Capital Minimization as a Hedging Objective

Dilip B. Madan
Robert H. Smith School of Business

Tata Institute of Fundamental Research

International Center for Theoretical Sciences

School in Mathematical Finance
January 16-27 2012, Mumbai, India

- The classical theory of financial markets does not explicitly define the level of cash reserves to be maintained in support of a trade,

- or what is the up-front profit that may be taken on the trade.

- Consequently, both the leverage provided and the rate of return are unclear.

- We provide explicit procedures for evaluating capital, profits, leverage and returns on trades.

- In the classical complete markets theory for financial markets the law of one price prevails and no arbitrage principles pin down a unique risk neutral or pricing measure.

- Economic agents can trade any amount of any financial product at the going market price that is its cost of replication.

- No cash reserves are needed as we have full replication with no residual risk to be held.

- Capital supporting trades are zero.

- Competition drops profits to zero as all products are provided at cost of replication.

- Rates of return and leverage are thereby undefined.

- Our departure from this classical theory comes in altering just one assumption in the definition of the financial market.

- All economic agents can still trade with the market at any desired amount, but now the terms of trade depend on the direction of trade.

- We therefore merely replace the law of one price by the law of two prices with a different price for buying from the market to that for selling to the market.

- Further departures from the classical model are possible.

- For example one may let prices depend on the size of the trade as well as the direction and we have done this in a later study of efficiency frontiers related to execution costs.

- We model markets directly with principles rooted in competition.

- In this regard we distinguish the modeling of markets from the modeling of objectives for agents in the economy.

- The classical Arrow and Debreu (1954) theory recognized this and the explicit recognition was apparent in the classical proofs for the existence of an equilibrium in which a special role is created for the Walrasian auctioneer as the only non-maximizing agent in the economy, focused entirely on getting markets to clear.

- Yes, markets are made up of people, but by virtue of the atomicity of individuals in markets, the modeling of markets diverges from the models for individual agents.

- In this regard we foretell that in our perspective, even though agents may maximize expected utility, this is not an appropriate objective for the market that is not a person, or an aggregate of persons.

- We model competitive markets as economizing on the commitment of capital, but first we have to define capital.

- Our model for two price markets follows Cherny and Madan (2009, 2010) and yields closed forms for bid and ask prices in terms of concave distortions of distribution functions.

- Here we obtain new expressions for these prices in terms of the derivative of the inverse of the distribution function.

- This derivative is seen to play a critical role measuring the risk exposure of a trade at a quantile level and both profits and capital are obtained as integrals over quantiles of charges for exposures measured this way.

- We shall describe the profit and capital charge at each quantile and explain the critical role played by the quantile sensitivity of the inverse distribution function.

- We go on to formulate our definition of up-front profit as the mid price of bid and ask less the cost of replication given by a risk neutral expectation.

- The capital charge for the issuance of a state contingent liability is defined as the difference between the ask and bid prices.

- Thus we obtain closed forms for both profit and capital and hence the leverage and rate of return on the trade.

- We also show that from the viewpoint of the market the hedging strategy should be chosen to minimize capital commitments.

- This gives us a new market based criterion for the choice of hedging strategies, distinct from the interests of individuals.

- The latter may be seen as maximizing expected utility or some other preference based criterion, but we model competitive markets as seeking to economize on the commitment of reserve capital.

- The paper presents four illustrative applications, here we present just one.

- The application we present considers the hedging of a call option on a stock driven by a variance gamma process.

- We show that on a one year contract one may reduce capital commitments by quarterly hedging and hedging monthly does not make a significant improvement over this.

- We also show that the maximization of expected utility may result in a significantly higher capital commitment when compared to implementing market objectives of capital reduction.

Two Price Markets

- The theory of two price markets and the closed form formulas for bid and ask prices using concave distortions was developed in Cherny and Madan (2010).

- In this approach the market is seen as a passive counterparty accepting the opposite side of zero cost trades proposed by economic agents or market participants.

- Cash flows to trades are modeled as random variables on a fixed probability space $(\Omega, \mathcal{F}, P)$ for a base probability measure that is in fact a risk neutral measure identified or selected by the economy.

- The market is modeled by describing all the cash flows the market will accept at zero cost as a counterparty.

- Since market participants may trade in any size this set of cash flows is closed under scaling by any positive multiple. Hence it constitutes a cone of random variables.

- If the market accepts two cash flows X, Y it will accept the sum and hence this set of cash flows is a convex cone.

- Finally, the market will accept at zero cost any cash flow that is nonnegative and so the convex cone contains the nonnegative cash flows.

- We therefore have a special structure for the zero cost cash flows acceptable to the market as a counterparty.

- This is that of a convex cone containing the nonnegative cash flows.

- The classical model for the market with its law of one price goes a step further and asserts that if a cash flow X is just acceptable to the market with $E^P[X] = 0$ then as we trade in both directions at the same price, $-X$ is also just acceptable and so the set of acceptable cash flows is identified with the half space defined by the condition $E^P[X] \geq 0$ or the set of all positive alpha trades as seen by the risk neutral measure P (Jensen (1968)).

- For two price markets we stop short of asserting the law of one price and hence the set of cash flows acceptable at zero cost is a proper convex cone containing the nonnegative cash flows.

- We denote by $\mathcal{A}$ this set of cash flows acceptable to the market at zero cost. It will be smaller than the classical set of positive alpha trades and characteristically if X is just acceptable then $-X$ will not be acceptable.

- One cannot reverse direction on the same terms.

- A more constructive and equivalent characterization of such cash flows (i.e. those that are given by convex cones containing the nonnegative cash flows) was provided by Artzner, Delbaen, Eber and Heath (1999).

- It was shown there that for any such set of acceptable risks $\mathcal{A}$, there exists a convex set $\mathcal{M}$ of probability measures $Q \in \mathcal{M}$, Q equivalent to P, (that are referred to as supporting measures in Cherny and Madan (2010) or test measures in Carr, Geman, Madan (2001)) with the property that $X \in \mathcal{A}$ if and only if

$$E^{Q}[X] \geq 0, all Q \in \mathcal{M}.$$

- By way of related literature we cite Bernardo and Ledoit (2000), Cochrane and Saa-Requejo (2000), Černý and Hodges (2000), and Jaschke and Küchler (2001).

Bid and Ask Prices

- For an arbitrary cash flow X Cherny and Madan (2010) show that in the absence of hedging assets the market bid $b(X)$ and ask $a(X)$ prices must satisfy

$$\begin{aligned} b(X) &= \inf_{Q \in \mathcal{M}} E^Q[X] \\ a(X) &= \sup_{Q \in \mathcal{M}} E^Q[X]. \end{aligned}$$

- These facts follow on noting that both $a(X) - X$ and $X - b(X)$ must belong to the set $\mathcal{A}$, and furthermore by competition in the market one seeks the lowest possible ask and the largest possible bid prices.

Bid and Ask with Hedging Assets

- In the presence of a collection $\mathcal{H}$ of zero cost hedging assets a special role is played by the risk neutral measures $\mathcal{R}$, defined as the set of all measures Q equivalent to P for which $E^Q[H] = 0$ for all $H \in \mathcal{H}$.

- In the presence of hedging assets we have that

$$\begin{aligned} b(X) &= \inf_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q[X] \\ a(X) &= \sup_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q[X]. \end{aligned}$$

- In the presence of hedging assets

$$\begin{aligned} b(X) &= \sup\{b : \exists\ H \in \mathcal{H} \text{ s.t. } X - b - H \in \mathcal{A}\} \\ a(X) &= \inf\{a : \exists\ H \in \mathcal{H} \text{ s.t. } a + H - X \in \mathcal{A}\}. \end{aligned}$$

- The lower and upper hedges denoted $\underline{H}$, $\overline{H}$ respectively satisfy

$$\begin{aligned} X - b(X) - \underline{H} &\in \mathcal{A} \\ a(X) + \overline{H} - X &\in \mathcal{A}. \end{aligned}$$

Acceptability Using Distortions

- We fix a concave distribution function $\Psi(u)$ defined on the unit interval that maps to the unit interval.

- A random variable X with distribution function $F(x)$ is then defined to be acceptable just if
$$\int_{-\infty}^{\infty} x d\Psi(F(x)) \geq 0.$$

- If attention is restricted to the class of acceptable distribution functions $F(x)$ or their inverses $G(u)$ then for the acceptability of G one must have a positive expectation under all change of measure densities on the unit interval $Z(u) \geq 0$, $\int_0^1 Z(u)du = 1$, for which $L' = Z$ satisfies $L(u) \leq \Psi(u)$ for $0 \leq u \leq 1$.

Bid and Ask With Distortions

- The bid and ask prices for X with a hedging space $\mathcal{H}$ are now given in terms of the distortion by

$$\begin{aligned} b(X) &= \sup_{H\in\mathcal{H}} \int_{-\infty}^{\infty} x d\Psi\left(F_{X-H}(x)\right) \\ a(X) &= \inf_{H\in\mathcal{H}} - \int_{-\infty}^{\infty} x d\Psi\left(F_{H-X}(x)\right). \end{aligned}$$

- In the absence of hedging assets one merely computes the relevant expectations with $H = 0$.

- The computation of the distorted expectation is facilitated in terms of an ordered sample from the relevant distribution function with $x_1 < x_2 < \cdots x_N$ as

$$\sum_{i=1}^{N} x_i \left(\Psi\left(\frac{i}{N}\right) - \Psi\left(\frac{i-1}{N}\right)\right).$$

Bid and Ask Prices in terms of the inverse distribution function

- We now develop expressions for the bid and ask prices in terms of the inverse distribution function $G(u)$ of a hedged cash flow X with distribution function $F(x)$.

