The Dynamics of Complex Polynomial Vector Fields in $\mathbb{C}$

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Objects of Study

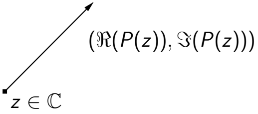
The set of **monic and centered polynomials** of degree $d \geq 2$

$$P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$$

parameterized by $\underline{a} = (a_0, \ldots, a_{d-2}) \in \mathbb{C}^{d-1}$.

The **associated vector field** in $\mathbb{C}$ is

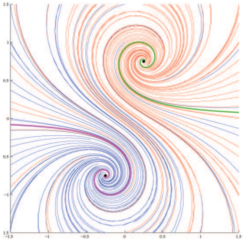
$$\xi_P = P(z)\frac{\mathrm{d}}{\mathrm{d}z}$$



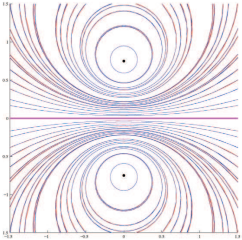
We study the **maximal trajectories** $t \mapsto \gamma(t, z_0)$ of ξ_P with $t \in \mathbb{R}$, i.e. maximal solutions to associated autonomous ODE

$$\dot{z} = P(z), \quad \gamma(0, z) = z$$

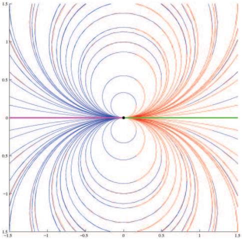
Quadratic Examples: $P(z) = z^2 + a$



$\zeta \notin \mathrm{i}\mathbb{R}$



$\zeta = \pm \mathrm{i}\sqrt{a}$



$\zeta = 0, \ \mathrm{mult}(\zeta) = 2$

Motivation (Holomorphic Dynamics)

Time-1 flow of $\dot{z} = f(z) - z$ approximates iteration of f

Important results utilizing vector fields:

- Parabolic bifurcations [Benziger, Shishikura, Oudkerk, Buff, Tan Lei, Epstein,. . .]
- Area of quadratic Julia sets [Buff, Cheritat]

Motivation ($\mathbb{R}^2$, $\mathbb{C}^2$ Vector Fields)

Open problem: Understand global dynamics of vector fields

$\mathbb{R}^2$ polynomial vector fields not completely understood:

- Hilbert's 16th problem (limit cycles)
  Perturbing $\mathbb{C}$ vector fields a common strategy
  [Alvarez, Gasull, Prohens; Llibre, Schlomiuk]

Higher order systems:

- $\mathbb{C}$ vector fields used to understand $\mathbb{C}^2$ and higher dimensional complex systems [Rousseau, Teyssier]

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Equilibrium Points of ξ_P - zeros of P

If ζ is a simple root of P with multiplier $\lambda = P'(\zeta)$ then ξ_P is holomorphically conjugate in a neighborhood of ζ to the linear vector field $\lambda z \frac{\mathrm{d}}{\mathrm{d}z}$.

Three cases:

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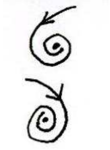
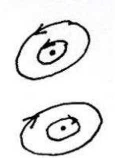
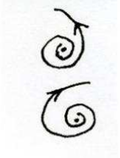
$\Im\lambda < 0$ $\Im\lambda > 0$			
	Sink	**Center**	**Source**

If ζ is a multiple root of P of multiplicity $m > 1$, then ξ_P has $m-1$ **attracting** and $m-1$ **repelling directions** at ζ.

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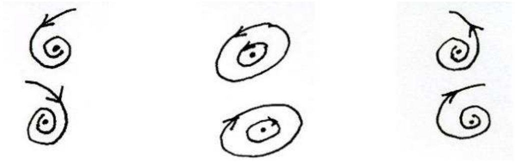

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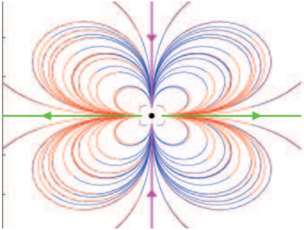
The Cubic Example $z^3 \frac{\mathrm{d}}{\mathrm{d}z}$

Multiple equilibrium point at 0 of multiplicity $m = 3$



Two attracting directions: $\pm i\mathbb{R}$
Two **attracting petals**

Two repelling directions: $\pm\mathbb{R}$
Two **repelling petals**

A **sepal** is the intersection of an attracting and repelling petal.
In this example: **four sepal zones**.

