Arbitrage-Free Pricing, Optimal Investment, and Equilibrium

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Plan of the course

Lecture I: Mathematical model of financial market. Arbitrage and 1st fundamental theorem.

Lecture 2: Arbitrage-free valuation. Completeness and 2nd fundamental theorem.

Lecture 3: Optimal investment.

Lecture 4: General equilibrium.

Part I

Mathematical model of financial market. Arbitrage and 1st fundamental theorem

Outline

Mathematical model of financial market

There are $d+1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The stocks' price process $S = (S_t)$ is a RCLL stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

Key assumption: trader's actions do not affect S ("small" economic agent).

Problem

Obtain conditions on S for model to be "viable".

Simple strategies

For a *simple* strategy with a process of *stocks' quantities*:

$$H_t = \sum_{n=1}^{N} \theta_n 1_{(t_{n-1}, t_n]},$$

where $\theta_n \in \mathbf{L}^0(\mathcal{F}_{t_{n-1}})$, the *wealth process*

$$X_t(H) = X_0 + \sum_{t_n \leq t} \theta_n (S_{t_n} - S_{t_{n-1}}).$$

Mathematical challenge: define $X(H)$ for *general* H.

Closability for simple strategies

Closability: the convergence of simple (H^n) to LCRL H in *ucp*

$$(H^n - H)^*_T = \sup_{t \in [0,T]} |H^n_t - H_t| \to 0$$

implies the existence of $X(H)$ such that

$$(X(H^n) - X(H))^*_T \to 0.$$

Theorem (Bichteler-Dellacherie, see Protter (2004))

Closability holds $\Leftrightarrow$ S *is a semimartingale.*

General strategies

Recall that S is a semimartingale if

$$S = M + A,$$

where M is a local martingale and A is a predictable process of bounded variation. For a semimartingale S we can extend the map

$$H \mapsto X(H)$$

from simple to general H arriving to *stochastic integrals*:

$$X_t(H) = X_0 + \int_0^t H_u dS_u.$$

Elegant setup: Emery's or semimartingale topology; see Protter (2004).

Economic viability

Conditions for economic viability of market model:

1. Price S is an outcome of an "equilibrium" (matching of demand and supply).
2. Any "rational" investor has an "optimal" *finite* strategy $\widehat{Q} = (\widehat{Q}_t)$.
3. There is a "rational" investor with an "optimal" *finite* strategy $\widehat{Q} = (\widehat{Q}_t)$.
4. The market S is "arbitrage-free".

Under suitable definitions of "terms" all these conditions are equivalent!

1st fundamental theorem

Let $\mathcal{Q}$ denote the family of *martingale measures* for S, that is,

$$\mathcal{Q} = \{\mathbb{Q} \sim \mathbb{P} : \quad S \text{ is a local martingale under } \mathbb{Q}\}$$

Theorem (1st FTAP)

$$\textit{Absence of arbitrage} \quad \Longleftrightarrow \quad \mathcal{Q} \neq \emptyset.$$

Free Lunch with Vanishing Risk (FLVR)

For 1st FTAP to hold true the following definition of arbitrage is needed (Delbaen and Schachermayer (1994)):

1. There is a set $A \in \Omega$ with $\mathbb{P}[A] > 0$.
2. For any $\epsilon > 0$ there is a strategy X such that
   2.1 X is *admissible*, that is, for some constant $c > 0$,
   $$X \geq -c.$$
   2.2 $X_0 \leq \epsilon$ (start with almost nothing)
   2.3 $X_T \geq 1_A$ (end with something)

Verification of the absence of arbitrage

Assume that $\mathcal{F}_t = \mathcal{F}_t^S$ (the information is generated by S). Then without loss in generality $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ is a canonical probability space of continuous functions $\omega = \omega(t)$ on $[0, T]$ and $S_t(\omega) = \omega(t)$. Suppose

$$S_t = S_0 + \int_0^t \mu_t du + W_t^{\mathbb{P}},$$

where $\mu_t = \mu((S_u)_{u \le t}, t)$ and $W^{\mathbb{P}}$ is a $\mathbb{P}$-Brownian motion.

Problem

Find (necessary and sufficient) conditions on $\mu = (\mu_t)$ for the absence of arbitrage (No FLVR).