- For this purpose we define the median $m = G(.5)$, or $F(m) = .5$. We now write

$$\begin{aligned} b(X) &= \int_{-\infty}^{\infty} x d\Psi(F(x)) \\ &= \int_0^1 G(u) d\Psi(u) \end{aligned}$$

$$\begin{aligned} a(X) &= -\int_{-\infty}^{\infty} x d\Psi(1 - F(-x)) \\ &= \int_0^1 G(u) d\Psi(1 - u) \end{aligned}$$

- We now partition the integrals over the unit interval into integrals over [0, .5] and [.5, 1] and then integrate by parts to get

$$
\begin{aligned}
b(X) &= \int_0^{\frac{1}{2}} G(u)d\boldsymbol{\Psi}(u) + \int_{\frac{1}{2}}^1 G(u)d\boldsymbol{\Psi}(u) \\
&= G(u)\boldsymbol{\Psi}(u)|_0^{\frac{1}{2}} \\
&\quad -\int_0^{\frac{1}{2}} \boldsymbol{\Psi}(u)dG(u) \\
&\quad +G(u)\left(\boldsymbol{\Psi}(u) - 1\right)|_{\frac{1}{2}}^1 \\
&\quad -\int_{\frac{1}{2}}^1 \left(\boldsymbol{\Psi}(u) - 1\right) dG(u) \\
&= m + \int_0^1 \left(\mathbf{1}_{u \geq \frac{1}{2}} - \boldsymbol{\Psi}(u)\right) dG(u)
\end{aligned}
$$

- Similarly we may write

$$
\begin{aligned}
a(X) &= \int_0^{\frac{1}{2}} G(u) d\Psi(1-u) + \int_{\frac{1}{2}}^1 G(u) d\Psi(1-u) \\
&= -G(u)\left(\Psi(1-u)-1\right)|_0^{\frac{1}{2}} \\
&\quad + \int_0^{\frac{1}{2}} \left(\Psi(1-u)-1\right) dG(u) \\
&\quad -G(u)\Psi(1-u)|_{\frac{1}{2}}^1 \\
&\quad + \int_{\frac{1}{2}}^1 \Psi(1-u) dG(u) \\
&= m + \int_0^1 \left(\Psi(1-u) - \mathbf{1}_{u\le\frac{1}{2}}\right) dG(u)
\end{aligned}
$$

- When a cash flow is increased by a constant, both the ask and bid prices rise by this constant as is clear from the general definitions of these prices.

- In the expressions provided this feature is captured by the move in the median.

- The rest of these expressions account for the charge related to the risk exposure.

- It is instructive in this regard to consider first a linear distribution function of the form

$$F(x) = \frac{(x-a)^+ - (x-b)^+}{(b-a)}$$

of a random variable uniformly distributed in the interval (a, b).

- The inverse distribution function is just

$$G(u) = a + (b-a)u$$

with the median being

$$m = \frac{a+b}{2}$$

and the risk charge embedded in the ask and bid prices is just proportional to the length of the interval of uncertainty with

$$\begin{aligned} b(X) &= \frac{a+b}{2} + (b-a)\int_0^1 \left(\mathbf{1}_{u \geq \frac{1}{2}} - \Psi(u)\right) du \\ a(X) &= \frac{a+b}{2} + (b-a)\int_0^1 \left(\Psi(1-u) - \mathbf{1}_{u \leq \frac{1}{2}}\right) du \end{aligned}$$

- More generally for a piecewise linear distribution function taking the level p_i at the point a_i with $p_0 = 0$ and $p_N = 1$ we have that

$$
\begin{aligned}
F(x) &= \sum_{i=1}^{N} \begin{pmatrix} p_{i-1} \\ +\,(p_i - p_{i-1})\,\frac{(x-a_{i-1})^+ - (x-a_i)^+}{(a_i - a_{i-1})} \end{pmatrix} \\
G(u) &= a_0 + \sum_{i=1}^{n} \begin{pmatrix} \frac{a_i - a_{i-1}}{p_i - p_{i-1}} \\ \times \left((u - p_{i-1})^+ - (u - p_i)^+ \right) \end{pmatrix}
\end{aligned}
$$

- The ask and bid prices are now given by

$$\begin{aligned} b(X) &= m + \sum_{i=1}^{N} \frac{a_i - a_{i-1}}{p_i - p_{i-1}} \\ &\quad \times \int_{p_{i-1}}^{p_i} \left(\mathbf{1}_{u \geq \frac{1}{2}} - \Psi(u) \right) du \\ a(X) &= m + \sum_{i=1}^{N} \frac{a_i - a_{i-1}}{p_i - p_{i-1}} \\ &\quad \times \int_{p_{i-1}}^{p_i} \left(\Psi(1-u) - \mathbf{1}_{u \leq \frac{1}{2}} \right) du \end{aligned}$$

  and we have an exposure to a sum of uniform variates with probabilities $p_i - p_{i-1}$ over the interval (a_{i-1}, a_i).

- The risk charge is then proportional to the interval length $(a_i - a_{i-1})$ and the charge is for the bid and ask respectively

$$\frac{1}{p_i - p_{i-1}} \int_{p_{i-1}}^{p_i} \left(\mathbf{1}_{u \geq \frac{1}{2}} - \Psi(u) \right) du,$$
$$\frac{1}{p_i - p_{i-1}} \int_{p_{i-1}}^{p_i} \left(\Psi(1-u) - \mathbf{1}_{u \leq \frac{1}{2}} \right) du.$$

- In the limit the charge is

$$\left(\mathbf{1}_{u \geq \frac{1}{2}} - \Psi(u) \right), \left(\Psi(1-u) - \mathbf{1}_{u \leq \frac{1}{2}} \right)$$

for the interval $dG(u)$.

- The derivative of the inverse distribution function measures the sensitivity of the cash flow to a change in the quantile and is a measure of the risk exposure at the particular quantile.

- A zero derivative representing no risk exposure and no risk charge.

- The particular distortion determines the charge per unit exposure for each quantile.

Profits and Capital

- We now define the level of cash reserves supporting trades and the amount of up-front profits that may be associated with positions in terms of bid and ask prices.

- Consider first case of profits.

- We view a random variable defined as a state contingent cash flow as produced at a cost given by the risk neutral expectation or $E^P[X]$.

- Now this cash flow could be sold to the two price market at the market bid price $b(X)$.

- We see the market as successfully selling it to counterparties at the market ask price $a(X)$ and thereby earning the spread.

- The market however is not a person and has no needs for funds and redistributes the spread among the two parties of the trade.

- Hence we receive from the market the bid $b(X)$ the counterparty pays the ask $a(X)$ but then the market distributes half the spread to each of us with the net cost being the mid price and our receipts also being equal to the mid price

$$m(X) = \frac{a(X) + b(X)}{2}.$$

- The profit on the trade is then

$$\pi(X) = m(X) - E^P[X].$$

- We now come to the definition of the level of supporting cash reserves.

- Again if X is produced and sold to market we expect to receive for it the bid price $b(X)$. If the trade was to be unwound and we have to buy back X from the market we would have to pay for it $a(X)$.

- To guard against such an unfavorable unwind we should hold reserves in the amount $a(X) - b(X)$.

- Therefore we define the cash reserve capital to be

$$\kappa(X) = a(X) - b(X).$$

- We now observe that an increase in a cash flow by a constant raises the mid price and the risk neutral expectation by the same amount leaving the profit unchanged.

- The same is true for the cash reserves as the ask and bid price rise by the same amount.

- Hence both the profit and the capital are dependent solely on the risk exposures embedded in the slope of the inverse distribution function.

- It is useful to identify these functions.

- Applying the same analysis to the expectation under P as we did for the distorted expectation we observe that

$$E^P[X] = m + \int_0^1 \left(\mathbf{1}_{u \geq \frac{1}{2}} - u \right) dG(u).$$

- We may now write

$$\pi(X) = \int_0^1 H(u)dG(u)$$
$$\kappa(X) = \int_0^1 K(u)dG(u)$$

where

$$H(u) = \left(\begin{array}{c} \frac{\Psi(1-u)-\mathbf{1}_{u<\frac{1}{2}}+\mathbf{1}_{u\geq\frac{1}{2}}-\Psi(u)}{2} \\ -\left(\mathbf{1}_{u\geq\frac{1}{2}} - u\right) \end{array} \right)$$
$$K(u) = \Psi(1-u) + \Psi(u) - 1$$

- The rate of return is then easily defined by

$$\rho(X) = \frac{\pi(X)}{\kappa(X)}.$$

- We observe that $K(u)$ is symmetric about $\frac{1}{2}$, with

$$K(u) = K(1-u),$$

and the function $H(u)$ is antisymmetric with

$$H(u) = -H(1-u).$$

- Furthermore we have that $H(0) = H(1/2) = H(1) = 0$. The function H is negative for $u < 1/2$ and positive for $H > 1/2$.

- We see that sensitivity of cash flows to quantiles above 0.5 are exposures to gain leading to profits while sensitivities to quantiles below 0.5 are loss exposures with a marked loss.

- We now consider some sample distortions.

- Cherny and Madan (2009) introduced a family of distortions indexed by a parameter γ that defined a decreasing sequence of sets of acceptable risks $\mathcal{A}_\gamma$ starting with the half space of positive expectation under P for $\gamma = 0$ and tending to arbitrage or the nonnegative cash flows as γ tends to infinity.

- Further the distortions were organized with an infinite derivative near zero and zero derivative near unity to incorporate a reweighting upwards to infinity for large losses and a reweighting downwards towards zero for large gains.

- Such a family of distortions incorporates both risk aversion in the market and an absence of gain enticement in the market.

