Two Price Economies

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Motivation

- The market for liquid assets as a counterparty in classical financial economics is viewed as accepting any amount of a financially traded asset at the going market price.

- The specific asset describes the cash flow accessed and for this cash flow there is just one price at which it one may buy or sell any amount desired by market participants who are usually seen as small and atomistic relative to the size of the market.

- Hence we have the law of one price prevailing in the classical model.

Two Price Markets

- We maintain this vision of the market as a counter-party.

- However, when modeling markets we now permit just one difference.

- This difference is to allow for the price to depend on the direction of trade.

- There are then two prices, one for buying from the market and another for selling to the market.

- The former may be termed the ask price while the latter is the bid price.

Structured Products

- There is a large segment of financial markets that creates financial products using the liquid markets for hedging risks and in these markets there are typically two prices, one for buying and the other for selling.

- The differences between these two prices can be quite large.

- The differences reflect the very real and substantial costs of holding unhedgeable risks in incomplete markets.

- To the extent one may hedge exposures using liquid markets the spreads may be decreased but not necessarily otherwise.

Related Work

- The motivations for our spreads in markets are more closely related to those in Bernardo and Ledoit (2000), Cochrane and Saa-Requejo (2000), Černý and Hodges (2000), Carr, Geman and Madan (2001) and Jaschke and Kuchler (2001).

- Relative to these papers the current contribution is to develop closed form formulas for bid and ask prices in general, and for put and call options in particular.

- These formulas are a consequence of our model for markets as a counterparty coupled with the parametrization of indices of acceptability accomplished in Cherny and Madan (2009).

Markets as a Counterparty and the Implied Theoretical Bid and Ask Prices

- We begin with the liquid market as a counterparty in classical finance.

- The space of traded cash flows we take to be all bounded random variables on a base probability space $(\Omega, \mathcal{F}, P)$.

- In the best possible case we have complete markets and a unique pricing measure Q equivalent to P.

- For any cash flow C with a nonzero price w we may form the difference $X = C - w$ that now has zero price and we may buy from or sell to market any amount of the cash flow X.

- In addition we may also buy at a higher price or sell for a lower price making the set of cash flows marketed in liquid markets, all cash flows with the property that $E^Q(X) \geq 0$.

Marketed Cash Flows

- The set of such classically marketed cash flows in liquid markets is well understood in finance and amounts to the collection of all positive alpha trades.

- From the perspective of incomplete markets where exact replication at a determinate cost of replication is not possible and residual risk must be held this collection of marketed cash flows is too wide and unrealistic.

- More exactly, if a cash flow C is priced at w and the market buys the cash flow $X = C - w$ then in incomplete markets the price w would be a bid price.

- If the market were instead to sell $-X = w - C$ then this would not be possible and one would expect to pay a spread s with the market willing to accept $(w + s) - C$.

- With market incompleteness the set of cash flows just marketable are no longer closed under negation.

Marketed Cash Flow Properties

- We note that the set of classically marketed cash flows is convex, and linear combinations of marketed cash flows are marketed.

- The set of classically marketed cash flows is also closed under scaling as we may scale positions given that in the classical model we may trade any amount.

- Our more general model for the marketed cash flows preserves these two properties and hence the set of marketed cash flows is still a convex cone.

- Furthermore, as all nonnegative cash flows are certainly marketed.

Marketed Cash Flows Formalized

- The set of marketed cash flows is thus a convex cone containing the nonnegative cash flows.

- It follows from Artzner, Delbaen, Eber and Heath (1999) that the set of marketed cash flows must consist of all cash flows X satisfying

$$E^Q[X] \geq 0 \text{ for any } Q \in \mathcal{D}$$

  for some convex set $\mathcal{D}$ of probability measures equivalent to P.

- We suppose in general that $\mathcal{D}$ contains a risk neutral measure and the set of marketed cash flows is smaller than the set of classically marketed cash flows.

Marketed Cash Flows over time

- Our model for the market is defined by specifying the class of marketed cash flows and we do not expect this set to be constant over time.

- In times of crisis when attitudes to holding residual risks change the cone may well contract, while it may get more lenient in boom times.

- It is therefore useful to allow the cone to vary in size with a parameter that may calibrate the stress level of the market with the cone contracting as stress levels increase.

- We recognize that risk attitudes and levels of risk aversion will influence the size of the cone of marketed risks but we do not model explicitly the preferences of participants but the state of the market as reflected in the cone of marketed risks or the size of the set of test measures $\mathcal{D}$.

Parametric Stress Levels

- To accomplish this variation in the set of marketed cash flows we follow Cherny and Madan (2009) and introduce the index of acceptability that allows us to speak of cash flows acceptable at level γ or equivalently marketed at level γ.

- An *index of acceptability* is a map α that associates with each bounded random variable X a number $\alpha(X)$ in the extended half line $[0, \infty]$, that is the level of acceptability of X.

- The map has the following four simple properties.

  - First, if X and Y are acceptable at a level γ, i.e. $\alpha(X)$ and $\alpha(Y)$ exceed γ then so does $\alpha(X + Y)$.

– Second, if X is acceptable at a level γ and Y dominates X then Y is also acceptable at level γ.

– Third, positive scalar multiples of random variables acceptable at a level γ are acceptable at this level.

– Finally, we have the technical condition of closure under convergence in probability for the set of random variables acceptable at a level γ.

Supporting Measures

- There is an inherent relationship between acceptability indices and families of probability measures.

- If α is an acceptability index, then for each $\gamma \geq 0$, there exists a set $\mathcal{D}_\gamma$ of probability measures absolutely continuous with respect to the original probability such that a random variable is acceptability at level γ if and only if it has a positive expectation under each measure from $\mathcal{D}_\gamma$:

$$\alpha(X) \geq \gamma \iff E^Q[X] \geq 0 \text{ for any } Q \in \mathcal{D}_\gamma.$$

- The sets $\mathcal{D}_\gamma$ increase in γ, i.e. $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma'}$ for $\gamma \leq \gamma'$. In terms of the family $(\mathcal{D}_\gamma)_{\gamma \geq 0}$, the index α gets the following description: $\alpha(X)$ is the largest value γ such that the expectation of X is positive with respect to each measure from $\mathcal{D}_\gamma$:

$$\alpha(X) = \sup\left\{\gamma \geq 0 : E^Q[X] \geq 0 \text{ for any } Q \in \mathcal{D}_\gamma\right\}$$

Ask Prices

- We now employ a fixed acceptability level γ for a fixed acceptability index α and consider the problem of pricing the risk of a random variable X, first in the absence of hedging opportunities and then in their presence.

- Consider first the sale or delivery by the market of a terminal cash flow represented by the random variable X.

- We expect that when the market sells a cash flow X it charges a minimal price a, motivated by competition.

- However, the resulting residual cash flow $a - X$ must be α-acceptable at level γ or in the class of level γ marketed cash flows.

- This minimal price is then the *ask price* of X. For $a - X$ to be acceptable at level γ the price a must exceed $E^Q[X]$ for all $Q \in \mathcal{D}_\gamma$ and hence the minimal price is given by

$$\begin{aligned} a_\gamma(X) &= \inf\{a : \alpha(a - X) \geq \gamma\} \\ &= \inf\left\{a : E^Q[a - X] \geq \gamma \text{ for any } Q \in \mathcal{D}_\gamma\right\} \\ &= \sup_{Q \in \mathcal{D}_\gamma} E^Q[X]. \end{aligned}$$

Bid Prices

- When the market buys X for a price b it is $X - b$ that must be acceptable at level γ and the maximal price is

$$b_\gamma(X) = \inf_{Q \in \mathcal{D}_\gamma} E^Q[X]. \tag{1}$$

In particular, we see that the bid price is dominated by the ask price.