Solution

Levi's theorem $\implies$ that the only possible martingale measure $\mathbb{Q}$ is such that

$$W_t^{\mathbb{Q}} = S_t - S_0,$$

is a $\mathbb{Q}$-Brownian motion. Then by 1st FTAP

$$\text{No FLVR} \quad \iff \quad \mathbb{P} \sim \mathbb{Q}.$$

One can show (easy!, see Jacod and Shiryaev (2003) for general results or this kind relying on Hellinger processes) that

$$\mathbb{P} \sim \mathbb{Q} \quad \iff \quad \int_0^T \mu_t^2 dt < \infty \quad \mathbb{P} + \mathbb{Q} \quad \text{a.s.}.$$

References

Freddy Delbaen and Walter Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3): 463–520, 1994.

Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, Springer-Verlag, Berlin, second edition, 2003.

Philip E. Protter. *Stochastic integration and differential equations*, Springer-Verlag, Berlin, second edition, 2004.

Part II

Arbitrage-free valuation. Completeness and 2nd fundamental theorem

Outline

Financial market

Pricing $=$ Replication

Black and Scholes formula

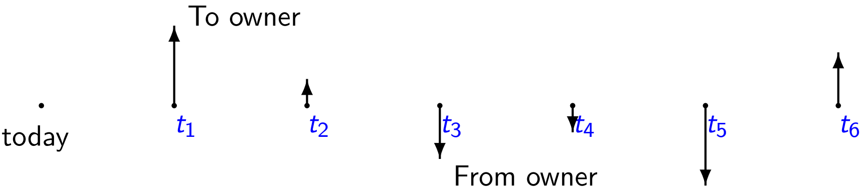
2nd fundamental theorem

References

Financial security

$$\boxed{\textit{Financial Security} = \textit{Cash Flow}}$$

Example (Interest Rate Swap)



Pricing problem: compute "fair" value of the security *today*.

Classification of financial securities

We classify all financial securities into 2 groups:

1. *Traded securities*: the price is *given* by the market.

$$\boxed{\textit{Financial model} = \textit{All traded securities}}$$

2. *Non-traded securities*: the price has to be *computed*.

Remark

This "black-and-white" classification is quite idealistic. Real life securities are usually "gray".

In this tutorial we shall deal with *Arbitrage-Free Pricing* methodology.

Arbitrage-free price

Inputs:

1. Financial model (collection of all traded securities)
2. A non-traded security.

Arbitrage strategy (intuitive definition) :

1. start with zero capital (*nothing*)
2. end with positive and non zero wealth (*something*)

Assumption

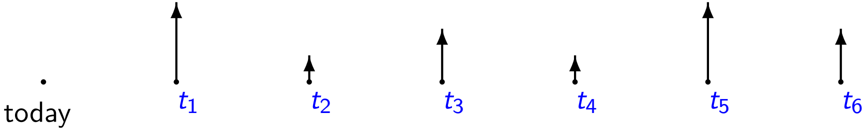
The financial model is arbitrage free.

Definition

An amount p is called an *arbitrage-free price* if, given an opportunity to trade the non-traded security at p, one is not able to construct an arbitrage strategy.

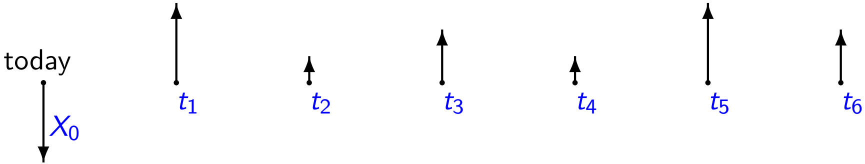
Replication

Cash flow of non-traded security:



Replicating strategy:

1. starts with some initial capital X_0
2. generates *exactly* the same cash flow in the future



Methodology of arbitrage-free pricing

Theorem

An arbitrage-free price p is unique if and only if there is a replicating strategy. In this case,

$$p = X_0,$$

where X_0 is the initial capital of a replicating strategy.

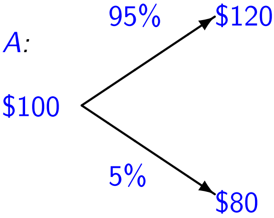
Main Principle:

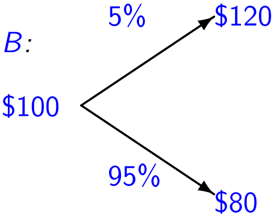
$$\boxed{\textit{Unique Arbitrage-Free Pricing} = \textit{Replication}}$$

Problem on two calls

Problem

Consider two stocks: A and B. Assume that





Consider call options on A and B with the same strike $K = \$100$.
Assume that $T = 1$ and $r = 5\%$.
Compute the difference $C^A - C^B$ of their arbitrage-free prices.