- An example of such a distortion is $MINMAXVAR$ for which

$$\Psi^{\gamma}(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}.$$

We may separate loss aversion λ and from absence of gain enticement γ and define

$$\Psi^{\lambda,\gamma}(u) = 1 - \left(1 - u^{\frac{1}{1+\lambda}}\right)^{1+\gamma}.$$

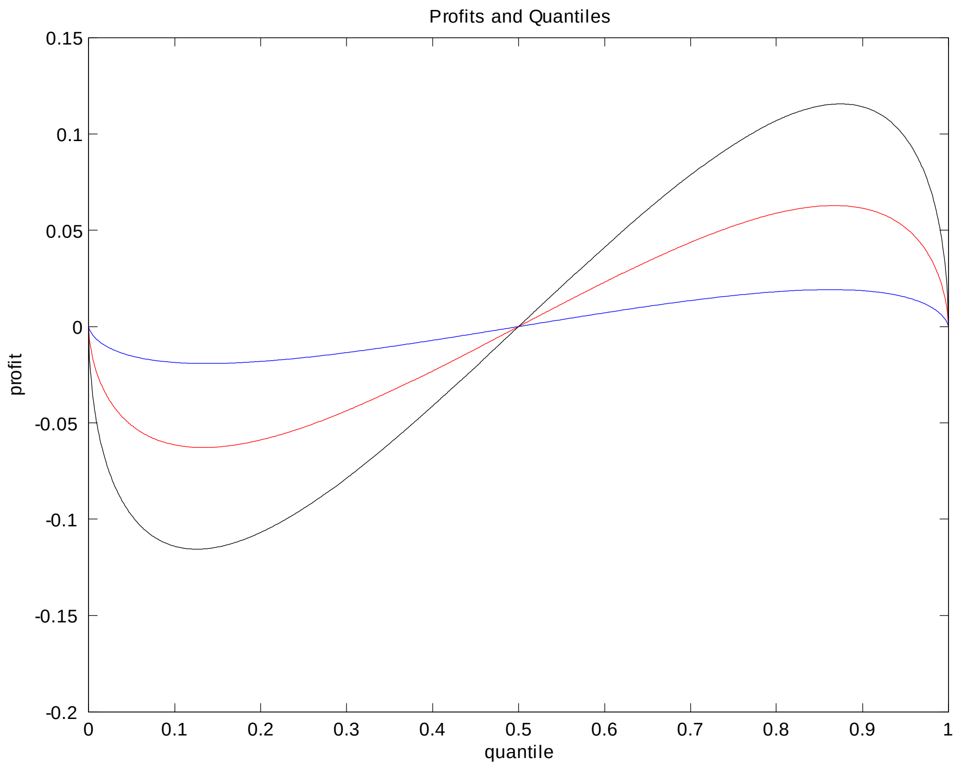

Figure 1: The profit charge on quantiles for MINMAX-VAR at three stress levels of 0.1, 0.25 and 0.5

- We display a sample of the functions H, K for the distortion $MINMAXVAR$.

Capital and Leverage

- We now report on the relationship between our capital assessments and the leverage being granted by such capital charges.

- For this purpose we develop first a measure of the scale of operations.

- In this regard we note that all our measures of profit, capital do not respond to constants.

- We consider a measure of scale that is equally unresponsive.

- Given that we finally work on the unit interval we use for the purpose of removing constants the deviation from the median m.

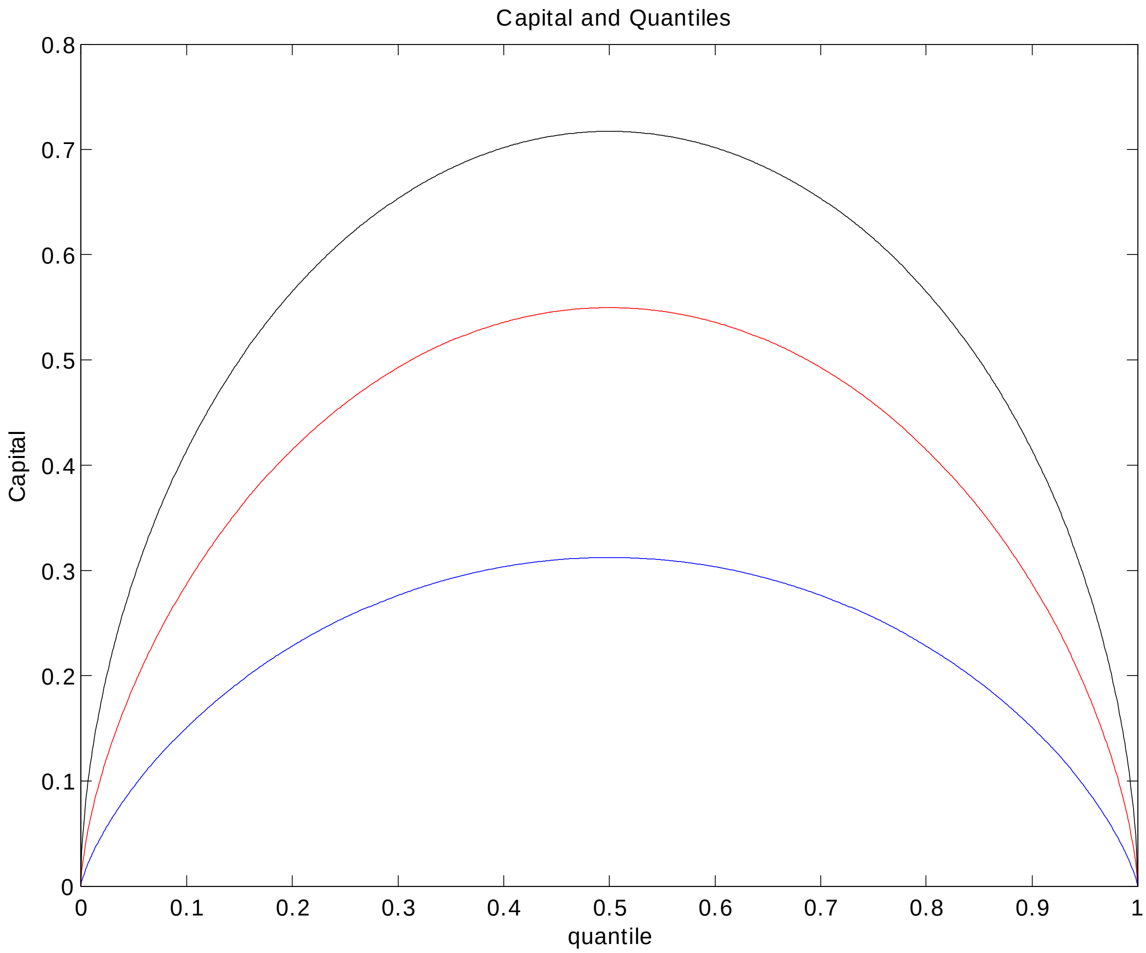

Figure 2: Capital charges for different quantile levels for MINMAXVAR at three stress levels of 0.1, 0.25 and 0.5.

- We take as our measure for scale the expectation of the deviation from the median or with density $f(x)$ we have

$$scale = \int_{-\infty}^{\infty} |x - m| f(x) dx.$$

- We now bring this computation back to the unit interval as follows.

$$\begin{aligned}\int_{-\infty}^{\infty}|x-m|f(x)dx &= \int_{-\infty}^{\infty}|y|f(m+y)dy \\ &= -\int_{-\infty}^{0}yf(m+y)dy \\ &\quad +\int_{0}^{\infty}yf(m+y)dy \\ &= \int_{-\infty}^{0}F(m+y)dy \\ &\quad +\int_{0}^{\infty}\left(1-F(m+y)\right)dy \\ &= \int_{0}^{\frac{1}{2}}udG(u) \\ &\quad +\int_{\frac{1}{2}}^{1}(1-u)dG(u).\end{aligned}$$

Define by

$$S(u) = u\mathbf{1}_{u\leq\frac{1}{2}} + (1-u)\mathbf{1}_{u\geq\frac{1}{2}}$$

then we have that

$$scale = \int_{0}^{1} S(u)dG(u).$$

- The leverage may be measured by the ratio of the scale to capital.

- This ratio is invariant to both shift and scale as both the numerator and denominator are invariant to shifts and they both scale and so the ratio is invariant.

- We may then measure the leverage with respect to a standard normal density for which $G(u)$ is the inverse of the standard normal distribution.

- We present a graph of $S(u)$ against $K(u)$ for various settings of the stress parameter γ.

- We observe that at $\gamma = .5$ one has stopped providing any leverage as the capital function $K(u)$ completely dominates the scale function $S(u)$.

- The domination occurs at $\gamma = 0.4405$.

- For lower values of γ some leverage is provided.

- For the values of γ at 0.025, 0.05, 0.1, 0.25, 0.4405 and 0.5 the leverage levels for a standard Gaussian are respectively 8.9305, 4.5045, 2.2908, 0.9605, 0.5751 and 0.5146 as obtained by integrating the kernels S, K against the inverse of the standard normal distribution function on the unit interval.

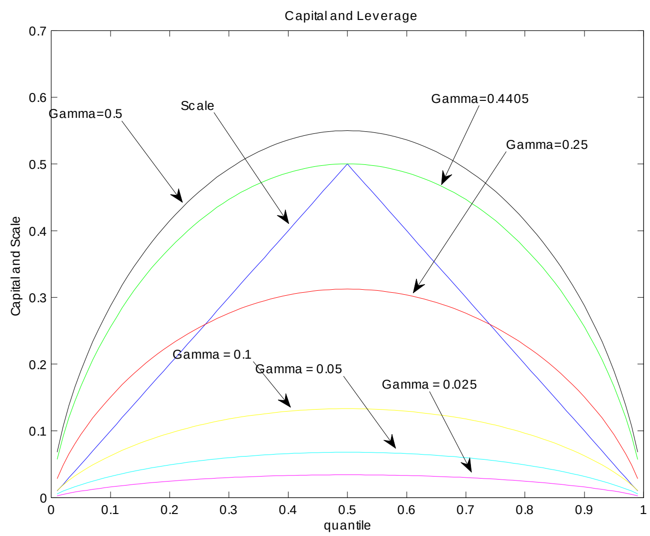

Figure 3: Graph of Capital Charges against Scale for various settings of the stress parameter in minmaxvar.

A Sample of Profit, Capital Leverage and Return Computations under geometric Brownian motion

- We begin with a case of exposure to loss and a negative profit. Let us consider the debt claim under the geometric Brownian motion model of Black and Scholes (1973) and Merton (1973) with

$$S = 100\,\mathrm{exp}(.2Z - .04/2)$$

where Z is a standard normal variate and the cash flow given by

$$X = Min(S, 80).$$

- For this cash flow the client takes loss and we take gain so we discount and the values are for minmaxvar at 0.75 as follows.

$$
\begin{aligned}
b(X) &= 77.4751 \\
a(X) &= 79.5226 \\
m(X) &= 78.4989 \\
E^P[X] &= 78.8100 \\
\pi(X) &= -0.3111 \\
\kappa(X) &= 2.0475 \\
\lambda(X) &= 0.5812 \\
\rho(X) &= -.1519.
\end{aligned}
$$

for the bid, ask, mid, expectation, profit, capital and return respectively.

- We present the rate of return for out of the money options under geometric Brownian motion with a 20% volatility for a variety of strikes and maturities.

- The computations are for the distortion $minmaxvar$ at the stress level of 0.25.

- We have 41 strikes ranging from 80 to 120 in steps of a dollar and 36 maturities from a quarter to 2 years in steps of .05.