Post Hedge Ask

- We now introduce a set of hedging cash flows $\mathcal{H}$.

- The hedging cash flows are those that are obtainable from pursuing different zero-cost hedging strategies, i.e. a random variable $H \in \mathcal{H}$ represents the cash flow received on following a corresponding zero-cost or self financed trading strategy in a collection of liquid assets traded in the financial markets.

- The *ask price* of a cash flow X is now defined as the smallest price such that there exists a hedging strategy with the property that when the market sells X at this price it also pursues a hedging strategy receiving a total residual cash flow that is α-acceptable at level γ.

- Mathematically, the γ level ask price is:

  $a_\gamma(X) = \text{inf}\{a : \text{ there exists } H \in \mathcal{H} \text{ such that } \alpha(a+H-X$

Post Hedge Bid

- Similarly, we define the γ level *bid price* as

  $b_\gamma(X) = \textsf{sup}\{b:$ there exists $H \in \mathcal{H}$ such that $\alpha(-b-H+$

- The *upper* and *lower hedges* $\overline{H}, \underline{H}$ are elements of $\mathcal{H}$ satisfying respectively

$$\alpha(a_\gamma(X) + \overline{H} - X) \geq \gamma,$$
$$\alpha(-b_\gamma(X) - \underline{H} + X) \geq \gamma.$$

Risk Neutral Measures

- We will focus our discussion on ask prices and the upper hedge.

- We assume that $\mathcal{H}$ is a linear space, which effectively means absence of transactions costs, short sale constraints, position limits and the law of one price holds for these assets with trading possible in both directions at the same price.

- In the presence of hedging assets an important role is played by a particular class of measures that we term risk neutral measures. We define the set of risk neutral measures:

$$\mathcal{R} = \{Q : Q \text{ is absolutely continuous with respect to } P \text{ and } E^Q[H] = 0 \text{ for any } H \in \mathcal{H}\}.$$

- We make the assumption that the level γ exceeds the level

$$\gamma_* = \textsf{sup}\{\alpha(H) : H \in \mathcal{H}\}.$$

Role of Assumption

- This assumption is clearly necessary since otherwise we can find a hedge whose level of acceptability exceeds γ, and then by scaling it up we can bring $\alpha(a + H - X)$ to the level γ for any price a.

- In this case, the ask price becomes minus infinity.

Theorem

Theorem 1 **(i)** *The sets* $\mathcal{D}_\gamma$ *and* $\mathcal{R}$ *have nonempty intersection.*

(ii) *The upper price is the maximum of expectations of* X *over the set of pricing kernels from the above intersection:*

$$a_\gamma(X) = \sup_{Q \in \mathcal{D}_\gamma \cap \mathcal{R}} E^Q(X).$$

(iii) *We have*

$$a_\gamma(X) = \inf_{H \in \mathcal{H}} -w_\gamma(H - X).$$

while the set of upper hedges coincides with the set of solutions of this maximization problem.

Remarks

- We remark that the condition $\gamma \geq \gamma_*$ is equivalent to the assertion that there is no hedging strategy $H \in \mathcal{H}$, for which $E^Q(H) \geq 0$ for all $Q \in \mathcal{D}_\gamma$.

- This is almost the same as the condition of No Strictly Acceptable Opportunities in Carr, Geman and Madan (2001).

- Thus, statement (i) is an analogue of the Fundamental Theorem of Asset Pricing in our context. Statement (ii) may also be seen as an extension of the result of Carr, Geman and Madan (2001) providing the dual formulation of the pricing problem.

- Additionally we note that if we enforce the set of marketed assets in the illiquid markets to be smaller than the classical one by ensuring that $\mathcal{D}_\gamma$ contains a risk neutral measure then $\gamma_* = 0$ and we just take $\gamma > 0$ for the cone of assets marketed to an illiquid market.

- In practice the restriction of the searches over risk neutral measures is intuitively sensible.

- This is because when searching for hedging strategies, if we do not have the risk neutrality condition, then the hedging assets become assets to invest in or to take short positions in with a view to earning returns as opposed to hedging.

- The only reason to take a position in hedging asset should be to improve the overall risk profile and not to make some extra expected return.

Parametric Models for Marketed Cash Flows and Closed Forms for Bid and Ask Prices

- Parametric and operational models for cones of marketed cash flows may be constructed when we restrict the property of being marketed or acceptable to be solely a function of the probability law of the risk at hand.

- Hence when testing for acceptability at a level γ for a cash flow X the only information we need is the distribution function F_X of the random variable X.

- In general the market may be concerned with the covariation of potential risks with risks already being held and just the probability law is not a sufficient guide to its market acceptability.

- An assessment based on the probability law may then be seen as a first approximation to be possibly further conditioned by other considerations.

Distortions and Acceptability

- This is a family of concave distribution functions on the unit interval $[0, 1]$. Each function Ψ^γ is zero at zero, unity at unity, increasing and concave in its argument.

- Furthermore for each u, the sequence $\Psi^\gamma(u)$ increases in γ.

- The index $\alpha(X)$ is then the biggest γ such that the expectation of X remains positive when we distort its distribution function by Ψ^γ :

$$\alpha(X) = \sup\left\{\gamma \geq 0 : \int_{-\infty}^{\infty} x d\Psi^\gamma(F_X(x)) \geq 0\right\}.$$

Distorted Expectation and Measure Changes

- One may observe that the distorted expectation

$$\int_{-\infty}^{\infty} x d\Psi^{\gamma}(F_X(x)),$$

  equals the expected value of X under a new probability measure $Q^{\gamma}(X)$, having density $\psi^{\gamma}(F_X(X))$ with respect to the original measure P, where $\psi^{\gamma}(u)$ is the $u-$derivative of $\Psi^{\gamma}(u)$.

- Note that for a concave distortion this new density reweights losses upwards when $F_X(X)$ is near zero and discounts gains when $F_X(X)$ is near unity.

- So acceptability via concave distortions works as follows: for a variable X, we have a family of measures $Q^{\gamma}(X)$, which distort the distribution function of X more and more to the left as γ increases.

- The value $\alpha(X)$ is then the maximal level of distortion such that this distorted expectation of X is still positive. One may think of the level γ as a stress level applied to the cash flow X being tested for acceptability, that is achieved if the stressed expectation is still positive.

Computation of Distorted Expectation

- The distorted expectation is easy to compute numerically once we have the distribution function of X. It is further simplified when we employ the empirical distribution function of a sample $x_1, \cdots x_N$. In this case

$$\int_{-\infty}^{\infty} x d\Psi^{\gamma}\left(F_X(x)\right) = \sum_{n=1}^{N} x_{(n)} \left(\Psi^{\gamma}\left(\frac{n}{N}\right) - \Psi^{\gamma}\left(\frac{n-1}{N}\right)\right)$$

where $x_{(n)}$ are the values x_n sorted in increasing order.

Examples of Distortions

- A particularly simple family of such concave distortions is given by

$$\Psi^{\gamma}(u) = 1 - (1-u)^{1+\gamma},\ u \in [0,1], \gamma \geq 0.$$

- For the corresponding index of acceptability α, the inequality $\alpha(X) \geq \gamma$ for an integer γ is equivalent to the property: expected value of the minimum of γ independent draws from the distribution of X is positive. For this reason this distortion was termed $MINVAR$, as it is related to the expectation of minimal outcomes across draws.