Pricing in Black and Scholes model

There are two traded assets: *savings account* with zero interest rate and *stock* with price process:

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right).$$

Here $W = (W_t)_{t \geq 0}$ is a Wiener process and

$\mu \in \mathbf{R}$: drift

$\sigma > 0$: volatility

Problem (Black and Scholes (1973))

Compute arbitrage-free price V_0 *of European put option with maturity* T *and payoff*

$$\Psi = \max(K - S_T, 0).$$

Replication in Black and Scholes model

Basic principle : $\boxed{\textit{Pricing} \ = \ \textit{Replication}}$

Replicating strategy :

1. has wealth evolution:

$$X_t = X_0 + \int_0^t \Delta_u dS_u,$$

   where X_0 is the initial capital and Δ_t is the number of shares at time t; $0 \leq X \leq K$.

2. generates *exactly* the same payoff as the option:

$$X_T(\omega) = \Psi(\omega) = \max(K - S_T(\omega), 0), \quad \mathbb{P}\text{-a.s.}.$$

Two standard methods: "direct" (PDE) and "dual" (martingales).

PDE method

Since $X_T = f(S_T)$ we look for replicating strategy in the form:

$$X_t = v(S_t, t)$$

for some deterministic $v = v(s, t)$. By Ito's formula,

$$dX_t = v_s(S_t, t)dS_t + (v_t(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 v_{ss}(S_t, t))dt.$$

But, (since X is a wealth process)

$$dX_t = \Delta_t dS_t,$$

where Δ_t (hedging delta) is the number of stocks at time t.

PDE method

Hence, $v = v(s,t)$ solves PDE:

$$\begin{cases} v_t(s,t) + \frac{1}{2}\sigma^2 s^2 v_{ss}(s,t) & = & 0 \\ v(s,T) & = & \max(K-s,0) \end{cases}$$

The arbitrage-free price and the hedging delta are given by

$$p = v(S_0, 0),$$
$$\Delta_t = v_s(S_t, t).$$

Martingale method

Observation: replication problem is defined "almost surely" and, hence, is invariant with respect to an equivalent choice of probability measure.

Convenient choice: *martingale measure* $\mathbb{Q}$ for S. We have

$$dS_t = S_t \sigma dW_t^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

Replication strategy: (by Martingale Representation Theorem)

$$X_t = X_0 + \int_0^t \Delta dS = \mathbb{E}^{\mathbb{Q}}[\Psi|\mathcal{F}_t].$$

Risk-neutral valuation: (no replication!)

$$p = X_0 = \mathbb{E}^{\mathbb{Q}}[\Psi].$$

Martingale method

The computation of hedging delta is conveniently done with *Clark-Ocone formula*:

$$\sigma S_t \Delta_t = \mathbb{E}^{\mathbb{Q}}[\mathbf{D}_t^{\mathbb{Q}}[\psi]|\mathcal{F}_t],$$

where $\mathbf{D}^{\mathbb{Q}}$ is the *Malliavin derivative* under $\mathbb{Q}$. For example, for European put

$$\mathbf{D}_t^{\mathbb{Q}}[\max(K - S_T, 0)] = -1_{\{S_T<K\}}\mathbf{D}_t^{\mathbb{Q}}[S_T] = -1_{\{S_T<K\}}\sigma S_T,$$

resulting in

$$\Delta_t = -\frac{1}{S_t}\mathbb{E}^{\mathbb{Q}}[1_{\{S_T<K\}}S_T]|\mathcal{F}_t] = -\widetilde{\mathbb{Q}}[S_T < K|\mathcal{F}_t],$$

where

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{S_T}{S_0}.$$

Complete financial model

There are $d+1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a *semimartingale* on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$.

Let $\mathcal{Q}$ denote the family of *martingale measures* for S, that is,

$$\mathcal{Q} = \{\mathbb{Q} \sim \mathbb{P} : \quad S \text{ is a local martingale under } \mathbb{Q}\}$$

Assumption

$\mathcal{Q} \neq \emptyset \quad \Longleftrightarrow \quad$ The model arbitrage-free (No FLVR).

Question

Is the model complete? In other words, does it allow replication of any non-traded derivative?

2nd fundamental theorem

Definition

The model is *complete* if for any random variable ψ with $0 \leq \psi \leq 1$ one can find a strategy with wealth process X such that $0 \leq X \leq 1$ and $X_T = \psi$.

Theorem (2nd FTAP)

$$\textit{Completeness} \quad \Longleftrightarrow \quad |\mathcal{Q}| = 1.$$

The theorem is stated in Harrison and Pliska (1983) and follows from an integral representation theorem in Jacod (1979).