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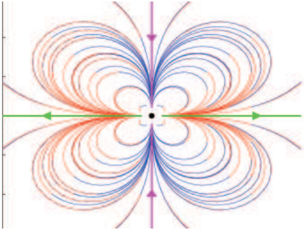

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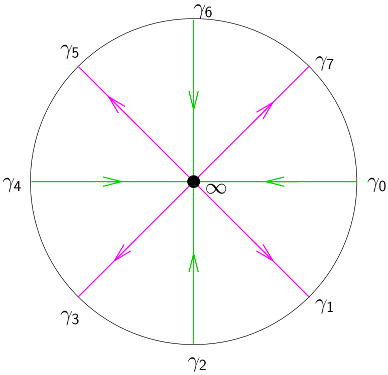
Trajectories at ∞

Can be shown that ∞ is a pole of order $d-2$ for ξ_P



- $2(d-1)$ trajectories γ_ℓ with asymptotic angles $\frac{2\pi\ell}{2(d-1)}$, $\ell \in \{0, 1, \ldots, 2d-3\}$
- ℓ even, **incoming** to ∞
- ℓ odd, **outgoing** from ∞

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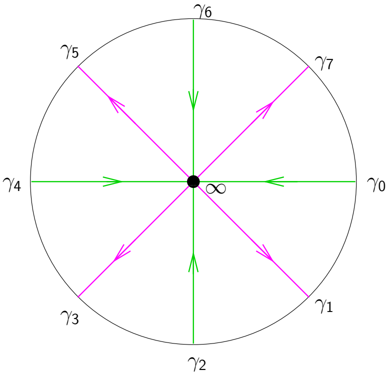

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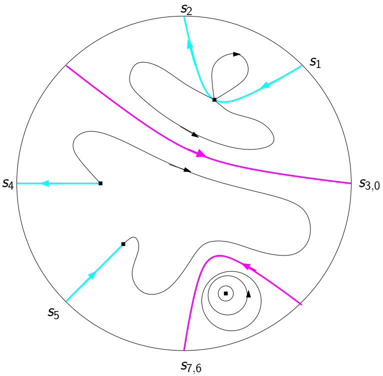

- **Separatrices** s_ℓ are maximal trajectories incoming to and outgoing from ∞
- labeled by $2(d-1)$ asymptotic angles
- s_ℓ may be **landing** or **homoclinic**

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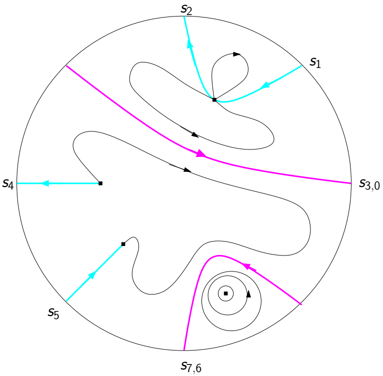

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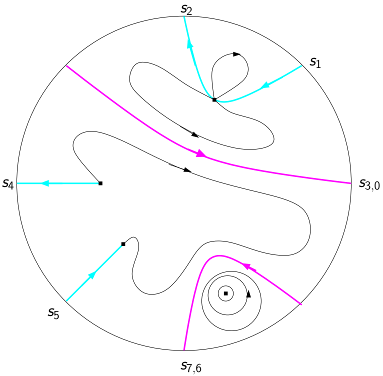

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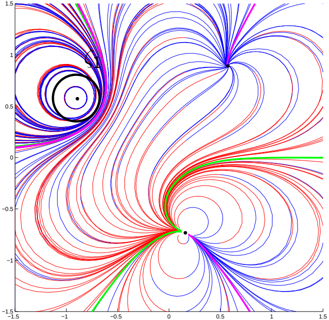
Example

$$P(z) = (z - \zeta_1)(z - \zeta_2)(z - \zeta_3)^2$$



- ζ_1 simple equilibrium pt.
- $P'(\zeta_1) = 10\mathrm{i} \Rightarrow \zeta_1$ center
- $P(z)\frac{\mathrm{d}}{\mathrm{d}z} \sim 10\mathrm{i}z\frac{\mathrm{d}}{\mathrm{d}z}$, $z \in V_{\zeta_1}$