- Distorted expectations are related to expectations under a change of measure with density $\psi^{\gamma}(F_X(X))$ with respect to the original probability P.

- One can observe that for the $MINVAR$ distortion the function ψ^γ is bounded by $1+\gamma$, which means that large losses are not reweighted up to infinitely large levels.

MINMAXVAR

- We consider this to be a drawback and introduce another distortion, which reweights large losses up to infinity and large gains down to zero. It is termed $MINMAXVAR$ and is given by

$$\Psi^{\gamma}(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}, \ u \in [0,1], \gamma \geq 0.$$

- We shall perform most computations in this paper using acceptability index with this $MINMAXVAR$ set of distortions.

- We may also generalize the distortion $MINMAXVAR$ to a parameter family of cones termed $MINMAXVAR2$ for which

$$\Psi^{\lambda,\gamma}(u) = 1 - \left(1 - u^{\frac{1}{1+\lambda}}\right)^{1+\gamma}, \ u \in [0,1], \ \lambda, \gamma \geq 0.$$

- The parameter λ controls the rate at which the density $\psi^{\lambda,\gamma}(u) = \mathbf{\Psi}^{\lambda,\gamma\prime}(u)$ approaches infinity as u tends to 0 while γ controls the rate at which this density approaches zero as u tends to unity.

- We refer to λ as a measure of risk aversion in the market while γ is a measure of the absence of gain enticement.

Two other distortions

- For the valuation of some sample structured products we also employ the acceptability indices based on $MAXVAR$ and $MAXMINVAR$ defined respectively by:

$$\begin{aligned}\Psi^{\gamma}(u) &= u^{\frac{1}{1+\gamma}},\ u \in [0,1], \gamma \geq 0, \\ \Psi^{\gamma}(u) &= \left(1-(1-u)^{1+\gamma}\right)^{\frac{1}{1+\gamma}},\ u \in [0,1], \gamma \geq 0.\end{aligned}$$

Supporting Measures

- Let us finally remark that for the acceptability index the set $\mathcal{D}_\gamma$ of supporting measures consists of measures Q such that

$$E\left[\left(\frac{dQ}{dP} - x\right)^+\right] \leq \mathbf{\Phi}^\gamma(x) \text{ for each } x \geq 0,$$

  where $\mathbf{\Phi}^\gamma$ is the conjugate dual to $\mathbf{\Psi}^\gamma$:

$$\mathbf{\Phi}^\gamma(v) = \sup_{u\in[0,1]} \left(\mathbf{\Psi}^\gamma(u) - uv\right).$$

- We may also shift attention to a base measure being the uniform distribution on the unit interval and identify random variables with the inverse of their distribution functions.

- The set of level γ acceptable random variables $G(u) = F_X^{-1}(u)$ are then those that have positive expectation for all measure changes with densities $Z(u)$

on the unit interval for which the distribution function, $H(u)$ with $H' = Z$, is first order stochastically dominated by the distortion $\Psi^{\gamma}(u)$. Equivalently $H(u) \leq \Psi^{\gamma}(u)$ for $u \in [0, 1]$.

Bid and Ask Prices using distortions

- We may now employ these distortions to obtain practically computable expressions for the ask and bid price as follows. For the ask price we have that

$$\alpha(a-X) \geq \gamma \iff \int_{-\infty}^{\infty} x d\Psi^{\gamma} d(F_{a-X}(x)) \geq 0 \iff a +$$

from which we arrive at the computationally feasible representation:

$$a_{\gamma}(X) = -\int_{-\infty}^{\infty} x d\Psi^{\gamma}(F_{-X}(x)).$$

- We now see the ask price directly as the negative of the distorted expectation of the cash flow $-X$. By a similar argument one infers that

$$b_{\gamma}(X) = \int_{-\infty}^{\infty} x d\Psi^{\gamma}\left(F_X(x)\right).$$

- It is shown in Madan (2009) that the concavity of Ψ^γ directly delivers the domination of bid prices by ask prices.

Hedging Criteria

- For the purpose of finding operational hedges we observe that in the context of cones defined by distortions we have the following representation of the functional w_γ

$$w_\gamma(X) = \int_{-\infty}^{\infty} x d\Psi^\gamma(F_X(x)).$$

- It follows from Theorem 1 (iii) that

$$a_\gamma(X) = \inf_{H \in \mathcal{H}} - \int_{-\infty}^{\infty} x d\Psi^\gamma(F_{H-X}(x)),$$

while the set of upper hedges coincides with the set of optimal solutions of this problem.

- The equivalent expression for in the case of bid prices is

$$b_\gamma(X) = \sup_{H \in \mathcal{H}} \int_{-\infty}^{\infty} x d\Psi^\gamma \left(F_{X-H}(x)\right).$$

Closed Forms for Bid and Ask Prices for Call and Put Options

- Let S be a random variable meaning time-T price of an underlying asset.

- Consider a call option $C = (S - K)^+$ and a put option $P = (K - S)^+$.

- Consider an acceptability index α based on a family (Ψ^γ) of concave distortions.

- We now give the expressions for the ask and bid prices of C and P in the absence of hedging assets,

- Recall that F_S denotes the distribution function of S.

- The ask and bid prices for C and P are given by:

$$
\begin{aligned}
a_\gamma(C) &= \int_K^\infty \Psi^\gamma(1 - F_S(x))dx, \\
b_\gamma(C) &= \int_K^\infty (1 - \Psi^\gamma(F_S(x)))dx, \\
a_\gamma(P) &= \int_0^K \Psi^\gamma(F_S(x))dx, \\
b_\gamma(P) &= \int_0^K (1 - \Psi^\gamma(1 - F_S(x)))dx.
\end{aligned}
$$

The Cone of Marketed Cash Flows Through the Financial Crisis of 2008

- We take data on bid and ask prices on S&P 500 index options at the end of each quarter for the years 2007 through 2009 and estimate the parameters of the cone λ, γ at each of these dates.

- The distribution function for the stock price at each maturity is taken from the $VGSSD$ model for the logarithm of the stock price.

- On this model the logarithm for the stock at unit time has the variance gamma distribution with parameters σ, ν, θ and the distribution at other time points t is obtained by scaling the unit time law by the scaling factor t^{ρ}.

- The unit time law comes from the variance gamma Lévy process (Madan and Seneta (1990), Madan, Carr and Chang (1998)) with three parameters controlling volatility, skewness and kurtosis.

The Variance Gamma Law

- The variance gamma process is defined by

$$X(t) = \theta g(t) + \sigma W(g(t))$$

  where $(W(t), t \geq 0)$ is a Brownian motion and $(g(t), t \geq 0)$ is a gamma process with unit mean and variance rate ν.

- The resulting composite process $X(t)$ is a pure jump process, with independent identically distributed increments and an infinitely divisible law at unit time. Skewness is controlled by the parameter θ while excess kurtosis is measured in the symmetric case by $3(1 + \nu)$.

- The model is calibrated to all strikes and maturities simultaneously and we estimate the six parameters $\sigma, \nu, \theta, \rho, \lambda, \gamma$ on each quarter end for the three years.

- There are then twelve sets of parameter estimates.

- We observe from the Table that though the cone opened up somehwat in September of 2007 it contracted in December of 2007 and narrowed significantly in March of 2008.

- Since then it has been opening up somewhat but is considerably smaller that it was in 2007.