- These are all assets with a gain exposure with a corresponding positive profit level and rate of return.

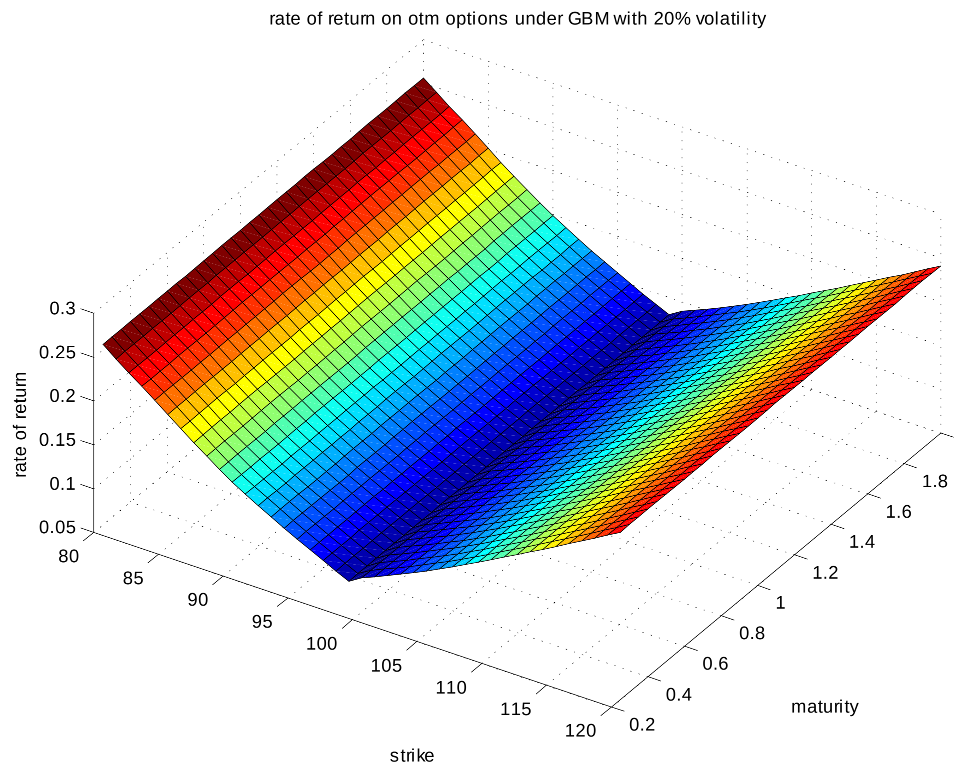

Figure 4: Graph of rates of return on a GBM risk neutral stock with a 20% volatility for a variety of strikes and maturities.

Hedging a Call option under the Variance Gamma process

- The risk neutral law for the stock is given by

$$S(t) = S(0)\exp\left(X(t) + \omega t\right)$$

where ω is the convexity correction term.

- The process $X(t)$ is given by

$$\begin{aligned} X(t) &= \theta g(t) + \sigma W(g(t)) \\ & \quad g(t) \text{ is the gamma process} \\ & \quad \text{with unit mean r} \\ & \quad \text{ate and variance rate } \nu. \\ \omega &= \frac{1}{\nu}\ln\left(1 - \theta\nu - \sigma^2\nu/2\right) \end{aligned}$$

- The computations were done for the VG process driving the stock with $\sigma = .2$, $\nu = .75$, and $\theta = -.3$.

- We take a one year 110 call on this path space and hedge each of the four quarters using stock positions that are interpolated at each quarter from the following points for the spot and the delta.

- The optimization criterion was the minimization of the post hedge ask price using $minmaxvar$ at the stress level 0.5.

- The initial delta was 0.2712.

- For the second quarter we have

stock	51.56	104.83	120.00
delta	0	0.2858	0.4445

- The third quarter positions are interpolated from

stock	41.69	105.63	131.10
delta	0	0.1796	0.7270

- The last quarter is given by

stock	35.85	105.06	140.87
delta	0	0.1063	0.8422

- In each case we extrapolated delta linearly for prices outside the interpolation interval and then floored the delta at zero and capped it at unity.

- The unhedged and hedged entities of interest are presented below.

- Also presented are the results for minimizing capital commitments and maximizing expected utility for a unit risk aversion coefficient.

- The graphs presents all four inverse distribution functions.

- For the minimum capital hedge and the maximization of expected utility the delta hedging was monthly in place of quarterly.

- Given the interest in minimizing the ask price and maximizing the bid price one may as well consider minimizing the sum of the ask price and the negative of the bid price which is the capital required.

- We see in this example that this particular expected utility maximization criterion over commits capital by some 18.89%.

	Unhedged	Hedged	Hedged Min Cap	Max Exp Ut
bid	3.2849	3.7775	4.2738	4.2098
ask	10.2536	8.6873	8.1053	8.7647
mid	6.7692	6.2324	6.2324	6.4873
exp	6.1025	6.0259	5.9944	6.0459
prf	0.6668	0.2065	0.1952	0.4414
cap	6.9686	4.9098	3.8314	4.5550
lev	0.8851	0.9571	0.9089	0.7496
ror	0.0957	0.0421	0.0509	0.0969

The inverse cdfs are presented in the following graph.

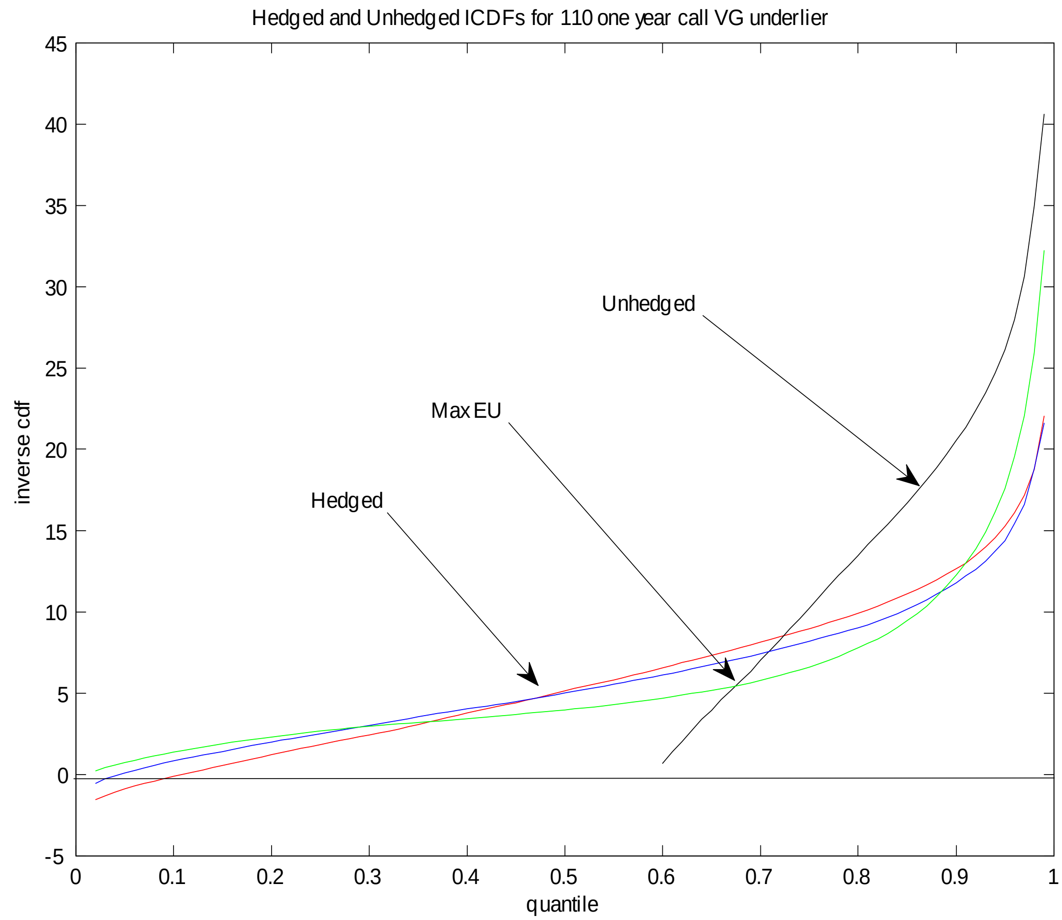

Figure 5: Graph of the inverse cdf for an unhedged and hedged call for an underlying VG process. The red curve is for a quarterly hedge minimizing the ask price while the blue curve is a monthly hedge minimizing the capital required. The green curve is from maximizing exponential expected utility with unit risk aversion.

Gamma Adjusted Deltas in Left Skewed Markets

- We consider here the hedging of quadratic exposures with a view to minimizing the capital charge for the residual risk exposure.

- Suppose the target cash flow $c(S)$ to be hedged over a time interval of size h is quadratic in the stock price with

$$c(S) = \delta(S - S_0) + \frac{\gamma}{2}(S - S_0)^2.$$

- Consider a hedge position θ in the stock with a residual cash flow of

$$\begin{aligned} r(S) &= \delta(S - S_0) + \frac{\gamma}{2}(S - S_0)^2 - \theta(S - S_0) \\ &= \frac{\gamma}{2}(S - S_0)^2 - (\theta - \delta)(S - S_0). \end{aligned}$$

- Suppose the underlying stock price motion $x = S - S_0$ has distribution $F(x)$ and the objective is to minimize the capital required defined by the difference between the ask and bid prices computed using the concave distortion $\mathbf{\Psi}$.

- To evaluate this capital we need to determine the distribution function of the residual cash flow as both the bid and ask prices are functions of this distribution function.

- Let C denote the residual cash flow. The probability $F_C(v)$ that $C \leq v$ is the probability that
$$\frac{\gamma}{2}x^2 - (\theta - \delta)\, x \leq v.$$
The minimum cash flow occurs at
$$x = \frac{\theta - \delta}{\gamma}$$
and is equal to
$$-\frac{(\theta - \delta)^2}{2\gamma}$$

- Hence

$$F_C(v) = 0 \text{ for } v \le -\frac{(\theta-\delta)^2}{2\gamma}.$$

- For $v > -\frac{(\theta-\delta)^2}{2\gamma}$ we solve for x the equation

$$\frac{\gamma}{2}x^2 - (\theta-\delta)\,x - v = 0$$

- with

$$x = \frac{\theta-\delta}{\gamma} \pm \sqrt{\left(\frac{\theta-\delta}{\gamma}\right)^2 + \frac{2v}{\gamma}}$$

- and the probability that $C \le v$ equals the probability that $S - S_0$ lies between the two solutions.