Risk-Neutral Valuation

Consider a European option with payoff Ψ at maturity T. The formula

$$V_0 = \mathbb{E}^{\mathbb{Q}}[\Psi],$$

where $\mathbb{Q} \in \mathcal{Q}$ is called *Risk-Neutral Valuation*.

Arbitrage-free models:

$$\boxed{\textit{Unique Arbitrage-Free Pricing} = \textit{Replication}}$$

Complete models: (no replication!)

$$\boxed{\textit{Arbitrage-Free Pricing} = \textit{Risk-Neutral Valuation}}$$

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Part III

Optimal investment

Outline

Introduction to optimal investment

Consider an economic agent (an investor) in an *arbitrage-free* financial model.

x: initial capital

Goal: invest x *"optimally"* up to maturity T.

Question

How to compare two investment strategies:

1. $x \longrightarrow X_T = X_T(\omega)$
2. $x \longrightarrow Y_T = Y_T(\omega)$

Clearly, we would prefer 1st to 2nd if $X_T(\omega) \geq Y_T(\omega)$, $\omega \in \Omega$. However, as the model is arbitrage-free, in this case, $X_T(\omega) = Y_T(\omega)$, $\omega \in \Omega$.

Introduction to optimal investment

Classical approach (Von Neumann - Morgenstern, Savage): an investor is *"quantified"* by

$\mathbb{P}$: "scenario" probability measure

$U = U(x)$: *utility function*

"Quality" of a strategy

$$x \longrightarrow X_T = X_T(\omega)$$

is then measured by *expected utility*: $\mathbb{E}[U(X_T)]$.
Given two strategies: $x \longrightarrow X_T$ and $x \longrightarrow Y_T$ the investor will prefer the 1st one if

$$\mathbb{E}[U(X_T)] \geq \mathbb{E}[U(Y_T)]$$

Introduction to optimal investment

Inputs:

1. Arbitrage-free financial model (all traded securities)
2. Risk-averse investor:

   x: initial wealth

   $\mathbb{P}$: "real world" probability measure

   $U = U(x)$: strictly increasing and strictly concave utility function

Output: an optimal investment strategy with wealth $x \longrightarrow \widehat{X}_T$ such that

$$\mathbb{E}[U(\widehat{X}_T)] = u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Here $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Merton's problem

First papers in continuous time finance: Merton (1969).

Black and Scholes model: *a savings account* and *a stock*.

1. We assume that the interest rate is 0.
2. The price of the stock:

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right).$$

Here $W = (W_t)_{t \geq 0}$ is a Wiener process and

$\mu \in \mathbf{R}$: drift

$\sigma > 0$: volatility

Merton's problem

The problem of optimal investment

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)]$$

becomes in this case a *stochastic control problem*:

$$u(x,t) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_{T-t})] = \sup_{\pi} \mathbb{E}[U(X^{\pi}_{T-t})],$$

where the *controlled* process X^{π} is the wealth process:

$$dX^{\pi} = X^{\pi}\pi(\mu dt + \sigma dW) \quad X^{\pi}_0 = x$$

and the *control* process π is the proportion of the capital invested in stock.

Merton's problem

Bellman equation:

$$u_t + \sup_{\pi} \left[\pi x \mu u_x + \frac{1}{2} \pi^2 \sigma^2 x^2 u_{xx} \right] = 0.$$

It follows that

$$\begin{cases} u_t(x,t) &= \frac{\mu^2 u_x^2}{2\sigma^2 u_{xx}}(x,t) \\ u_{xx}(x,t) &< 0 \\ u(x,T) &= U(x) \end{cases}$$

and the optimal proportion:

$$\widehat{\pi}(x,t) = -\frac{\mu u_x}{\sigma^2 x u_{xx}}(x,t).$$

Merton's problem

In Merton (1969) the system was solved for the case, when

$$U(x, \alpha) = \frac{x^\alpha - 1}{\alpha} \quad (\alpha < 1).$$

Here

$$-\frac{U'(x)}{xU''(x)} = \frac{1}{1-\alpha} \quad (= \text{const!})$$

This key property is "inherited" be the solution:

$$\frac{u_x}{xu_{xx}}(x, t) = \text{const.}$$

Merton's problem

After this substitution the first equation in the system becomes

$$u_t = \text{const}\ x^2 u_{xx}$$

and could be solved analytically.
The optimal strategy (Merton's point):

$$\widehat{\pi} = \frac{\mu}{(1-\alpha)\sigma^2}.$$

Merton's problem

In general case, we define the conjugate function

$$v(y,t) = \sup_{x>0} [u(x,t) - xy]$$

The function v satisfies

$$v_t = \text{const } y^2 v_{yy}$$
$$v(y,T) = V(y) := \sup_{x>0} [U(x) - xy]$$

Methodology: compute v first and then find u from the inverse duality relationship:

$$u(x,t) = \inf_{y>0} [v(y,t) + xy]$$

Model of a financial market

There are $d + 1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

Assumption (No Arbitrage or No FLVR)

$$\mathcal{Q} \neq \emptyset$$

where $\mathcal{Q}$ is the family of martingale measures for S.