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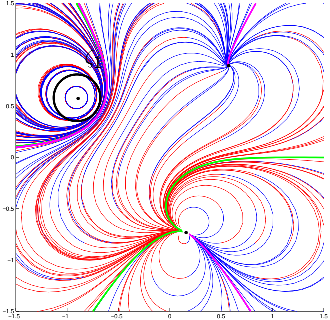

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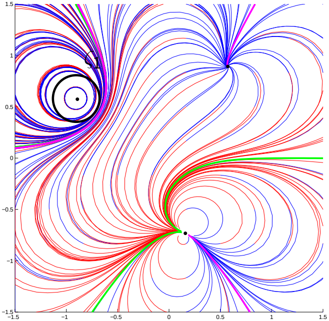

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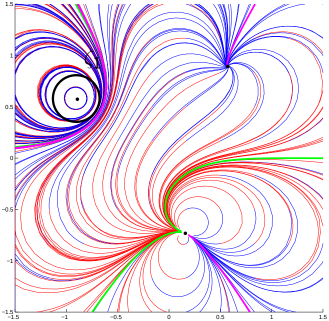

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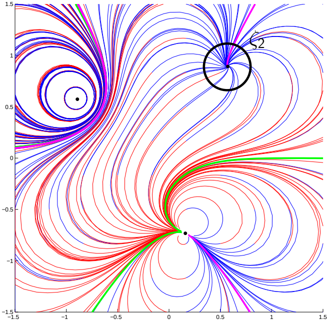

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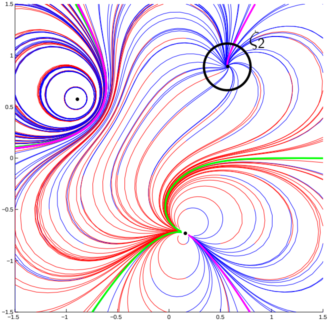

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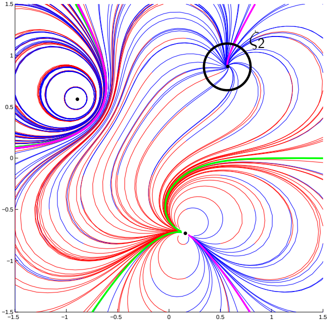

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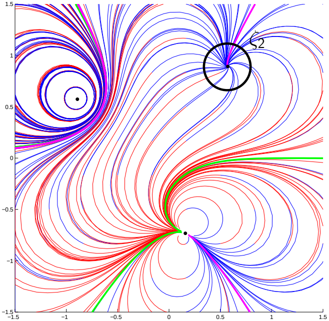

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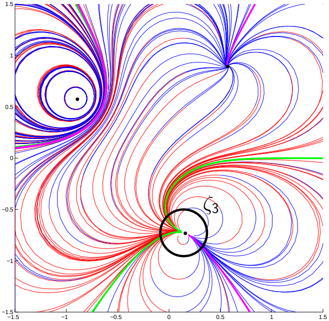

- ζ_3 double equilibrium pt.
- $\lambda = \mathrm{Res}(1/P, \zeta_3) = .096 + .128i$
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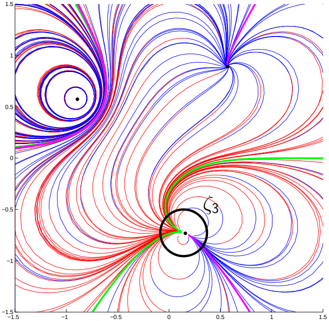

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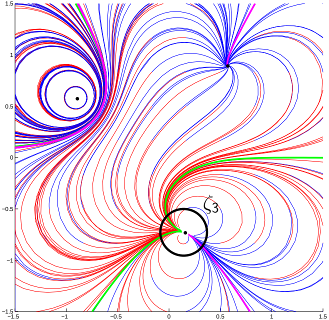

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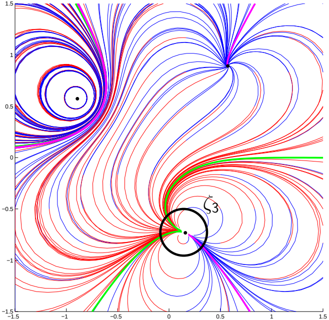