- We then have that

$$F_C(v) = F\left(\frac{\theta-\delta}{\gamma} + \sqrt{\left(\frac{\theta-\delta}{\gamma}\right)^2 + \frac{2v}{\gamma}}\right) - F\left(\frac{\theta-\delta}{\gamma} - \sqrt{\left(\frac{\theta-\delta}{\gamma}\right)^2 + \frac{2v}{\gamma}}\right)$$

- We define

$$\eta = -\frac{\theta-\delta}{\gamma}$$

and write

$$F_C(v) = F\left(-\eta + \sqrt{\eta^2 + \frac{2v}{\gamma}}\right) - F\left(-\eta - \sqrt{\eta^2 + \frac{2v}{\gamma}}\right)$$

- The lower bound is now

$$-\frac{\gamma\eta^2}{2}.$$

- Now $-C \leq \gamma\eta^2/2$ and the distribution function for $-C$ is $\Pr(-C \leq v)$ and

$$F_{-C}(v) = 1 \text{ for } v \geq \gamma\eta^2/2$$

- for $v < \gamma\eta^2/2$ we have that

$$\begin{aligned} F_{-C}(v) &= F\left(-\eta - \sqrt{\eta^2 - \frac{2v}{\gamma}}\right) \\ &\quad +1 - F\left(-\eta + \sqrt{\eta^2 - \frac{2v}{\gamma}}\right) \end{aligned}$$

- It follows that the ask price is

$$a = -\int_{-\infty}^{\frac{\gamma\eta^2}{2}} v d\Psi \begin{pmatrix} F\left(-\eta - \sqrt{\eta^2 - \frac{2v}{\gamma}}\right) \\ +1 - F\left(-\eta + \sqrt{\eta^2 - \frac{2v}{\gamma}}\right) \end{pmatrix}$$

- and the bid price is

$$b = \int_{-\frac{\gamma\eta^2}{2}}^{\infty} v d\Psi \begin{pmatrix} F\left(-\eta + \sqrt{\eta^2 + \frac{2v}{\gamma}}\right) \\ -F\left(-\eta - \sqrt{\eta^2 + \frac{2v}{\gamma}}\right) \end{pmatrix}$$

- Hence the capital is

$$
\begin{aligned}
c &= -\int_{-\infty}^{\frac{\gamma\eta^2}{2}} v d\Psi \begin{pmatrix} F\left(-\eta - \sqrt{\eta^2 - \frac{2v}{\gamma}}\right) \\ +1 - F\left(-\eta + \sqrt{\eta^2 - \frac{2v}{\gamma}}\right) \end{pmatrix} \\
&\quad - \int_{-\frac{\gamma\eta^2}{2}}^{\infty} v d\Psi \begin{pmatrix} F\left(-\eta + \sqrt{\eta^2 + \frac{2v}{\gamma}}\right) \\ -F\left(-\eta - \sqrt{\eta^2 + \frac{2v}{\gamma}}\right) \end{pmatrix}
\end{aligned}
$$

- Now we make the change of variable

$$
w = \frac{v}{\gamma}
$$

- to observe that

$$
c = -\gamma \begin{pmatrix} \int_{-\infty}^{\frac{\eta^2}{2}} w d\Psi \begin{pmatrix} F\left(-\eta - \sqrt{\eta^2 - 2w}\right) \\ +1 - F\left(-\eta + \sqrt{\eta^2 - 2w}\right) \end{pmatrix} \\ + \int_{-\frac{\eta^2}{2}}^{\infty} w d\Psi \begin{pmatrix} F\left(-\eta + \sqrt{\eta^2 + 2w}\right) \\ -F\left(-\eta - \sqrt{\eta^2 + 2w}\right) \end{pmatrix} \end{pmatrix}
$$

- For the optimal value for η we just need to minimize the expression for unit γ or we need to minimize

$$
\begin{aligned}
\Lambda(\eta) \;=\; & -\int_{-\infty}^{\frac{\eta^2}{2}} w d\Psi \begin{pmatrix} F\left(-\eta-\sqrt{\eta^2-2w}\right) \\ +1-F\left(-\eta+\sqrt{\eta^2-2w}\right) \end{pmatrix} \\
& -\int_{-\frac{\eta^2}{2}}^{\infty} w d\Psi \begin{pmatrix} F\left(-\eta+\sqrt{\eta^2+2w}\right) \\ -F\left(-\eta-\sqrt{\eta^2+2w}\right) \end{pmatrix}
\end{aligned}
$$

  and this expression just depends on the underlying stock price distribution.

- One may therefore determine the optimal η from the risk neutral distribution of the stock and then we hedge with gamma adjustment using the delta position

$$\theta = \delta - \eta\gamma.$$

- We illustrate the function Λ as a function of η for the variance gamma model calibrated to options on the SPX on July 15 2010.

- For weekly monitoring with VG parameters

  $.2196, .3317, -.3596$ we graph the unit gamma capital as a function of the adjustment factor η. The minimum in this case occurs at $\eta = 4.6071$.

- We observe the basic intuition underlying the gamma adjustment for delta hedging when markets are skewed downwards.

- Residual risks, even if they are symmetric are not symmetrically priced and since the downside exposure is more expensive the optimal delta should be adjusted downwards in the presence of some gamma risk.

- We now take up issues of dynamic hedging in incomplete markets.

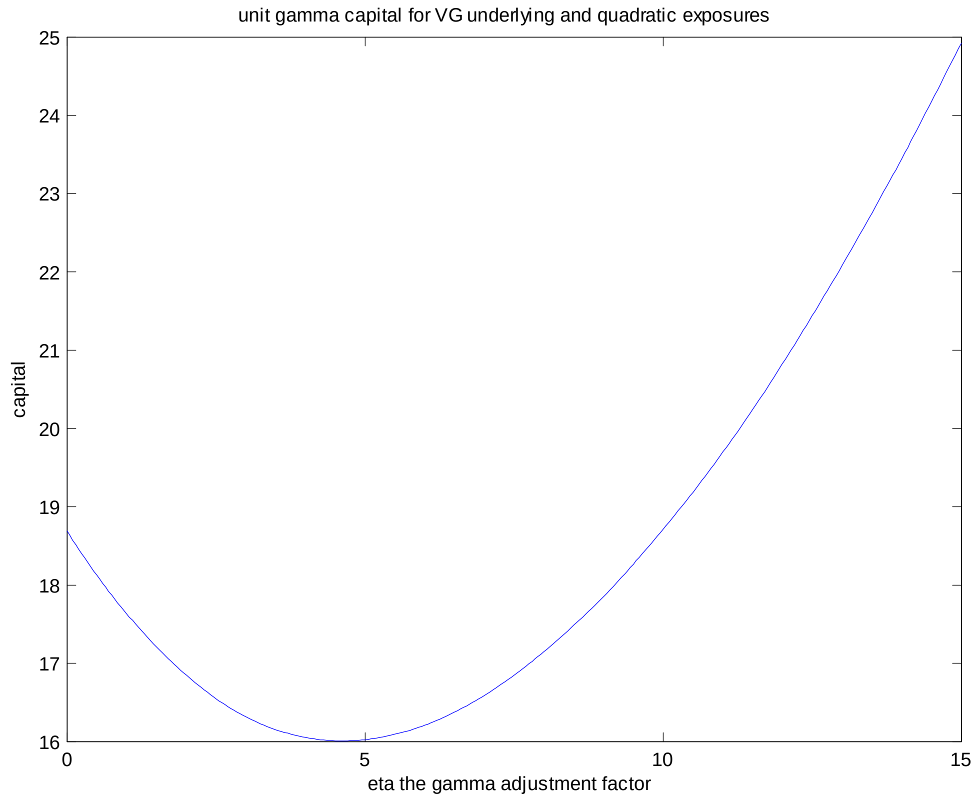

Figure 6: Capital as a function of the gamma adjustment factor.

A multi period model

- We now consider dynamic models for capital and the implementation of hedges that minimize the capital required to hold the remaining risk or uncertainty that is yet to be resolved in a trade.

- This requires a dynamic definition of bid and ask prices from which we will get our dynamic definition of capital.

- The hedges are then chosen to successively minimize this dynamically constructed measure of capital.

- With a view to understanding dynamically consistent sequences of bid and ask prices we consider a discrete time multiperiod model with dates $t = \{0, 1, 2, \cdots, T\}$.

- For our construction of dynamically consistent sequences of bid and ask prices we follow Cohen and Elliott (2010).

- We therefore suppose the underlying risk is that of a finite state Markov chain X_t, for $X_0 = 0$ and $t = 1, 2, \cdots, T$ with

$$X_t \in \{e_1, e_2, \cdots e_N\}.$$

  The process X may be represented as

$$X_t = E[X_t | \mathcal{F}_{t-1}] + M_t$$

  where M is a martingale difference process.

- We shall construct our dynamically consistent sequence of bid and ask prices as nonlinear expectations using Theorem 6.1 of Cohen and Elliott (2010) as solutions of a backward stochastic difference equation for a suitably chosen driver $F(\omega, t, Y, Z)$.

Non Linear Expectations

- In the context of discrete time Markov chains Cohen and Elliott (2010) have defined dynamically consistent translation invariant nonlinear expectation operators $\mathcal{E}(.|\mathcal{F}_t)$ defined on the family of subsets $\left\{\mathbb{Q}_t \subset L^2(\mathcal{F}_T)\right.$

- Our dynamic bid and ask prices will be examples of such operators.

- For completeness we recall the definition of an $\mathcal{F}_t-$consistent nonlinear expectation for$\{\mathbb{Q}_t\}$.

