymbol{\rho})) \otimes_R X.$$

Thus we may regard that u has values in $Ad(X) = V(Ad(\boldsymbol{\rho})) \otimes_R X$.

§**11. Selmer condition.**
We also have

$$\begin{aligned}u_\pi(\sigma\tau) &= u'_\pi(\sigma\tau)\boldsymbol{\rho}(\sigma\tau)^{-1}\\ &= \boldsymbol{\rho}(\sigma)u'_\pi(\tau)\boldsymbol{\rho}(\sigma\tau)^{-1} + u'_\pi(\sigma)\boldsymbol{\rho}(\tau)\boldsymbol{\rho}(\sigma\tau)^{-1}\\ &= Ad(\boldsymbol{\rho})(\sigma)u_\pi(\tau) + u_\pi(\sigma).\end{aligned}$$

Hence u_π is a 1-cocycle unramified outside Np. It is easy to see the injectivity of $\pi \mapsto [u_\pi]$:

$$\left\{\pi \in \Phi(R[X])\middle|\pi \bmod X \approx \boldsymbol{\rho}\right\} / \approx_X \hookrightarrow H^1(\mathbb{Q}, Ad(X)).$$

For $V_p^\pm(Ad(X)) = V_p^\pm(Ad(\boldsymbol{\rho}))\otimes_R X$, from $\mathrm{Tr}(u_\pi) = 0$ and unramifiedness of $\boldsymbol{\delta}$, we get

$$\begin{aligned}u_\pi(I_p) \subset V_p^+(Ad(X)) \Leftrightarrow\\ u'_\pi(I_p) \subset V_p^+(Ad(X)) \Leftrightarrow \delta_\pi(I_p) = 1.\end{aligned}$$

If $l|N$, we have $\boldsymbol{\rho}|_{I_l} = \epsilon_l \oplus 1$ and $\pi|_{I_l} = \epsilon_l \oplus 1$. Thus $\pi|_{I_l}$ factors through the image of I_l in the maximal abelian quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ which is isomorphic to $\mathbb{Z}_l^\times$. Thus $u_\pi|_{I_l}$ factors through $\mathbb{Z}_l^\times$. Since $p \nmid \varphi(N)$, $p \nmid l-1$, which implies $u_\pi|_{I_l}$ is trivial; thus u_π unramified everywhere outside p.

§12. Passing to the limit, conclusion.

Since $W = \varprojlim_n W/\mathfrak{m}_W^n$ for the finite rings $W/\mathfrak{m}_W^n$, we have $W^\vee = \varinjlim_n (W/\mathfrak{m}_W^n)^\vee$, which is a discrete R-modules, which shows $R[W^\vee] = \bigcup_n R[(W/\mathfrak{m}_W^n)^\vee]$.

We put the profinite topology on the individual $R[(W/\mathfrak{m}_W^n)^\vee]$. On $R[W^\vee]$, we give a injective-limit topology. Thus a map $\phi : X \to R[W^\vee]$ is continuous if $\phi^{-1}(R[(W/\mathfrak{m}_W^n)^\vee]) \to R[(W/\mathfrak{m}_W^n)^\vee]$ is continuous for all n with respect to the induced topology from X on the source and the profinite topology on the target.

From this, any deformation (continuous with respect to having values in $GL_2(R[W^\vee])$ gives rise to a continuous 1-cocycle with values in the discrete G-module $V(Ad(\pi))^\vee$. In this way, we get

$$(\Omega_{R/W} \otimes_R W)^\vee \cong \mathrm{Hom}_R(\Omega_{R/W}, W^\vee) \cong \mathrm{Sel}_{\mathbb{Q}}(Ad(\rho)).$$