Economic agent or investor

x: initial capital

U: utility function for consumption at the maturity T such that

1. $U : (0, \infty) \to \mathbf{R}$
2. U is strictly increasing
3. U is strictly concave
4. The Inada conditions hold true:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Problem of optimal investment

The goal of the investor is to maximize *the expected utility of terminal wealth*:

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0$$

Here $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Assumption

The value function is finite:

$$u(x) < \infty, \quad x > 0.$$

Two main approaches

1. *Bellman equation*.
2. *Duality and martingales*. Basic idea: as

$$\mathbb{E}[U(\widehat{X}_T(x))] = \max_{X \in \mathcal{X}(0)} \mathbb{E}[U(\widehat{X}_T(x) + X_T)]$$

we have that for any $X \in \mathcal{X}(0)$

$$\mathbb{E}[U'(\widehat{X}_T(x))X_T] = 0$$

Hence, there is $\mathbb{Q} \in \mathcal{Q}$ such that

$$U'(\widehat{X}_T(x)) = \mathrm{const}\frac{d\mathbb{Q}}{d\mathbb{P}}$$

Investment in complete models

Complete model: $|\mathcal{Q}| = 1$

Define the functions

$$V(y) = \max_{x>0} \left[U(x) - xy \right], \quad y > 0.$$

$$v(y) = \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}}) \right) \right], \quad y > 0$$

Theorem

$$u(x) = \inf_{y>0} [v(y) + xy]$$

Investment in complete models

Theorem

The following conditions are equivalent:

1. *The dual value function* $v = v(y)$ *is finite:*
$$v(y) < \infty, \quad y > 0$$
2. *The primal value function* $u = u(x)$ *is strictly concave and satisfies the Inada conditions.*

Moreover, in this case, $\widehat{X}(x)$ *exists for any* $x > 0$ *and*

$$\widehat{X}_T(x) = -V'\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right), \quad y = u'(x).$$

Investment in complete markets

The optimal terminal wealth $\widehat{X}_T(x)$ is uniquely determined by the equations:

$$\widehat{X}_T(x) = -V'(y\frac{d\mathbb{Q}}{d\mathbb{P}})$$
$$\mathbb{E}_{\mathbb{Q}}[\widehat{X}_T(x)] = x$$

The optimal number of stocks $\widehat{H}_t(x)$ at time t is given by the integral representation formula:

$$\widehat{X}_t(x) = \mathbb{E}_{\mathbb{Q}}[\widehat{X}_T(x)|\mathcal{F}_t] = x + \int_0^t \widehat{H}_u(x)dS_u.$$

Back to Merton's problem

For Black and Scholes model we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\frac{\mu}{\sigma} W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2} T) = \exp(-\frac{\mu}{\sigma} W_T^{\mathbb{Q}} + \frac{1}{2}\frac{\mu^2}{\sigma^2} T),$$

where

$$W_t^{\mathbb{Q}} = W_t + \frac{\mu}{\sigma} t,$$

is the $\mathbb{Q}$-Brownian motion. We deduce

$$\widehat{H}_t(x) S_t = \frac{\mu}{\sigma^2} R_t(x),$$

where $R(x)$ is the *risk-tolerance wealth process* defined as the wealth process replicating the payoff:

$$R_T(x) := -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}.$$

Basic questions for incomplete models

1. Does the optimal investment strategy $X(x)$ exist?
2. Does the value function $u = u(x)$ satisfy the *standard* properties of a utility function? In other words,
   2.1 Is u strictly concave?
   2.2 Do Inada conditions

$$u'(0) = \infty, \quad u'(\infty) = 0$$

   hold true?

Basic questions for incomplete models

3. Does the conjugate function

$$v(y) = \sup_{x>0}\{u(x) - xy\}, \quad y > 0,$$

have the representation:

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})],$$

where

$$V(y) = \sup_{x>0}\{U(x) - xy\}, \quad y > 0?$$

Asymptotic elasticity

Recall that the *elasticity* for U is defined as

$$E(U)(x) = \frac{xU'(x)}{U(x)}$$

The crucial role is played by the *asymptotic elasticity*:

$$AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}.$$

We always have $AE(U) \leq 1$.