- A system of operators

$$\mathcal{E}(.|\mathcal{F}_t) : L^2(\mathcal{F}_T) \to L^2\left(\mathcal{F}_t\right),\ 0 \le t \le T$$

  is an $\mathcal{F}_t-$consistent nonlinear expectation for$\{\mathbb{Q}_t\}$ if it satisfies the following properties:

- 1. For $Q, Q' \in \mathbb{Q}_t$, if $Q \geq Q'$ $\mathbb{P}-a.s.$ componentwise, then

  $$\mathcal{E}(Q|\mathcal{F}_t) \geq \mathcal{E}(Q'|\mathcal{F}_t)$$

  $\mathbb{P}-a.s.$ componentwise, with for each i,

  $$e_i\mathcal{E}(Q|\mathcal{F}_t) = e_i\mathcal{E}(Q'|\mathcal{F}_t)$$

  only if $e_iQ = e_iQ'$ $\mathbb{P}-a.s.$

- 2. $\mathcal{E}(Q|\mathcal{F}_t) = Q$ $\mathbb{P}-a.s.$ for any $\mathcal{F}_t-$measurable Q.

- 3. $\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)$ $\mathbb{P}-a.s.$ for any $s \leq t$

- 4. For any $A \in \mathcal{F}_t$, $\mathbf{1}_A\mathcal{E}(Q|\mathcal{F}_t) = \mathcal{E}(\mathbf{1}_AQ|\mathcal{F}_t)$ $\mathbb{P}-a.s.$

- Furthermore the system of operators is dynamically translation invariant if for any $Q \in L^2(\mathcal{F}_T)$ and any $q \in L^2(\mathcal{F}_t)$,

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + q.$$

Backward Stochastic Difference Equations

- Such dynamically consistent translation invariant nonlinear expectations may be constructed from solutions of Backward Stochastic Difference Equations.

- These are equations to be solved simultaneously for processes Y, Z where Y_t is the nonlinear expectation and the pair (Y, Z) satisfy

  $$Y_t - \sum_{t \le u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \le u < T} Z_u M_{u+1} = Q$$

  for a suitably chosen adapted map $F : \Omega \times \{0, \cdots, T\} \times \mathbb{R}^K \times \mathbb{R}^{K \times N} \to \mathbb{R}^K$ called the driver and for Q an $\mathbb{R}^K$ valued $\mathcal{F}_T$ measurable terminal random variable.

- We shall work generally with the case $K = 1$. For all t, (Y_t, Z_t) are $\mathcal{F}_t$ measurable.

- For a translation invariant nonlinear expectation the driver F must be independent of Y and must satisfy the normalisation condition $F(\omega, t, Y_t, 0) = 0$.

- For a continuous time analysis one has to involve backward stochastic differential equations El Karoui and Huang (1997), Cohen and Elliott (2010b).

- For the construction of a translation invariant nonlinear expectation the driver will be independent of Y.

$$F_b(\omega, t, Y_t, Z_t) = \int_{-\infty}^{\infty} x d\Psi(\Theta_t(x))$$

  where

$$\Theta_t(x) = \Pr\left(Z_t M_{t+1} \leq x\right).$$

- While for the ask price we define

$$F_a(\omega, t, Y_t, Z_t) = -\int_{-\infty}^{\infty} x d\Psi\left(1 - \Theta_t(-x)\right).$$

- One may obtain dynamically consistent sequences of bid and ask prices as nonlinear expectations by solving the backward stochastic difference equations associated with these drivers. In fact we have for $X \in L^2(\mathcal{F}_T)$ that

$$\begin{aligned} \mathcal{B}_t(X) &= Y_t^b \\ \mathcal{A}_t(X) &= Y_t^a \end{aligned}$$

where

$$Y_t^b = Y_{t+1}^b + F^b(\omega, t, Y_t^b, Z_t^b) - Z_t^b M_{t+1}$$

- Taking t conditional expectations we see that

$$\begin{aligned} Y_t^b &= E_t\left[Y_{t+1}^b\right] + F^b(\omega, t, Y_t^b, Z_t^b) \\ &= E_t\left[Y_{t+1}^b\right] + \int_{-\infty}^{\infty} x d\Psi(\Theta_t^b(x)) \\ \Theta_t^b(x) &= \Pr\left(Z_t^b M_{t+1} \leq x\right) \\ &= \Pr\left(Y_{t+1}^b - E_t\left[Y_{t+1}^b\right] \leq x\right), \end{aligned}$$

for the bid processand similarly for the ask process

$$
\begin{aligned}
Y_t^a &= E_t\left[Y_{t+1}^a\right] + F^a(\omega, t, Y_t^b, Z_t^b) \\
&= E_t\left[Y_{t+1}^a\right] - \int_{-\infty}^{\infty} x d\Psi(1 - \Theta_t^a(-x)) \\
\Theta_t^a(x) &= \Pr\left(Z_t^a M_{t+1} \le x\right) \\
&= \Pr\left(Y_{t+1}^a - E_t\left[Y_{t+1}^a\right] \le x\right).
\end{aligned}
$$

These equations constitute our recursion for the bid and ask price sequences.

- We observe from these equations that one may make the computations even when we do not have a finite state Markov chain and do not identify the process for $Z_t^u, u = a, b$ and we just use the recursions to compute the values for Y_t^u, $u = a, b$ and in our applications this is what we will do.

Time Stepping Tenor

- We now consider this recursion in a two period setting to help fine tune the details of the procedure.

- Consider then a risk A_2 representing an asset with uncertainty resolved at time 2.

- We shall compare treating the two periods as a single period and computing the bid price with the iteration of two steps of one period.

- For simplicity we suppose that A_2 evolves as a Lévy martingale so that

$$\begin{aligned} A_2 &= A_1 + X_2 \\ A_1 &= A_0 + X_1 \end{aligned}$$

  where X_1, X_2 are independent draws from a single zero mean Lévy process at unit time.

- If we apply our procedure for the two periods in one step we obtain for the bid price

$$B = A_0 + b(A_2 - A_0)$$

where we use for $b(X)$ the distorted expectation of the zero mean random variable X.

- We know that $b(A_2 - A_0) < 0$ and hence that $B < A_0$.

- Now consider the computation in two steps of one period. This will give at time 1 the value

$$b_1 = A_1 + b\,(A_2 - A_1)$$

and we know from the stationarity the law of the increment

$$A_2 - A_1 = X_2$$

that

$$\begin{aligned} b(A_2 - A_1) &= b(X_2) \\ &= c \end{aligned}$$

It follows that

$$b_1 = A_1 + c$$

- Going one more time step we get

$$\begin{aligned} b_0 &= A_0 + c + b(X_1) \\ &= A_0 + 2c \end{aligned}$$

- Now

  $$b(A_2 - A_0) = b(X_1 + X_2) \geq b(X_1) + b(X_2) = 2c$$

  and hence we get that

  $$B \geq b_0.$$

- We get a better bid price going in one step compared to the two short steps of one period.

- However if we think of the penalty $b(A_2 - A_0)$ as a capital charge it is for two periods while the two one step charges are for one period. The correct comparison is then between

  $$2b(A_2 - A_0) \text{ and } 2c$$

  and we could well have

  $$2b(A_2 - A_0) < 2c$$

  making the one step procedure the one with the higher bid.

- These considerations suggest that we take the timing factor into account in defining the penalty.

- Note further that the continuous time penalty is in the form

$$\int_s^T F(\omega, t, Y_t, Z_t)dt.$$

- For a time step of h we revise our recursion to

- $$\begin{aligned} Y_t^b &= E_t\left[Y_{t+1}^b\right] + F^b(\omega, t, Y_t^b, Z_t^b)h \\ &= E_t\left[Y_{t+1}^b\right] + h\int_{-\infty}^{\infty} x d\Psi(\Theta_t^b(x)), \end{aligned}$$

  for the bid processand similarly for the ask process

$$\begin{aligned} Y_t^a &= E_t\left[Y_{t+1}^a\right] + F^a(\omega, t, Y_t^b, Z_t^b)h \\ &= E_t\left[Y_{t+1}^a\right] - h\int_{-\infty}^{\infty} x d\Psi(1 - \Theta_t^a(-x)). \end{aligned}$$

Dynamic Hedging for Capital Conservation

- Given that the bid and ask price processes are nonlinear expectations we have that both the profit and the capital are nonlinear expectations.

- We now take for the stock an underlying exponential Lévy process with a given risk neutral law.

- For our time step we shall work with one week and then we report on dynamic hedging where each week the hedge position is chosen to minimize the capital reserve to be held against the remaining uncertainty yet to be resolved.

- We shall also report on the Black Scholes deltas computed at the implied volatility and the capital reserves to be held against such a hedging strategy.

- We shall observe that hedging at Black Scholes implied volatilities is considerably more expensive than the minimum capital hedge.

- We shall hedge a one year 120 call and a one year 80 put and in each case we compute the profit, the capital, the rate of return and the leverage as a function of the level of the spot at week's end.

- We present graphs of these functions at the end of 3, 6, and 9 months along with the bid, ask, mid, and risk neutral prices initially and the asscoiated profit, capital, return and leverage all computed using our dynamically consistent recursion.

- In addition we present graphs for the minimum capital delta as a function of the spot price also at 3, 6 and 9 months.

- The initial spot price is 100 the interest rate is 0.0379 the dividend yield is 0.0229 and the stock price dynamics calibrated to one year SPX options on July 15 2010 are given by the variance gamma process with $\sigma = 0.2397$, $\nu = 2.2765$ and $\theta = -0.2109$.

- For each of 52 week ends we determine a nonuniform grid between the lower and upper stock price with a probability of a tenth of a percent of being outside this interval.

- The grid is constructed with 100 levels for the stock price at each week end.

- The grid construction is based on the procedure described in Mijatović and Pistorius (2010).

- We know the payoff at week 52 and all prices, bid, ask and risk neutral equal this payoff at week 52.

- For each week end from week 51 to week 1 we work back recursively computing the bid, ask and risk neutral expectation at each grid point by simulating the process forward for one week and then computing the required expactations and distorted expectations.

- The distortion used is minmaxvar and the stress level is 0.25.

- At each grid point we numerically solve an optimization problem to minimize the capital reserve and determine the delta position in the stock for this grid point.

- The final output consists of three matrices of size 51 by 100 that contain the bid, ask and risk neutral expectation at each grid point when there is no hedge.

- In the presence of an optimized delta hedge we have one more matrix and this is capital minimizing delta if the stock is at the particular grid level and the particular week end.

- Finally we simulate for one week the stock from the initial start level of 100 to compute the initial bid, ask and risk neutral price.

- We then get the initial profit, capital, return and leverage.

- The one step ahead bid ask and risk neutral values in the simulation are interpolated from the stored grid point values as we work back through the recursion.

- The results for the unhedged and minimum capital hedges are presented in two subsections.