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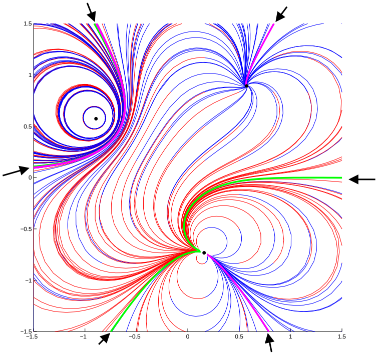

- ∞ pole of order 2
- There are $2(d-1) = 6$ trajectories meeting at infinity

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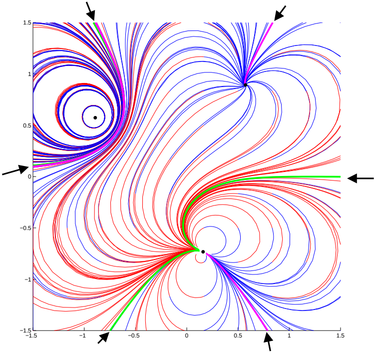

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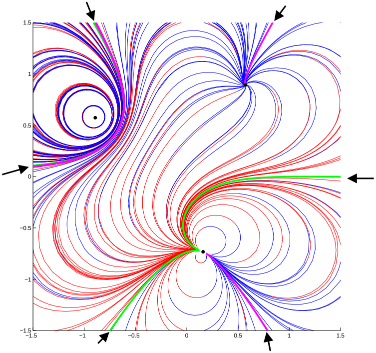

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Work So Far

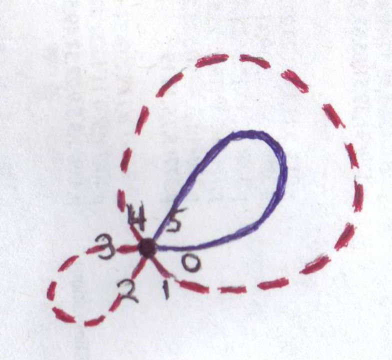
$\mathbb{C}$-vector fields: Partial results exist

- Local Dynamics [Jenkins, Strebel, Sverdlove, Brickman and Thomas, Gasull, Garijo, Jarque, Needham, Hájek,. . .]
- Global Topology [Neumann, Federov, Andronov]
- Bifurcations of rational vector fields under rotations [Muciño-Raymundo]

Douady, Estrada, Sentenac [DES]: classified structurally stable $\mathbb{C}$-polynomial vector fields

The Classification Problem

"**What are the objects of a given type, up to equivalence?**"

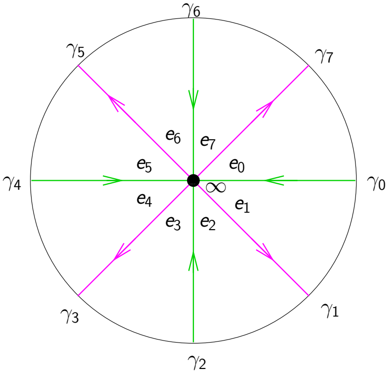


$$(3, 1+i, 3i) \in \mathbb{R}^1_+ \times \mathbb{H}^2_+$$

Combinatorial Invariant

Analytic Invariants

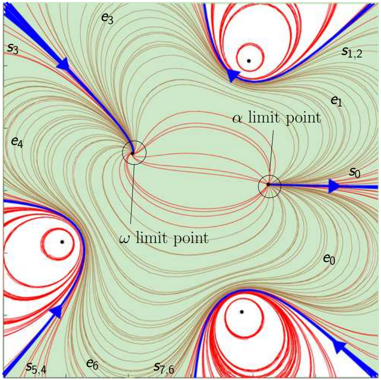
Accesses to ∞



- $2d - 2$ accesses to ∞
- An **end** e_ℓ is infinity with access between $\gamma_{\ell-1}$ and γ_ℓ

Zones

Three types of connected components of $\mathbb{C} \setminus (\{\text{seps}\} \cup \{\text{eq pts}\})$.