- Volatility and skewness have risen through the crisis as represented by σ and θ, though they have reduced a bit towards the end of 2009.

Table 1

VGSSD and Cone Parameter Values Through the Financial Crisis

Date	sigma	nu	theta	rho	lambda	gamma
20070330	0.1134	1.0226	-0.0917	0.5268	0.0559	0.0437
20070629	0.1321	0.9760	-0.1024	0.5237	0.0521	0.0369
20070928	0.1393	0.9368	-0.1441	0.5550	0.0309	0.0322
20071231	0.1672	0.7176	-0.1994	0.5289	0.0980	0.0842
20080331	0.1553	0.5199	-0.2464	0.5130	0.2327	0.1674
20080630	0.1689	0.6301	-0.2141	0.5153	0.0928	0.0643
20080930	0.2430	0.6174	-0.2890	0.3998	0.1132	0.1774
20081231	0.2682	0.6401	-0.3968	0.5357	0.1427	0.0655
20090327	0.3387	0.6542	-0.3930	0.5051	0.0734	0.1181
20090630	0.2433	0.8827	-0.2493	0.5985	0.0127	0.0481
20090930	0.2259	0.8793	-0.2229	0.5941	0.0020	0.0189
20091229	0.2269	1.0759	-0.1969	0.6513	0.0922	0.1272

Benefits of Delta Hedging

- We know that hedging at a Black-Merton-Scholes delta helps remove the effects of drift, and we shall quantify these benefits, but we additionally ask if this activity also improves the residual cash flow by making it more acceptable even in the absence of drift.

- To answer this question we consider an underlying exponential Lévy process driving the stock for which exact replication provably fails and there is always a residual cash flow. We consider the improvements that may be possible in the residual cash flow induced by hedging at a Black Scholes delta with a flat volatility.

- We measure the quality of the cash flow by the level of the ask price using the distortion $minmaxvar$ at stress level .75 for downside puts and stress level .5 at upside calls, recognizing that calibrated stress levels for puts are higher than for calls.

Stock Price Process

- We take the underlying process for the stock to be driftless and a martingale and so there are no issues associated with accessing drifts in the residual cash flow. The stock price process $(S(t), t \geq 0)$ with drift rate μ may be expressed in terms of the variance gamma process by

$$\begin{aligned} S(t) &= S(0) \exp\left(X(t) + (\mu + \omega)t \right) \\ \omega &= -\log E\left[\exp(X(1)) \right]. \end{aligned}$$

- We first report results evaluating the risk profile benefits of delta hedging where the drift rate is set to zero.

Options and Parameters

- We analyse two maturities 0.25 and 0.5 years and four strikes of $80, 90, 110, 120$ where the first two are puts and the latter two are calls with the spot starting at 100.

- This gives us 8 options for our analysis.

- We next have to consider possible values for the parameters of the underlying Lévy process and we consider three levels for σ the volatility of the Brownian motion of 0.1, 0.15, and 0.2.

- For the excess kurtosis parameter ν we take three levels of 0.25, 0.5 and 1.0. For the skewness parameter θ we take 5 levels of $-.1$, $-.05$, 0, .05, and .1. This gives us 45 cases.

- We then have a total 360 cases.

Hedging Volatility

- For each case we determine on a simulated path space the optimal hedging volatility or the flat volatility that minimizes the ask price computed as the negative of the distorted expectation of the residual cash flow at the stated stress levels for the stress function $minmaxvar$.

- We will then report on how the spread of the unhedged over the hedged ask price responds to the structure of the underlying Lévy process. The hedges are all at a Black-Merton-Scholes delta with a flat volatility and daily rehedging.

Effects of Delta Hedge on ask prices

- We find that beyond the effects of removing drifts the activity of delta hedging at even a flat Black Scholes volatility improves the quality of the residual cash flow by raising its acceptability or for a fixed acceptability permitting a reduction in the ask price.

- The average price reduction enabled by delta hedging is 15% with an interquartile range from 7% to 26%.

- The average increase in the hedge volatility, that is the ratio of the flat volatility used in hedging to minimize the ask price to the true volatility, is a factor of 2.2 with an interquartile range from .77 to 2.44.

Regression Sensitivities

- In order to analyse the effects of strike, maturity, and the parameters of the Lévy process on the price reductions possible we ran a fixed effects model on the 360 cases studied. We allowed for a a dummy variable for the strike at $80, 120$ relative to $90, 110$. We used a dummy variable for the longer maturity and the higher levels of σ, ν, and θ. The results are presented in Table 2.

- We observe a significant reduction overall with a lower reduction for tail strikes and an increased reduction for longer maturities.

- The reduction increases with volatility, falls with excess kurtosis and is not significant to movements in the level of statistical skewness, that is usually small.

Table 2
Fixed Effect Regression Model
For Price Improvements

variable	Coeff	t-stat
constant	0.1791	7.03
strike dummy	-0.0687	-4.48
σ up one level	0.0328	1.75
σ up two levels	0.0613	3.26
ν up one level	-0.0543	-2.89
ν up two levels	-0.0985	-5.24
θ down two levels	-0.0141	-0.58
θ down one level	-0.0002	-0.01
θ up one level	-0.0042	-0.17
θ up two levels	-0.0096	-0.39
maturity dummy	0.0661	4.31
R2	18.23	

Table 3

Ask Prices for Six Month Option

Hedge	90P	110C
No Hedge	4.9460	4.8087
Single Delta	4.4358	4.0076
Weekly	3.3703	3.1779
Daily	3.2987	3.0999

TABLE 4
Estimated CGMY parameters
on daily index return data
12/12/2002-12/27/2007

	C	G	M	Y
SPX	5.6014	168.29	185.32	0.7404
FTSE	4.3387	68.039	91.011	1.0880
SX5E	3.4139	78.471	96.360	0.7841
GDAXI	12.483	78.484	93.218	0.5487
IBEX	1.2005	76.511	104.41	0.9533
N225	0.3085	69.368	120.92	1.3051
HSI	12.716	112.16	114.13	0.5444
NDX	2.5781	112.97	121.80	0.9529
DJX	0.2050	108.95	109.36	0.1247

Post Hedge Cash Flows

- To observe the difference we consider the case of a six month 90 put with $\sigma = 0.15$, $\nu = 0.25$, $\theta = 0$ where the one delta hedge vol was 0.18 while the daily delta hedge vol was 0.21.

- We graph the probability distribution of the residual cash flow with one delta and a daily delta both on a flat volatility.

- The resulting distribution in red is clearly more acceptable and has a lower ask price attaining a 30% reduction from 4.35 to 3.40.

- This study was extended to consider also a 110 six month call and we also graphed the distribution for weekly hedging along with daily hedging.

- These graphs are also presented.