Unhedged Results

- We first report on the initial values of the variables of interest for the 80 put and the 120 call in the absence of any dynamic hedging.

- The variables of interest are the bid, ask, the mid price, rne the risk neutral expectation, scale the expected absolute deviation from the median, prf the profit, cap the capital reserve,ret the return and lev the leverage provided. These are presented in Table

1.

Table 1
Initial Unhedged Values

Variable	80 Put	120 Call
bid	5.0738	1.6162
ask	5.5572	1.8419
mid	5.3155	1.7291
rne	5.1966	1.6988
scale	0.1259	0.0670
prf	0.1189	0.0303
cap	0.4834	0.2256
ret	0.2460	0.1343
lev	0.2605	0.2971

- We next present graphs of the profit, capital, return and leverage at 3, 6 and 9 months as a function of the level of the spot price.

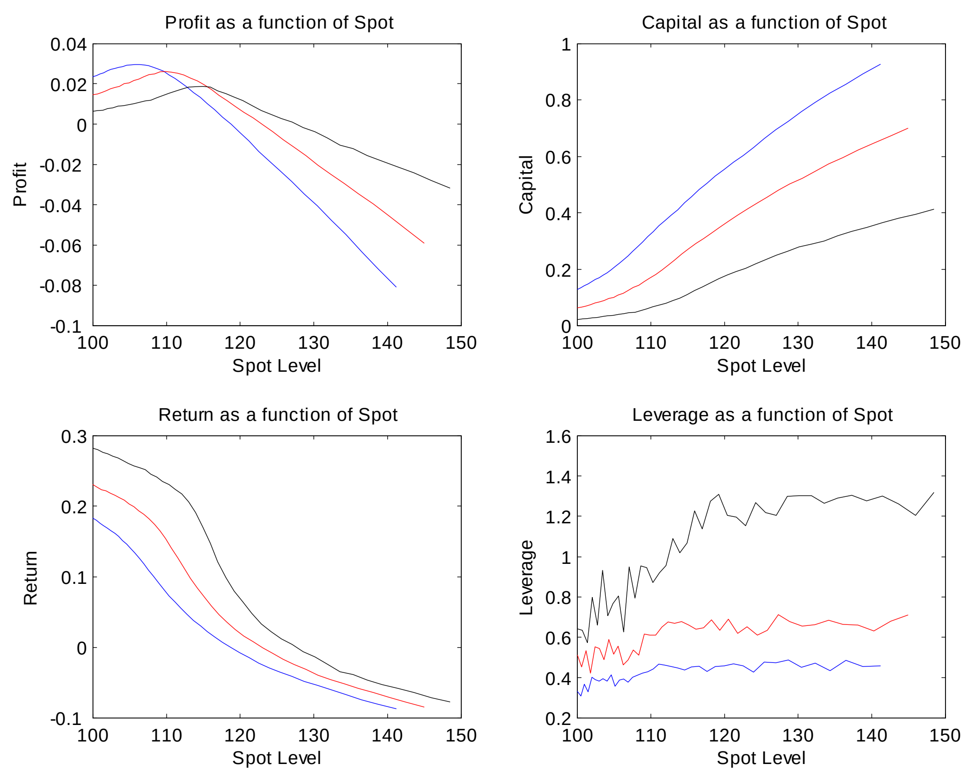

Figure 7: 120 one year Call Profits, Capital Return and Leverage as a function of the spot. At three months in blue, six months in red and 9 months in black.

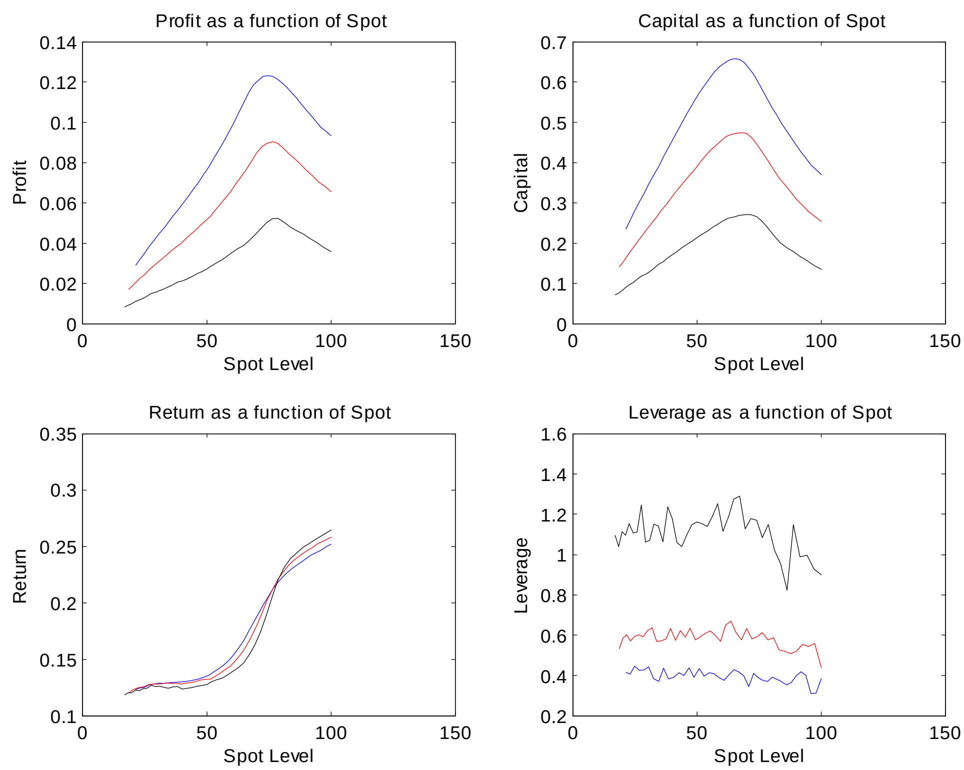

Figure 8: 80 one year put profit, capital, return and leverage as a function of the spot. At three months in blue, six months in red and 9 months in black.

Minimum capital hedge results

- We report next on the results on implementing at week end a delta position that minimizes the post hedge reserve capital charge against the remaining uncertainty. Again we first present the results on the variables of interest at the initial date in Table 2.

Table 2

Minimum Capital Delta Hedge

Variable	80 Put	120 Call
bid	4.6723	1.5221
ask	4.9465	1.6834
mid	4.8094	1.6028
rne	4.7471	1.5644
scale	0.0681	0.0476
prf	0.0623	0.0384
cap	0.2741	0.1613
ret	0.2273	0.2382
lev	0.2485	0.2948

- We next present graphs of the profit, capital, return and leverage at 3, 6 and 9 months for the 120 one year Call and the 80 one year Put as functions of the level of the spot.

Additionally we present a graph for the delta positions taken to hedge the remaining risk with a view to minimizing reserve capital charges.

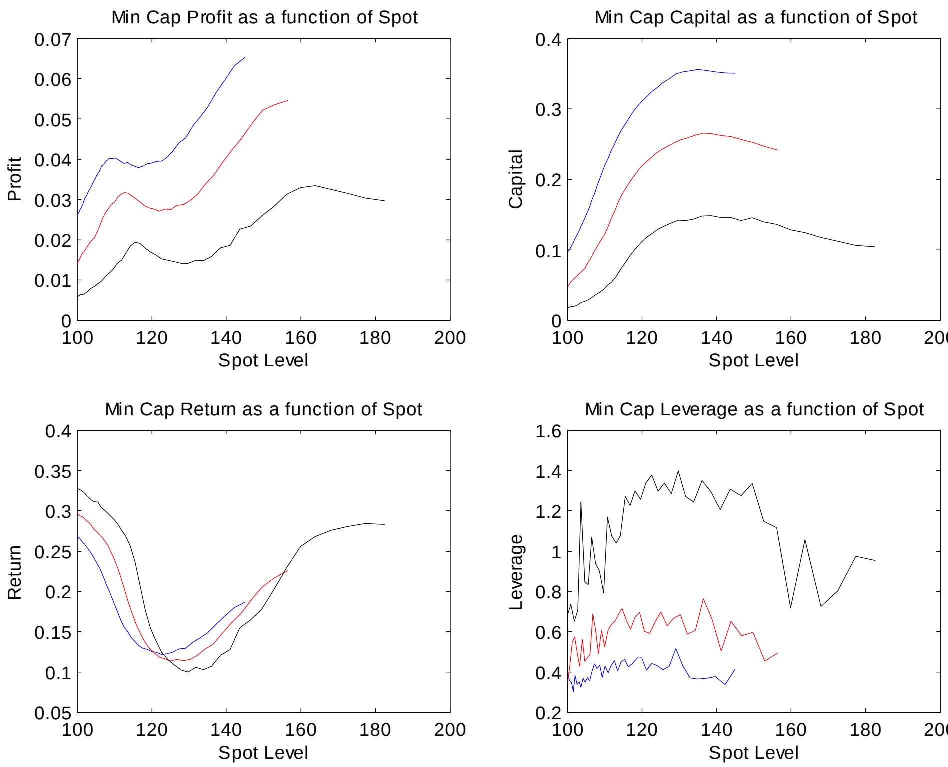

Figure 9: Profit, Capital Return and Leverage with a minimum capital dynamic hedge as functions of the spot for a one year 120 call. At three months in blue, six months in red and nine months in black.