Assumption

$$AE(U) < 1.$$

Minimal market independent condition

Theorem (K. and Schachermayer (1999))

The following conditions are equivalent :

1. $AE(U) < 1$.
2. *For any financial model the "qualitative" properties 1–3 hold true.*

In addition, in this case

$$AE(u) \leq AE(U) < 1.$$

Remark

The condition $AE(U) < 1$ is similar to Δ_2-condition in the theory of Orlicz spaces.

Necessary and sufficient conditions

Theorem (K. and Schachermayer (2003))

The following conditions are equivalent for given financial model:

1. *For any* $y > 0$ *there is* $\mathbb{Q} \in \mathcal{Q}$ *such that*

$$\mathbb{E}[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)] < \infty.$$

2. *The "qualitative" properties 1–3 hold true.*

Dual space of supermartingales

The lower bound in

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$$

is, in general, not attained. However, if we extend the space of density processes of martingale measures to the space $\mathcal{Y}(y)$ of strictly positive supermartingales Y such that

1. $Y_0 = y$
2. XY is a supermartingale for any $X \in \mathcal{X}(x)$

then (without any extra assumptions!) we have

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)]$$

and the lower bound above is attained by $\widehat{Y}(y) \in \mathcal{Y}(y)$. *This is even more convenient for computations!*

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Part IV

Equilibrium

Outline

Dynamic (Radner) equilibrium

Inputs:

- M agents, utility functions U_m for consumption at common maturity T, initial random endowments Λ_m.
- Interest rate $r = (r_t)$; hereafter $r = 0$.

Output: financial market with J stocks having prices $S = (S_t^j)$ where the agents' optimal strategies (stock's quantities) $H^m = (H_t^{m,j})$ satisfy the *clearing condition* (zero-net supply):

$$\sum_{m=1}^{M} H_t^{m,j} = 0, \quad t \in [0, T], \ j = 1, \dots, J.$$

Construction of Radner equilibrium

Two steps procedure: (see Dana and Jeanblanc (2003))

1. Find static (Arrow-Debreu) equilibrium with pricing measure $\mathbb{Q}$: welfare theorems $+$ fixed point.
2. Find (any!) J-dimensional local martingale $S = (S_t^j)$ under $\mathbb{Q}$ such that the S market is complete $\Leftrightarrow$ Any $\mathbb{Q}$-local martingale M is a stochastic integral under S:

$$M = M_0 + \int H dS.$$

The second item is easy and the answer is *a priori* YES or NO because it does not depend on a choice of $\mathbb{Q} \sim \mathbb{P}$!.

Static (Arrow-Debreu) equilibrium

Inputs:

- M agents, utility functions U_m for consumption at common maturity T, initial random endowments Λ_m.
- Interest rate $r = 0$.

Output: pricing measure $\mathbb{Q}$ such that if the agents can trade *any* ($\mathbb{Q}$-integrable) contingent claim ξ at the price

$$p = \mathbb{E}^{\mathbb{Q}}[\xi]$$

then their optimal positions $(\widehat{\Lambda}_m)$ satisfy the clearing condition:

$$\sum_{m=1}^{M} \widehat{\Lambda}_m = \Lambda = \sum_{m=1}^{M} \Lambda_m.$$

Assumptions on agents

(A1) The initial random endowments are strictly positive:

$$\Lambda_m > 0$$

and the total initial (= terminal) wealth $\Lambda = \sum_m \Lambda_m$ has all (positive and negative) moments:

$$\mathbb{E}[\Lambda^p + \left(\frac{1}{\Lambda}\right)^p] < \infty, \quad p \geq 0.$$

(A2) Each utility function $U_m = U_m(x)$ is strictly increasing, strictly concave, and twice continuously differentiable on $(0, \infty)$. Moreover, it has a bounded relative risk aversion, that is, for some $c > 0$,

$$\frac{1}{c} \leq A_m(x) = -\frac{xU_m''(x)}{U_m'(x)} \leq c, \quad x \in (0, \infty).$$

Existence of Radner and Arrow-Debreu equilibrium

Theorem
Under (A1) and (A2) there exists an Arrow-Debreu equilibrium.

(A3) There exists a complete financial market (with $J < \infty$ stocks).