$\alpha\omega$-zones

- $\zeta_\alpha \neq \zeta_\omega$
- landing and homoclinic separatrices
- even and odd ends

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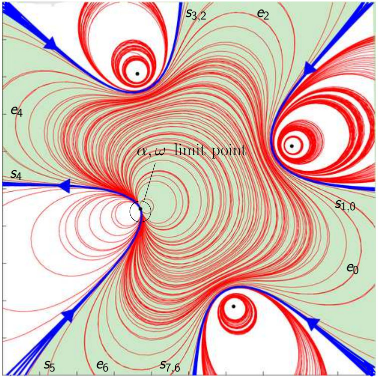

Sepal zones

- $\zeta_\alpha = \zeta_\omega$
- landing and homoclinic separatrices
- all even or all odd ends

Zones

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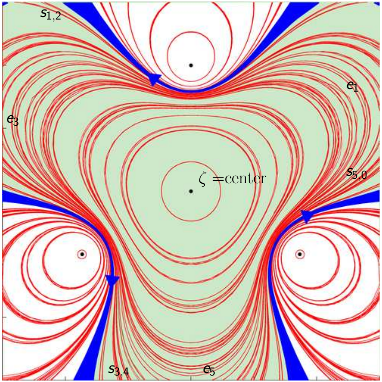


Center zones

- center contained in Z
- homoclinic separatrices
- all even or all odd ends

Rectifying Coordinates

In any simply connected domain avoiding zeros of P,

$$\phi(z) = \int_{z_0}^{z} \frac{dw}{P(w)}.$$

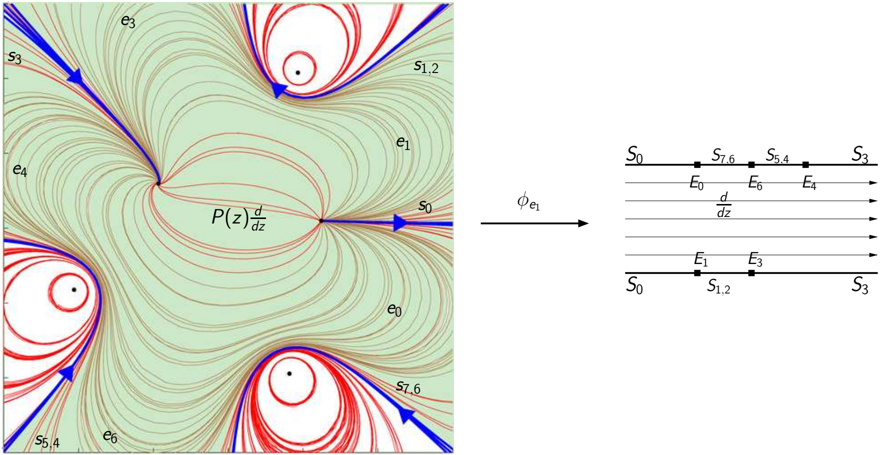
Note that

$$\phi_* (\xi_P) = \phi' (z) \, P (z) \, \frac{\mathrm{d}}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}.$$

The coordinates $w = \phi(z)$ are called **rectifying coordinates**.

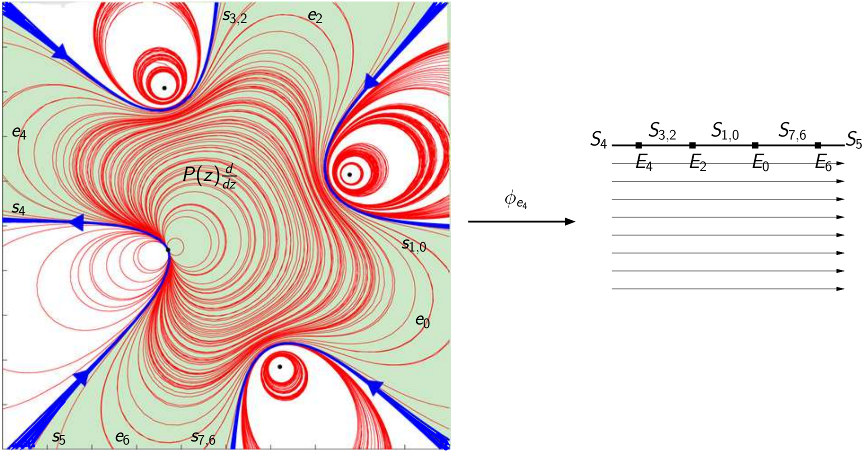
Rectifying Coordinates

Rectified $\alpha\omega$-zones are horizontal strips.