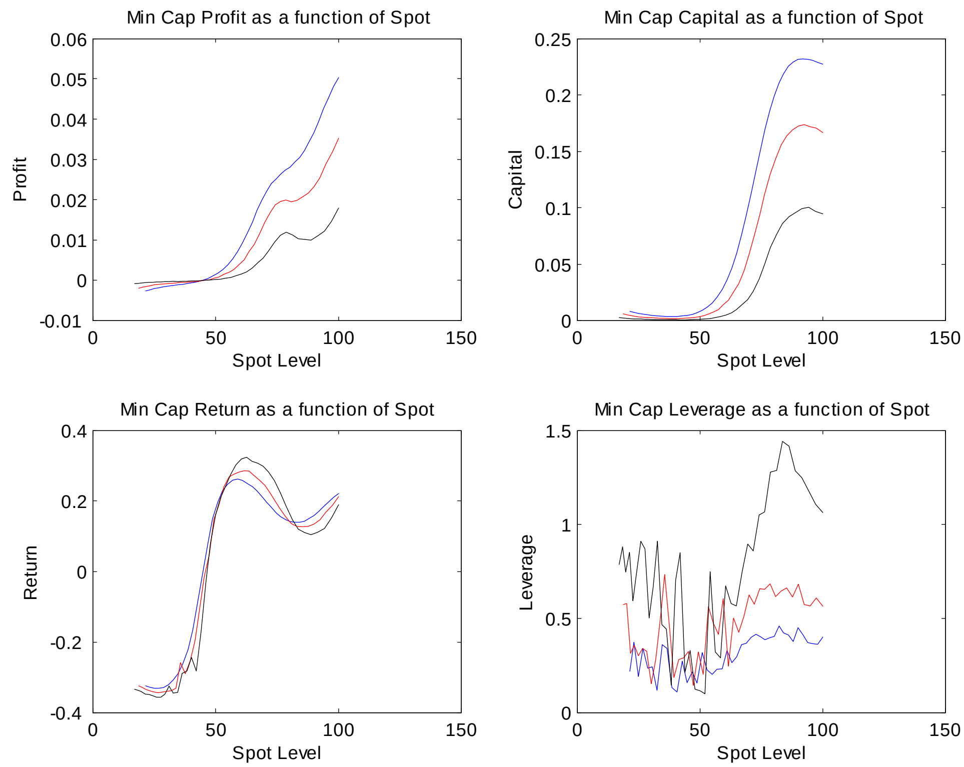

Figure 10: Profit, Capital Return and Leverage with a minimum capital dynamic hedge as functions of the spot for a one year 80 put. At three months in blue, six months in red and nine months in black.

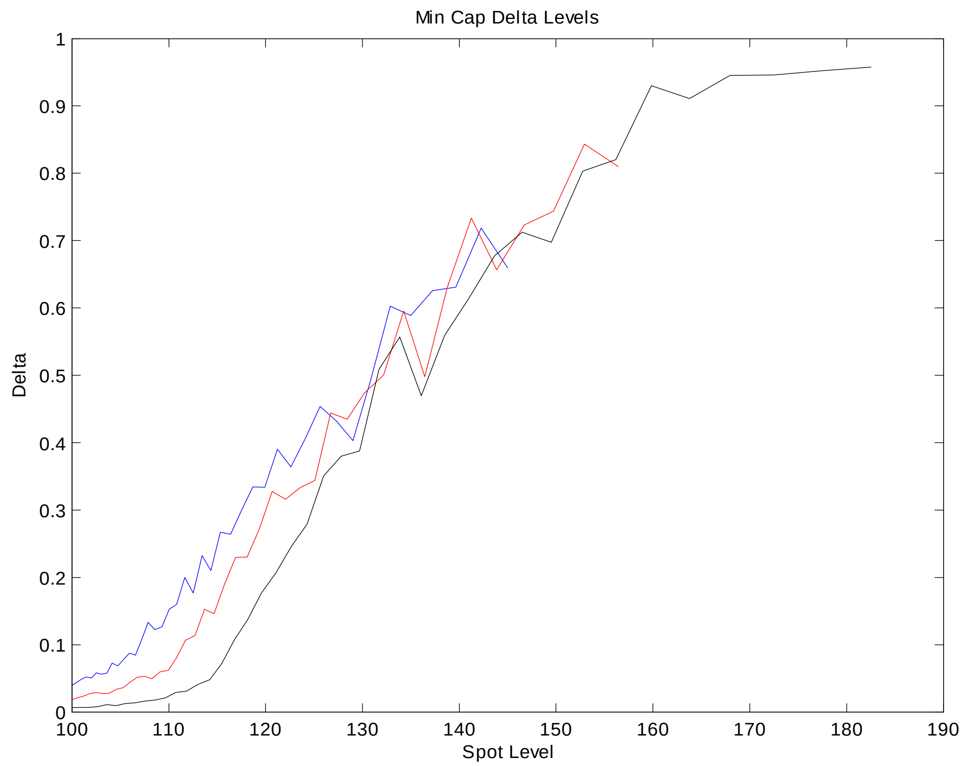

Figure 11: Minimum capital delta positions as functions of the spot for the 120 one year call. At three months in blue, 6 months in red and 9 months in black.

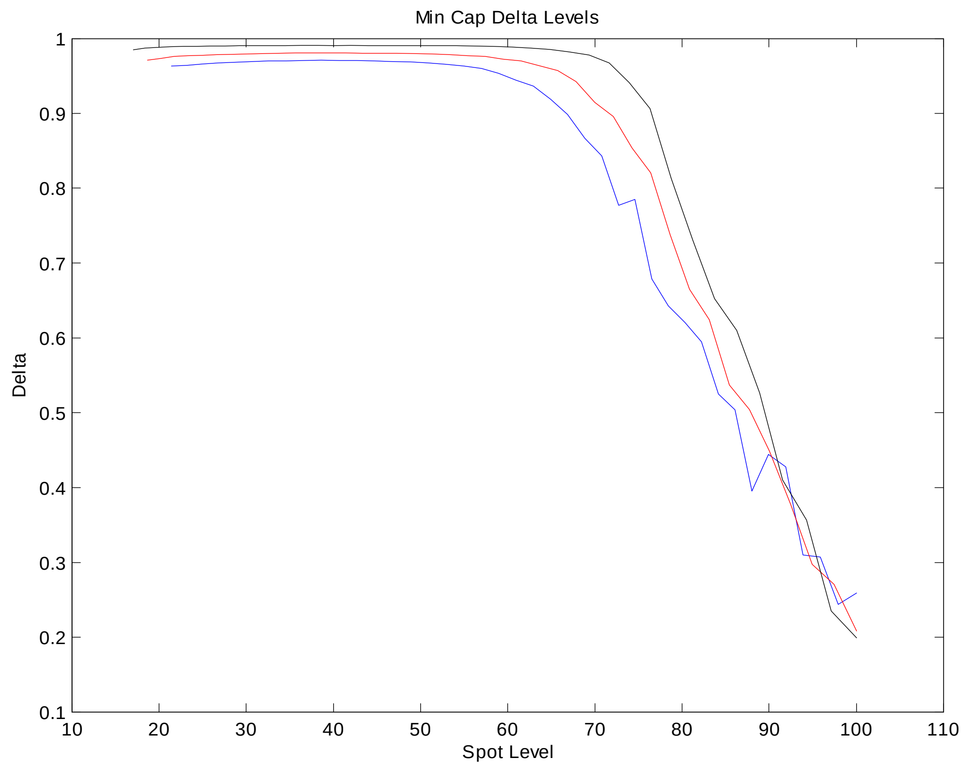

Figure 12: Minimum capital delta positions as functions of the spot for the 80 one year put. At three months in blue, 6 months in red and 9 months in black.

- We also computed the dynamic levels of profit, capital, return and leverage for both the 120 one year call and 80 one year put using Black Scholes deltas computed at the implied volatility.

- The Black Scholes call deltas were considerably higher and the put deltas were lower in absolute value than what is observed under the optimal minimum capital hedge.

- We report in Table 3 the initial values for hedging at Black Scholes implied. We observe that the capital levels are sufficiently higher for the Black Scholes delta hedge.

Table 3

Black Scholes Delta Hedge

Variable	80 Put	120 Call
bid	3.8532	2.3318
ask	4.8601	3.8047
mid	4.3566	3.0683
rne	4.3386	2.9177
scale	0.1478	0.0904
prf	0.0180	0.1506
cap	1.0069	1.4728
ret	0.0179	0.1023
lev	0.1468	0.0641

Conclusion

- The theory of two price markets as recently developed in Cherny and Madan (2010) provides closed forms for bid and ask prices for a state contingent cash flow based on hedged residual cash flows being acceptable to markets.

- The concept of acceptability used was defined in Artzner, Delbaen, Eber and Heath (1999) and we employ an operational form based on concave distortions introduced in Cherny and Madan (2009).

- It is argued that when markets are viewed as passive impersonal counterparties sharing the spreads they earn with their trading counterparties then the profit on a trade may be seen as difference between the mid quote and the risk neutral expectation. This theoretical perspective permits the definition of upfront profits on trades.

- Furthermore, the cash reserve needed to unwind a sale to market at bid is seen to be the difference between the ask and the bid prices. We therefore also have a foundation for capital reserves and hence both leverage and rates of return on trades.

- New expressions are developed for the bid and ask prices in terms of the sensitivity of the inverse distribution function to the quantile level.

- This sensitivity turns out to be a measure of risk exposure at the quanitle level and both profits and capital are quantile based charges integrated over the quantiles for this risk exposure.

- The profit charge is positive for gain quantiles above the median and is negative for loss quantiles below the median.

- Capital charges are positive at all quantiles but fall to zero in the extremes and are highest near the median.

- The theory is illustrated on unhedged exposures in the Black Merton Scholes model, followed by variance swaps, call options for variance gamma underliers and capital minimizing revisions for delta hedging in left skewed markets.

- The competitive pressures to minimize ask prices and maximize bid translate into market objective functions to economize capital. It is observed that the maximization of expected utility as a proxy criterion for the market may result in uneconomic capital levels and hedges should be designed to economize capital.

- We present first in the static case and then in the dynamic case the argument for adjusting Black Scholes implied call deltas downwards for a gamma exposure in a left skewed market when the objective for the hedge is the conservation of capital.

- For similar reasons the absolute value of put deltas should be increased to accomodate the costs imposed on residual risks by the market skewness.

- The gamma adjustment factor in the static case is shown to be just a function of the risk neutral distribution that can be calibrated from the option surface.

- This could be periodically adjusted as skewness level are observed to move around.

- In the dynamic case one may precompute at the date of trade initiation a matrix of delta levels as a function of the underlying for the life of the trade and subsequently one just has to look up the matrix for the hedge.

- Additionally the matrix could be periodically recalibrated.

- Also constructed are matrices for the capital reserve, the profit, leverage and rate of return remaining in the trade as a function of the spot at a future date in the life of the trade.

- These could be periodically recalibrated using the procedures outlined.

- The dynamic computations constitute an application of the theory of nonlinear expectations as described in Cohen and Elliott (2010) for a finite state stochastic difference equation framework.

- Bid and ask prices are computed as nonlinear expectations using a penalty driver for the periodic risk that comes from our static model.

- The drivers used are given by our static model of concave distortions.