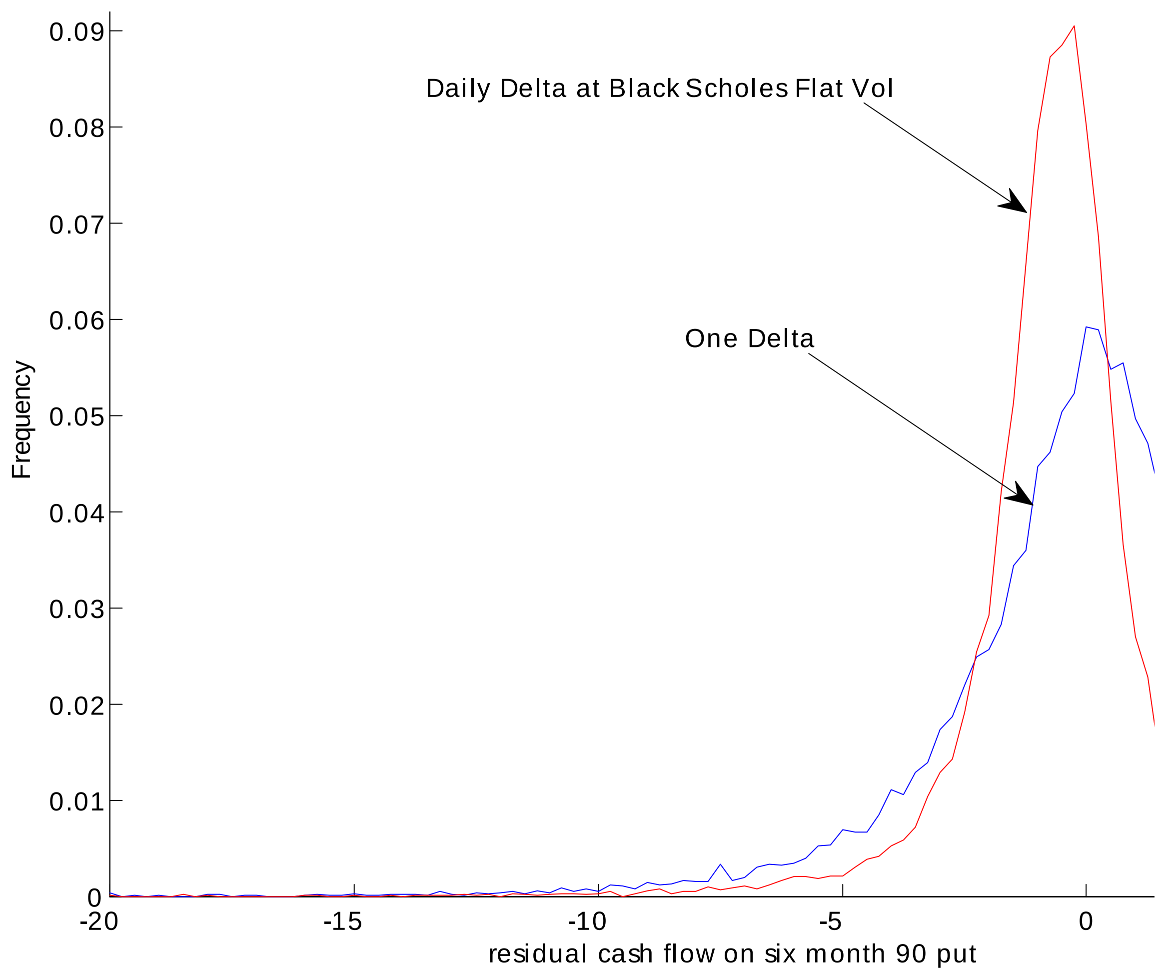

Figure 1: Residual Cash Flow Probability Distribution with initial Delta and Daily Delta.

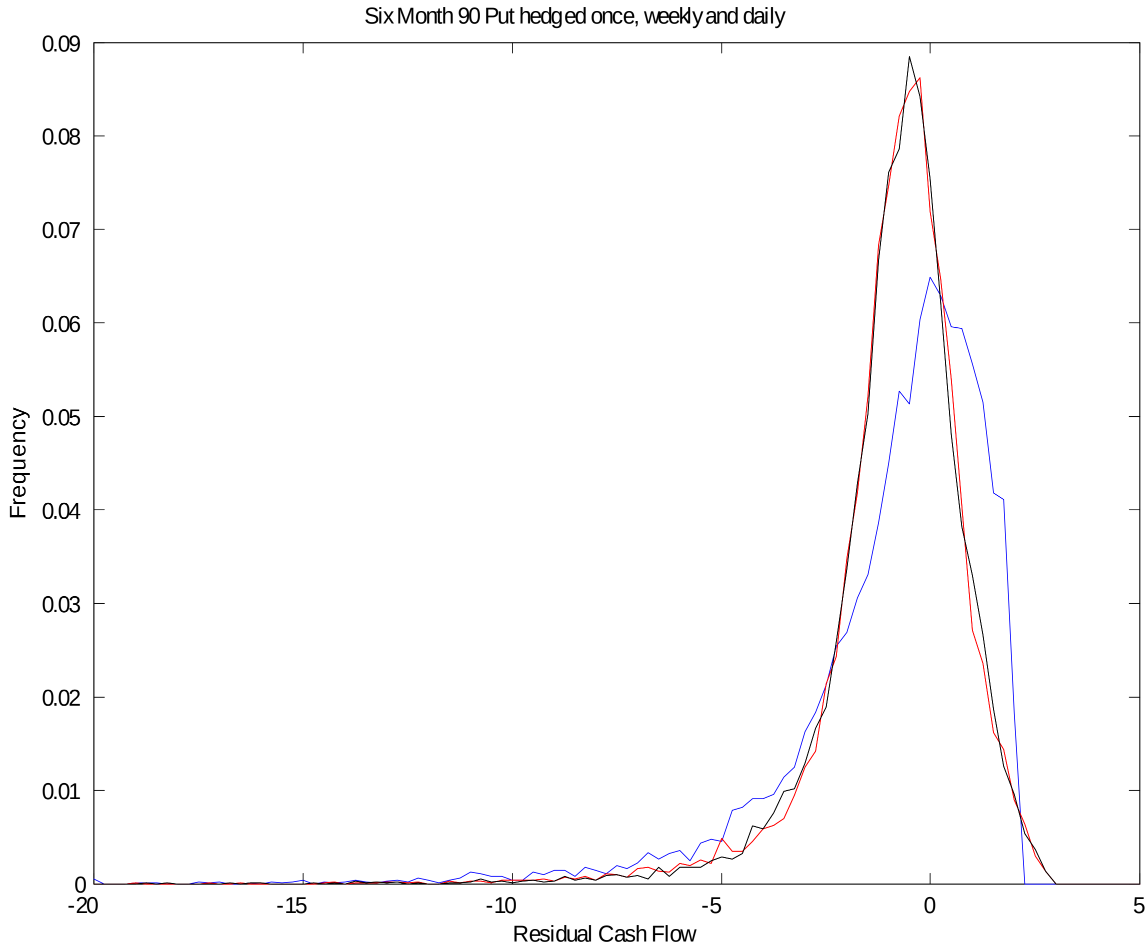

Figure 2: Residual Cash Flow Probability Distribution for six month 90 put with one delta in blue, a weekly delta in red and a daily delta in black.