Remark
(A3) is a property of $(\Omega, (\mathcal{F}_t), \mathbb{P})$ and does not depend on a choice of *equivalent* $\mathbb{P}$.

Theorem
Under (A1), (A2), and (A3) there exists a Radner equilibrium.

Pareto optimal allocation

Theorem (1st welfare)

Any Arrow-Debreu equilibrium $\mathbb{Q}$ results in the optimal positions $(\widehat{\Lambda}_m)$ for the agents which are Pareto optimal.

Definition

Random variables $\alpha = (\alpha^m)_{1 \leq m \leq M}$ form *a Pareto allocation* if there is no other allocation $\beta = (\beta^m)_{1 \leq m \leq M}$ of the same total endowment:

$$\sum_{m=1}^{M} \beta^m = \sum_{m=1}^{M} \alpha^m,$$

which leaves all agents better off:

$$\mathbb{E}[U_m(\beta^m)] > \mathbb{E}[U_m(\alpha^m)] \quad \text{for all } 1 \leq m \leq M.$$

Pareto optimal allocation

Key observation: Given the total endowment Λ, the set of all possible Pareto optimal allocations is *finite-dimensional* and is parameterized by the interior of the simplex.

Denote by Σ^M the simplex and by $R = R(w, x)$ the representative agent utility:

$$\Sigma^M = \{w \in [0,1]^M : \sum_{m=1}^M w^m = 1\},$$

$$R(w, x) = \sup_{x^1+\cdots+x^M = x} \sum_{m=1}^M w^m U_m(x^m), \quad w \in \operatorname{int} \Sigma^M, x \in \mathbf{R}.$$

Pareto optimal allocation

Theorem ($\approx$ 2nd welfare)

The following statements are equivalent

1. *The allocation* $\alpha = (\alpha^m)_{m=1,\dots,M}$ *is Pareto optimal.*
2. *There is a (deterministic) vector* $w \in \operatorname{int} \Sigma^M$ *such that*

$$w^m U'_m(\alpha^m) = \frac{\partial R}{\partial x}(w, \sum_{m=1}^{M} \alpha^m), \quad m = 1, \dots, M.$$

Moreover, such a vector w *is defined uniquely.*

Remark

More common form of 2nd welfare theorem: a Pareto allocation is an Arrow-Debreu allocation for *some (non-zero) supply*.

Endogenous Radner equilibrium

Inputs:

- M agents, utility functions U_m for consumption at common maturity T, initial random endowments Λ^m.
- Interest rate $r = (r_t)$; hereafter $r = 0$.
- J stocks with terminal dividends $\psi = (\psi^j)$ (stocks are fixed in advance or *endogenously*)

Output: prices $S = (S_t^j)$ with terminal values

$$S_T = \psi,$$

such that the agents' optimal strategies (stock's quantities) $H^m = (H_t^{m,j})$ satisfy the *clearing condition* (with zero net supply):

$$\sum_{m=1}^{M} H_t^{m,j} = 0, \quad t \in [0,T],\ j = 1,\ldots,J.$$

Construction of equilibrium

Two steps procedure :

1. Find static (Arrow-Debreu) equilibrium, that is, find a pricing measure $\mathbb{Q}$ such that in the case when economic agents can trade *any* payoff ξ at the price

$$p = \mathbb{E}^{\mathbb{Q}}[\xi]$$

   then the clearing condition holds (the total wealth does not change). Method: fixed point.
2. Define $S_t = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t]$, $t \in [0, T]$, (ψ is the terminal dividend) and verify *endogenous* completeness of the S-market.

Martingale Integral Representation

$(\Omega, \mathcal{F}_T, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$: a complete filtered probability space.

$\mathbb{Q}$: an equivalent probability measure.

$S = (S_t^j)$: J-dimensional martingale under $\mathbb{Q}$.

We want to know whether any local martingale $M = (M_t)$ under $\mathbb{Q}$ admits an integral representation with respect to S, that is,

$$M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0, T],$$

for some predictable S-integrable process $H = (H_t^j)$.

- Completeness in Mathematical Finance.
- Jacod's Theorem (2nd FTAP): the integral representation holds iff $\mathbb{Q}$ is the only martingale measure for S.
- Easy to verify if S is given in terms of *local characteristics*.

Martingale Integral Representation

For verification of endogenous completeness in Radner equilibrium we need the following version.

Inputs: random variables $\zeta > 0$ and $\psi = (\psi^j)$

- The density of the martingale measure $\mathbb{Q}$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\,\zeta.$$

- ψ is the terminal value for S:

$$S_t = \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t], \quad t \in [0, T].$$

Problem

Determine (easily verifiable) conditions on ζ and ψ so that the martingale representation property holds under $\mathbb{Q}$ and S.