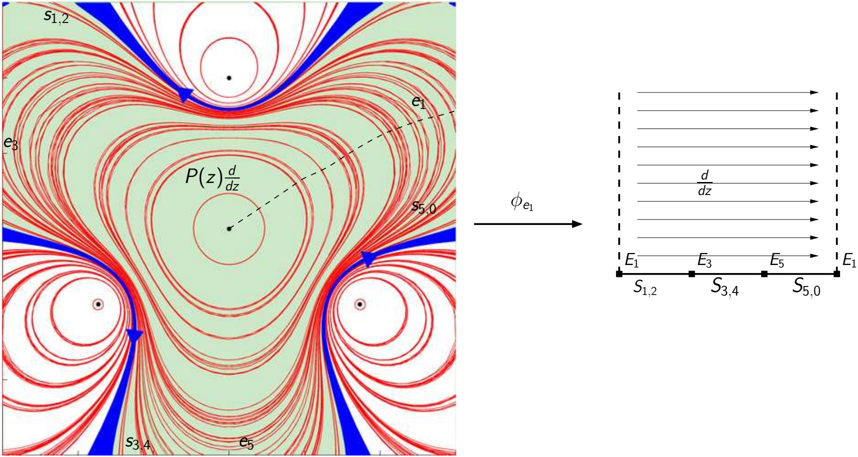
Rectifying Coordinates

Rectified sepal zones are upper or lower half planes.



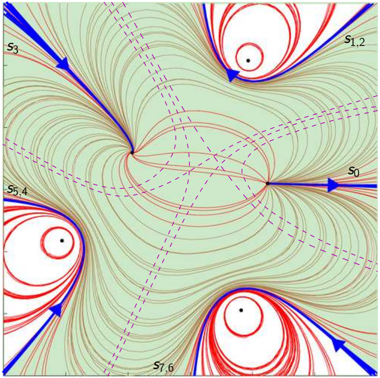
Rectifying Coordinates

Rectified center-zones are half-infinite cylinders.

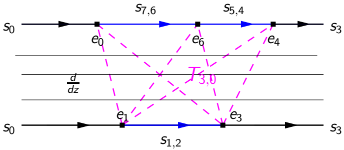


Transversals

Closed geodesics in $\hat{\mathbb{C}} \setminus \{\text{equilibrium pts}\}$ for $\frac{|dz|}{|P(z)|}$ through ∞:



There are h homoclinic separatrices



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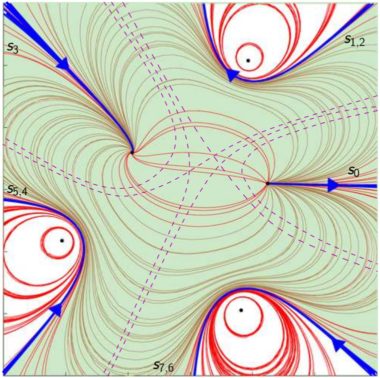

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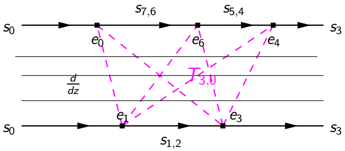

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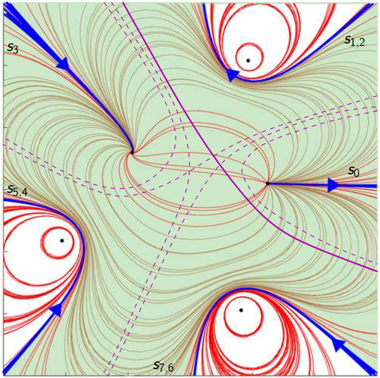

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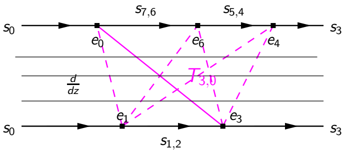