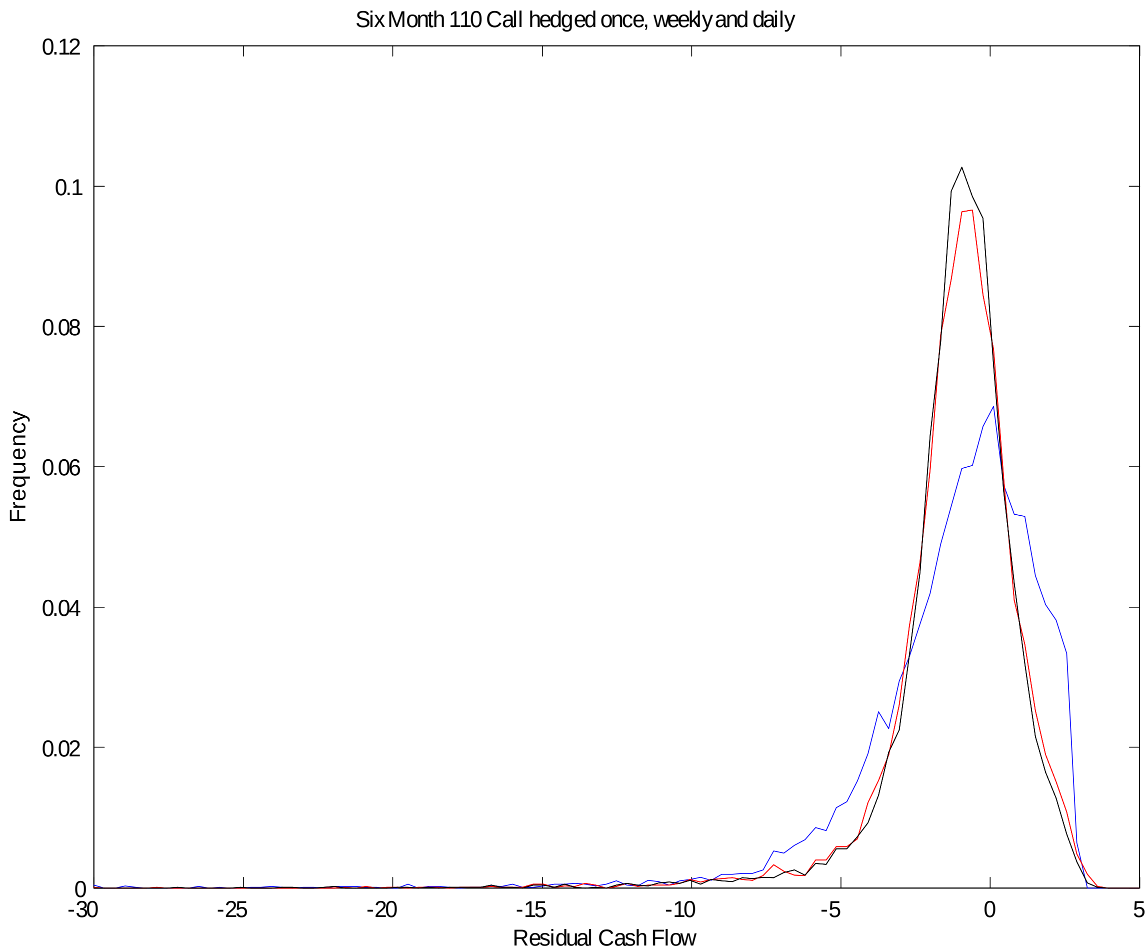

Figure 3: Residual Cash Flow Frequency Distribution for six month 110 call with one delta (blue), a weekly delta (red) and a daily delta (black).

Drift Removal Benefits

- Finally we document briefly the benefits that follow from the ability of Black Scholes delta hedging at a flat volatility to contribute towards removing the possible negative effects of drift.

- For this purpose we consider hedging the 110 call in a path space where the drift rate is 0.1, 0.25, and 0.5 respectively. The parameters describing the risk are $\sigma = 0.15$, $\nu = 0.25$, and $\theta = 0.1$. In this case we did not optimize the hedge volatility but just used 0.5.

- We find the unhedged ask prices at the stress level 0.5 to be respectively 9.3274, 16.3011, and 32.1304 while the daily delta hedged price is relatively uniform at respectively 3.0459, 2.6905, 3.4779.

- The benefits of eliminating the effects of drift on the final profit and loss are quite extensive and clear. It is probably this aspect of the hedge that drives the underlying popularity of Black Scholes delta hedging.

Pricing a Sample of Structured Products

- We consider here daily cash flow swaps, which are contracts of the form

$$F = \sum_{t=1}^{T} f(S_t, S_{t-1}),$$

where S_t is the price of the underlying asset at time t and T is the maturity of the swap.

- There are three examples of payoff functions considered here, termed a *daily cliquet*, *large realized variance swap*, and *large total variation swap*. These are defined as follows. For a daily cliquet with strikes $k_d < 1 < k_u$, we have

$$f(S_t, S_{t-1}) = (S_t - k_d S_{t-1})^+ - (S_t - k_u S_{t-1})^+.$$

A large realized variance swap with strike k is defined by

$$f(S_t, S_{t-1}) = (\ln(S_t/S_{t-1}))^2 I(|\ln(S_t/S_{t-1})| > k).$$

A large total variation swap with strike k corresponds to

$$f(S_t, S_{t-1}) = |S_t - S_{t-1}| I(|S_t - S_{t-1}| > k).$$

The Hedges

- We calculate as before, both the unhedged and hedged prices. In the latter case we consider the underlying asset and the *variance swap* as hedge instruments and in our notation,

  $$\mathcal{H} = \{H | H = h_1(S_1 - S_0) + h_2[(\ln(S_1/S_0))^2 - v] : h_1, h_2 \in$$

  where v is the (one-day) price of the variance swap.

The Base Measure

- In the calculations below we use two different candidates for the physical measure P.

- The first one is just the empirical distribution of $\ln(S_t/S_{t-1})$ over the last four years. However, the empirical distribution might not include possible large moves.

- In order to capture them, we consider another candidate for the measure P as the CGMY distribution fitted to the return history (for the description of the CGMY distribution we refer to Carr, Geman, Madan, and Yor, 2003). We then sample 10000 outcomes from this fitted distribution to get a larger sample of possible outcomes. The results of fitting the CGMY distribution are reported in Table 4.

- We also present in Figure (4) the graphs of the binned return data, the fitted CGMY densities, and the reference fitted Gaussian density for each of the nine indices employed. These are SPX, FTSE, SX5E, GDAXI, IBEX, N225, HSI, NDX, and DJX. The data period was December 12 2002 to December 27 2007.

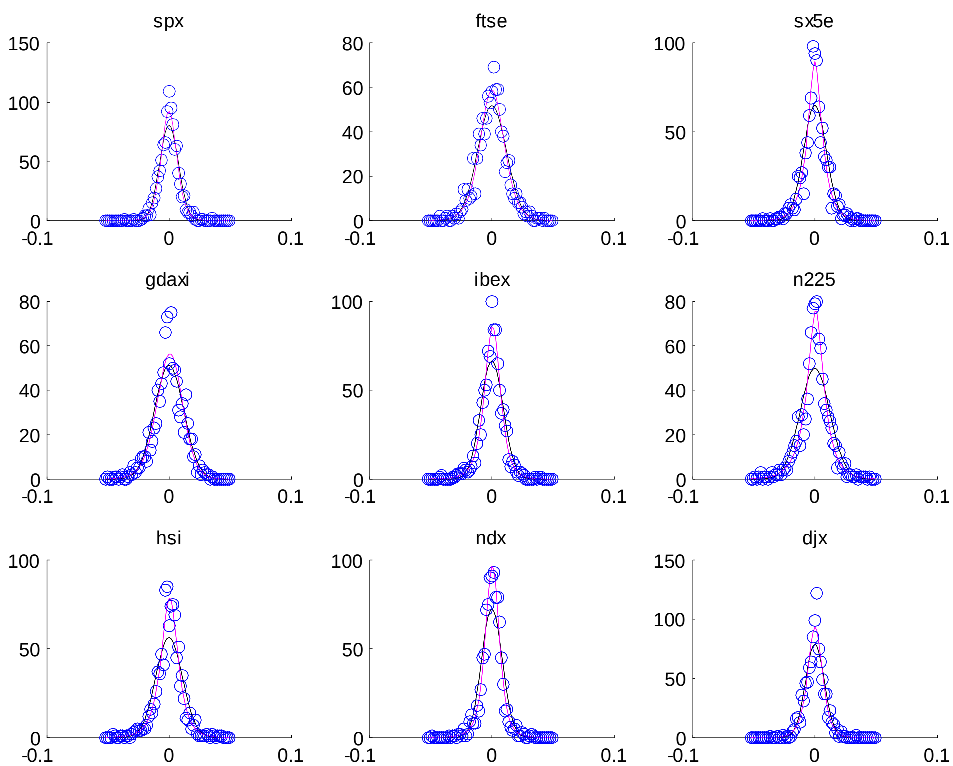

Figure 4: Graph displaying CGMY model densities fit to daily return data on nine indices over four years.