Assumptions

We present results from K. and Predoiu (2011).
The random variables $\psi = S_T$ and $\zeta = \text{const}\frac{d\mathbb{Q}}{d\mathbb{P}}$ are given by

$$\begin{aligned} \psi^j &= F^j(X_T), \quad j = 1, \dots, J, \\ \zeta &= G(X_T), \end{aligned}$$

where

- $F^j = F^j(x)$ and $G = G(x)$ are deterministic functions.
- $X = (X^i_t)$ is a d-dimensional diffusion:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, T],$$

with drift and volatility functions $b^i = b^i(t, x)$ and $\sigma^{ij} = \sigma^{ij}(t, x)$.

Assumptions on functions

1. The functions $F = F(x)$ and $G = G(x)$ are weakly differentiable and have exponential growth:

$$|\nabla F| + |\nabla G| \leq N e^{N|x|}$$

2. The Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i}\right)$ has rank d almost surely under the Lebesgue measure on $\mathbb{R}^d$.

In Anderson and Raimondo (2008) and Hugonnier et al. (2010) in item 2, the Jacobian matrix needs to have full rank only on some open set (counter-example in our setting).

Assumptions on the diffusion X

1. The drift vector $b = b(t, x)$ is bounded, analytic with respect to t, and measurable with respect to x.
2. The volatility matrix $\sigma = \sigma(t, x)$ is bounded, analytic with respect to t, uniformly continuous with respect to x:

$$|\sigma(t, x) - \sigma(t, y)| \leq \omega(|x - y|).$$

for some strictly increasing function $\omega = (\omega(\epsilon))_{\epsilon > 0}$ such that $\omega(\epsilon) \to 0$ as $\epsilon \downarrow 0$, and has a bounded inverse:

$$|\sigma^{-1}(t, x)| \leq N \quad \text{(uniform ellipticity for } \sigma\sigma^*).$$

- Counter-example on t-analyticity condition in $\sigma = \sigma(t, x)$.
- In Anderson and Raimondo (2008) X is a Brownian motion.
- In Hugonnier et al. (2010) the functions $b = b(t, x)$ and $\sigma = \sigma(t, x)$ are analytic with respect to both t and x.

Main result

Theorem

Under the conditions above the probability measure $\mathbb{Q}$ with the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{G(X_T)}{\mathbb{E}[G(X_T)]},$$

and the $\mathbb{Q}$-martingale

$$S_t = \mathbb{E}^{\mathbb{Q}}[F(X_T)|\mathcal{F}_t], \quad t \in [0, T],$$

with values in $\mathbb{R}^J$ are well-defined and any local martingale M under $\mathbb{Q}$ is a stochastic integral with respect to S.

Application to finance

Recall that a Pareto pricing measure corresponding to weights $w \in \operatorname{int} \Sigma^M$ has the form:

$$\zeta = \text{const} \frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const} \frac{\partial R}{\partial x}(w, \Lambda),$$

where

$$\Lambda = \sum_{m=1}^{M} \Lambda^m$$

is the total terminal wealth and

$$R(w, x) = \max_{x^1 + \cdots + x^M = x} \sum_{m=1}^{M} w^m U_m(x^m),$$

is the representative agent's utility.

Assumptions on agents and stocks

- The total terminal wealth of the agents $\Lambda = e^{H(X_T)}$, where $H = H(x)$ is Lipschitz continuous.
- Each utility function $U_m = U_m(x)$ is strictly increasing, strictly concave, and twice continuously differentiable on $(0, \infty)$. Moreover, for some $c > 0$,

$$\frac{1}{c} \leq A_m(x) = -\frac{xU_m''(x)}{U_m'(x)} \leq c, \quad x \in (0, \infty).$$

- The terminal stocks' values

$$S_T^j = \psi^j = F^j(X_T),$$

  where $F = F(x)$ is continuously differentiable, has exponential growth, and its Jacobian matrix has full rank on $\mathbb{R}^d$.

Endogenous completeness

Theorem

Under the conditions above, for any Pareto weight $w \in \operatorname{int}\Sigma^M$, a Pareto pricing measure $\mathbb{Q}$ with the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = const\frac{\partial R}{\partial x}(w, \Lambda)$$

and the $\mathbb{Q}$-martingale

$$S_t = \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0, T],$$

are well-defined and the S-market is complete.

Theorem

Under the conditions above there exists endogenous Radner equilibrium.

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