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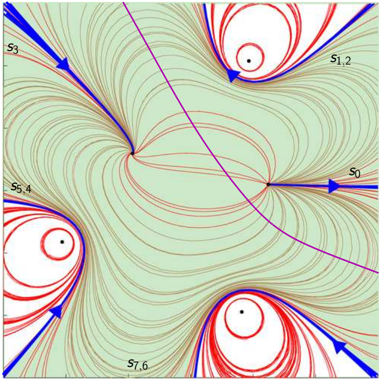
Analytic invariants: $(s+h)$-tuple in $\mathbb{H}^s \times \mathbb{R}_+^h$:



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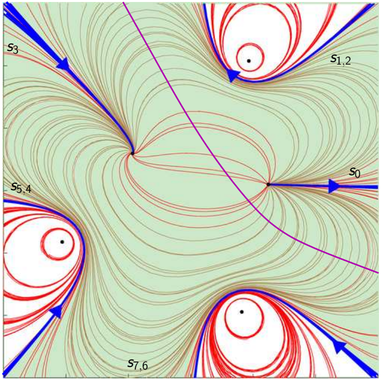

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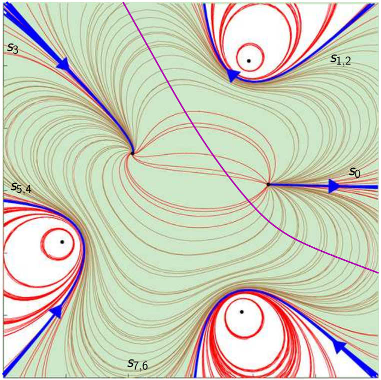
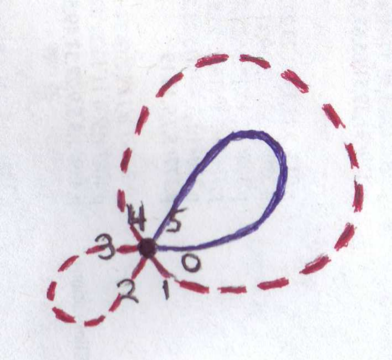


To each homoclinic separatrix is assigned $\int_{s_{k,j}} \frac{dz}{P(z)} > 0$

To each distinguished transversal is assigned $\int_T \frac{dz}{P(z)} \in \mathbb{H}$

Invariants for Polynomial Vector Fields



Combinatorial Invariant

$$(3, 1+i, 3i) \in \mathbb{R}^1_+ \times \mathbb{H}^2_+$$

Analytic Invariants

Classification Problem

Proposition

The combinatorial invariants $\mathcal{C}$ and analytic invariants $\mathcal{A}$ are **complete set of invariants**. *That is, the map*

$$\Phi : \Xi_d \to \bigcup (\mathcal{C}, \mathcal{A})$$

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Theorem (Branner, D.)

Given admissible combinatorial and analytic data, there exists a unique $\xi_P \in \Xi_d$ having the given invariants.

Parameter Space

The **space of polynomial vector fields** of degree d is

$$\Xi_d = \{\xi_P\} \simeq \mathbb{C}^{d-1}$$

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Decompose Ξ_d into disjoint loci $\mathcal{C}$ such that all $\xi_P \in \mathcal{C}$ have the same combinatorial invariant.

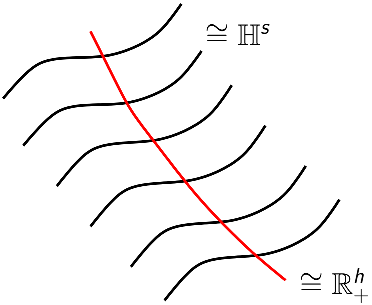
Two goals:

- Examine structure of each locus
- Determine how the loci fit together (bifurcations)

Properties of $\mathcal{C}$

Theorem (D.)

Each $\mathcal{C}$ is homeomorphic to $\mathbb{H}^s \times \mathbb{R}_+^h$, and $\mathcal{C}$ is naturally foliated by $\mathbb{C}$-analytic manifolds of complex dimension s.



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Each $\mathcal{C}$ is an $\mathbb{R}_+$ cone with $\underline{0} \in \mathbb{C}^{d-1}$ $\left(z^d \frac{\mathrm{d}}{\mathrm{d}z} \in \Xi_d\right)$ as base point.

$P(z) = \prod_{j=1}^{d}(z - \zeta_j)$ and $P_c(z) = \prod_{j=1}^{d}(z - c\zeta_j)$ in same class.