Results

- For each of the three contracts described above, we consider as underliers our nine indices. For the acceptability index, we consider 4 different candidates: AIMIN, AIMAX, AIMAXMIN, and AIMINMAX based respectively on $MINVAR$, $MAXVAR$, MAX and $MINMAXVAR$.

- We calculate the ask and bid prices with no market and with the hedge space $\mathcal{H}$ given by both choices of the measure P. The results are presented in Tables 5–7.

- The strikes used for the daily cliquet were $k_d = 0.97$, $k_u = 1.03$.

- The realized variance was paid out only when the absolute return exceeded a half a percent, and the result is in basis points.

- The total variation was positive only if it exceeded 5 units of the currency.

- The stress level γ was chosen to be 0.5 for MINVAR, 0.25 for MAXVAR, and 0.15 for both MAXMINVAR and MINMAXVAR.

- We observe from the tables that for the cliquet the bid-ask spreads for the hedged prices are quite low for the simulated model, while they are higher for the sample data and are still higher for the unhedged prices.

- The model prices are lower than the prices from the data sample possibly reflecting the lower activity in the 3% range for the model.

- This is further confirmed with higher prices for the realized variance and the total variation.

Table 5. Results on Daily Cliquet

			spx	ftse	sx5e	gdaxi	ibex	n225	hsi	ndx	djx
	No Hedge	Sample	38.30	164.51	105.13	156.38	319.51	428.97	466.36	50.73	3.41
	Ask	Model	37.92	162.06	102.90	151.13	308.50	436.97	449.40	50.19	3.35
	No Hedge	Sample	33.28	139.87	86.49	123.32	255.80	339.71	380.32	42.38	3.00
MinVar	Bid	Model	34.53	147.06	91.53	133.04	276.65	365.10	404.42	43.99	3.09
Stress .5	Hedge	Sample	35.76	153.81	96.51	142.19	295.10	391.81	430.19	45.89	3.18
	Ask	Model	34.72	148.09	92.79	135.33	279.02	368.74	407.94	44.50	3.10
	Hedge	Sample	33.28	139.87	86.49	123.32	255.80	339.71	380.32	42.38	3.00
	Bid	Model	34.53	147.06	91.53	133.04	276.65	365.10	404.42	43.99	3.09
	No Hedge	Sample	36.96	158.68	100.60	149.02	307.40	410.24	447.82	48.31	3.29
	Ask	Model	36.79	157.16	98.91	144.69	297.75	416.96	434.19	47.99	3.25
	No Hedge	Sample	33.10	140.29	86.90	126.18	264.17	342.18	385.88	41.37	2.96
MaxVar	Bid	Model	32.99	140.37	85.90	124.05	262.69	358.78	381.17	41.07	2.92
Stress .25	Hedge	Sample	35.22	151.46	94.98	139.25	289.45	382.80	420.89	45.06	3.14
	Ask	Model	34.71	148.20	92.94	135.64	279.29	369.31	407.82	44.50	3.10
	Hedge	Sample	33.76	143.27	89.00	128.30	266.89	352.76	392.01	42.90	3.03
	Bid	Model	34.55	147.08	91.42	132.93	276.62	365.28	404.09	43.97	3.09
	No Hedge	Sample	37.45	160.83	102.27	151.69	311.78	416.93	454.48	49.19	3.33
	Ask	Model	37.20	158.99	100.40	147.07	301.76	424.28	439.85	48.79	3.29
	No Hedge	Sample	32.72	138.38	85.44	123.51	258.93	333.14	378.35	40.60	2.93
MaxMinVar	Bid	Model	32.59	138.68	84.53	121.79	259.18	352.35	375.43	40.25	2.88
Stress .15	Hedge	Sample	35.42	152.30	95.52	140.32	291.45	386.01	424.31	45.37	3.15
	Ask	Model	34.71	148.15	92.93	135.66	279.19	369.18	407.90	44.50	3.10
	Hedge	Sample	33.59	142.04	88.08	126.48	262.80	348.12	387.72	42.71	3.02
	Bid	Model	34.54	147.07	91.43	132.88	276.61	365.01	404.23	43.98	3.09
	No Hedge	Sample	37.37	160.42	101.96	151.16	310.78	415.31	453.04	49.03	3.33
	Ask	Model	37.12	158.62	100.11	146.61	300.98	422.94	438.66	48.63	3.28
	No Hedge	Sample	32.81	138.81	85.76	124.05	259.90	334.66	379.76	40.76	2.94
MinMaxVar	Bid	Model	32.68	139.06	84.83	122.26	259.97	353.72	376.61	40.41	2.89
Stress .15	Hedge	Sample	35.38	152.08	95.36	140.01	290.75	385.20	423.49	45.32	3.15
	Ask	Model	34.71	148.13	92.90	135.61	279.15	369.08	407.84	44.49	3.10
	Hedge	Sample	33.63	142.25	88.24	126.78	263.42	348.89	388.53	42.77	3.02
	Bid	Model	34.55	147.08	91.45	132.92	276.65	365.16	404.29	43.99	3.09

Table 6. Results on Realized Variance Payoffs

			spx	ftse	sx5e	gdaxi	ibex	n225	hsi	ndx	djx
	No Hedge	Sample	0.7892	0.9513	1.6294	2.0718	1.1411	1.9352	1.2029	1.8919	0.7597
	Ask	Model	0.7877	0.8916	1.5665	1.9681	1.1389	1.9303	1.2049	1.7964	0.7562
	No Hedge	Sample	0.6265	0.7634	1.3224	1.6826	0.9144	1.5664	0.9618	1.5334	0.6035
MinVar	Bid	Model	0.6249	0.7132	1.2702	1.5980	0.9169	1.5720	0.9668	1.4569	0.6007
Stress .5	Hedge	Sample	0.6608	0.8011	1.3546	1.7137	0.9495	1.5952	0.9944	1.5641	0.6372
	Ask	Model	0.6614	0.7495	1.3037	1.6302	0.9525	1.5996	1.0014	1.4860	0.6375
	Hedge	Sample	0.6265	0.7634	1.3224	1.6826	0.9144	1.5664	0.9618	1.5334	0.6035
	Bid	Model	0.6249	0.7132	1.2702	1.5980	0.9169	1.5720	0.9668	1.4569	0.6007
	No Hedge	Sample	0.8462	1.1146	1.8690	2.3318	1.2486	2.0275	1.2746	2.0082	0.8221
	Ask	Model	0.8548	1.0202	1.8023	2.2383	1.2991	2.0498	1.3382	1.8956	0.8344
	No Hedge	Sample	0.4640	0.5484	0.9481	1.2104	0.6678	1.1608	0.7075	1.1346	0.4462
MaxVar	Bid	Model	0.4625	0.5169	0.9121	1.1475	0.6643	1.1689	0.7022	1.0821	0.4441
Stress .25	Hedge	Sample	0.6477	0.7871	1.3421	1.7015	0.9361	1.5837	0.9818	1.5518	0.6244
	Ask	Model	0.6477	0.7359	1.2908	1.6177	0.9390	1.5884	0.9882	1.4743	0.6237
	Hedge	Sample	0.6288	0.7659	1.3236	1.6841	0.9171	1.5670	0.9639	1.5343	0.6055
	Bid	Model	0.6273	0.7151	1.2711	1.5988	0.9183	1.5739	0.9684	1.4572	0.6027
	No Hedge	Sample	0.8188	1.0420	1.7636	2.2167	1.2001	1.9798	1.2397	1.9505	0.7926
	Ask	Model	0.8227	0.9619	1.6964	2.1173	1.2252	1.9886	1.2768	1.8450	0.7970
	No Hedge	Sample	0.3781	0.4228	0.7413	0.9574	0.5355	0.9687	0.5784	0.9419	0.3619
MaxMinVar	Bid	Model	0.3750	0.4053	0.7148	0.9037	0.5252	0.9790	0.5593	0.9035	0.3585
Stress .15	Hedge	Sample	0.6527	0.7924	1.3470	1.7063	0.9412	1.5882	0.9867	1.5566	0.6293
	Ask	Model	0.6529	0.7410	1.2958	1.6225	0.9441	1.5929	0.9933	1.4789	0.6289
	Hedge	Sample	0.6281	0.7651	1.3232	1.6835	0.9162	1.5669	0.9633	1.5340	0.6049
	Bid	Model	0.6267	0.7146	1.2710	1.5987	0.9181	1.5732	0.9680	1.4572	0.6022
	No Hedge	Sample	0.8089	1.0281	1.7407	2.1884	1.1851	1.9571	1.2247	1.9279	0.7829
	Ask	Model	0.8127	0.9494	1.6744	2.0900	1.2096	1.9662	1.2606	1.8239	0.7872
	No Hedge	Sample	0.3836	0.4294	0.7524	0.9716	0.5434	0.9818	0.5868	0.9547	0.3671
MinMaxVar	Bid	Model	0.3805	0.4115	0.7255	0.9172	0.5330	0.9919	0.5676	0.9157	0.3637
Stress .15	Hedge	Sample	0.6524	0.7920	1.3466	1.7059	0.9408	1.5879	0.9863	1.5563	0.6289
	Ask	Model	0.6525	0.7406	1.2954	1.6222	0.9437	1.5926	0.9929	1.4786	0.6285
	Hedge	Sample	0.6288	0.7658	1.3238	1.6841	0.9168	1.5675	0.9639	1.5346	0.6056
	Bid	Model	0.6273	0.7153	1.2716	1.5993	0.9187	1.5739	0.9686	1.4577	0.6029

Table 7. Results on Total Variation Payoff

			spx	ftse	sx5e	gdaxi	ibex	n225	hsi	ndx	djx
	No Hedge	Sample	11.3107	54.1705	42.4635	67.7910	112.9533	207.9990	176.8989	23.6941	0.3600
	Ask	Model	11.5203	50.1056	41.9073	68.7053	108.6655	205.1170	163.3263	23.6366	1.0085
	No Hedge	Sample	8.1553	40.1079	30.7981	48.0466	82.4132	154.8801	130.9343	17.7039	0.2763
MinVar	Bid	Model	8.4590	36.2574	30.3891	50.0223	79.0754	156.9029	118.5010	17.7988	0.7425
Stress .5	Hedge	Sample	10.6781	52.0641	40.6782	64.9280	106.5695	195.0155	165.5637	22.5391	0.3417
	Ask	Model	10.6806	46.9825	39.4804	64.5850	101.8281	183.6402	152.0211	21.9295	0.9405
	Hedge	Sample	8.1553	40.1079	30.7981	48.0466	82.4132	154.8801	130.9343	17.7039	0.2763
	Bid	Model	8.4590	36.2574	30.3891	50.0223	79.0754	156.9029	118.5010	17.7988	0.7425
	No Hedge	Sample	10.7003	52.5262	40.8481	64.7661	109.1663	200.0666	170.0844	22.3956	0.3437
	Ask	Model	11.0899	48.9779	40.9707	67.0128	105.8322	195.3654	158.5390	22.5177	0.9735
	No Hedge	Sample	7.3908	37.2196	28.9393	46.3034	78.1091	142.5240	120.1894	16.1463	0.2499
MaxVar	Bid	Model	7.4435	32.0499	27.0138	44.3802	70.0371	137.4448	104.7074	15.7281	0.6525
Stress .25	Hedge	Sample	10.0864	49.6951	39.0005	62.8168	103.8599	189.3005	159.7005	21.2689	0.3238
	Ask	Model	10.0911	44.2299	37.0370	60.6283	95.8107	176.9075	143.3115	20.6767	0.8862
	Hedge	Sample	8.2344	40.2245	31.4507	50.3643	85.8077	157.8223	134.6838	17.7608	0.2744
	Bid	Model	8.4581	35.8467	30.1060	49.7206	78.1005	157.8166	118.7712	17.8182	0.7339
	No Hedge	Sample	10.8793	52.9595	41.3152	65.6734	110.2239	202.3475	172.0095	22.8024	0.3486
	Ask	Model	11.1958	49.1407	41.1117	67.3153	106.3327	198.1470	159.5289	22.8374	0.9815
	No Hedge	Sample	6.7437	34.3511	26.6957	42.8540	72.2922	131.5831	110.6218	15.0152	0.2328
MaxMinVar	Bid	Model	6.7069	28.5612	24.1827	39.7958	62.7924	126.7232	93.6751	14.4479	0.5881
Stress .15	Hedge	Sample	10.3028	50.6910	39.6412	63.5877	104.9201	191.5558	162.0830	21.7446	0.3307
	Ask	Model	10.3048	45.2661	37.9426	62.0996	98.0703	179.6866	146.4578	21.1291	0.9062
	Hedge	Sample	8.1154	39.3392	30.5167	48.3181	82.8482	155.2856	131.7869	17.5949	0.2727
	Bid	Model	8.4008	35.5422	29.8164	49.2306	77.5509	155.6511	117.6573	17.7557	0.7309
	No Hedge	Sample	10.7900	52.5656	41.0049	65.1892	109.4205	200.8388	170.6962	22.6388	0.3462
	Ask	Model	11.0979	48.6895	40.7431	66.7165	105.3840	196.6352	158.0877	22.6588	0.9729
	No Hedge	Sample	6.8134	34.6475	26.9332	43.2332	72.9026	132.7346	111.6201	15.1490	0.2347
MinMaxVar	Bid	Model	6.7788	28.8776	24.4433	40.2218	63.4701	127.9197	94.7054	14.5877	0.5943
Stress .15	Hedge	Sample	10.2543	50.4554	39.4393	63.2199	104.3684	190.7548	161.3833	21.6589	0.3296
	Ask	Model	10.2648	45.0669	37.7799	61.8417	97.6590	178.7198	145.8663	21.0620	0.9027
	Hedge	Sample	8.1631	39.5746	30.7070	48.6417	83.3374	156.0282	132.4415	17.6865	0.2739
	Bid	Model	8.4412	35.7444	29.9869	49.5025	77.9796	156.4859	118.2599	17.8306	0.7346