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Remark

All combinatorics achievable by bifurcations of $z^d \frac{\mathrm{d}}{\mathrm{d}z}$.

Enumeration Problem

Goal: Find closed-form expression for $\mathbf{c_d} := \sharp\mathcal{C}$ in $\mathbf{\Xi_d}$

Douady, Estrada, and Sentenac proved this is the Catalan number C_{d-1} for the **structurally stable** vector fields in Ξ_d.

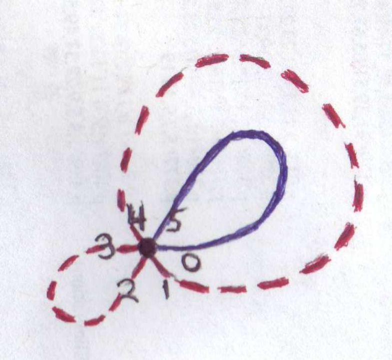
Theorem (D.)

$$c_d = [z^d]G(z) = \sum_{n=0}^{d-1} \frac{(2-2d)_n(1-d)_n 2^n}{(2)_n n!}$$
$$= {}_2F_1\left([2-2d, 1-d]; [2]; 2\right)$$

where $(x)_n = (x)(x+1)\cdots(x+n-1)$ *is the Pochhammer symbol*

Enumeration Problem: Idea of Proof

A **bracketing problem**: count configurations of pairings of square $[\cdots]$ and round $(\cdots)$ parentheses in the string $0\ 1\ \ldots 2d-3$.



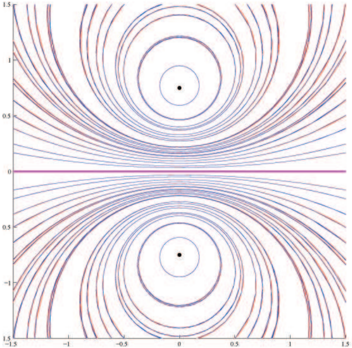
(0 [1 [2 3] 4] 5)

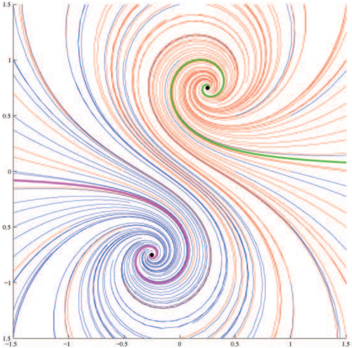
Combinatorial Invariant

Bracketing

Bifurcations

"**What are the qualitative changes when parameters vary?**"





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Given a point $\underline{a} \in \Xi_d$*, determine all possible combinatorics in a neighborhood* $V_{\underline{a}}$*.*

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A class is either **structurally stable** or in the **bifurcation locus**.

Structurally Stable Vector Fields

Theorem (Tan, D.)

For fixed multiplicity, if s_ℓ lands at $\zeta_{\underline{a}}$ at $\underline{a}$, then s_ℓ lands at $\zeta_{\underline{a}'}$ for $\underline{a}' \in V_{\underline{a}}$.

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Vector fields with no multiple equilibrium points and no homoclinc separatrices are **structurally stable**.

Vector fields having a homoclinic separatrix or multiple point belong to the **bifurcation locus**.

Bifurcations for fixed multiplicities: Type I

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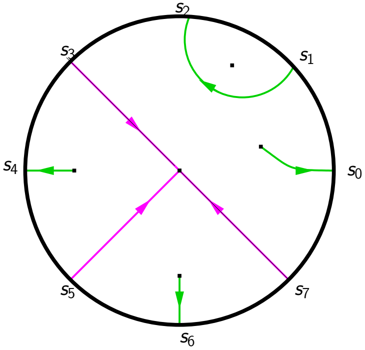
$\underline{a} \in \Xi_5 \cong \mathbb{C}^4$



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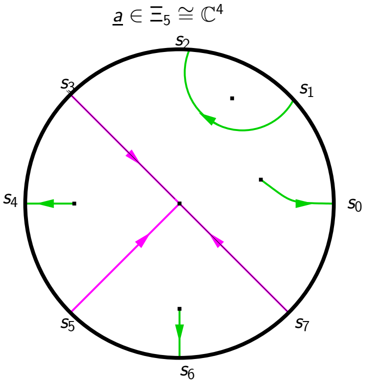

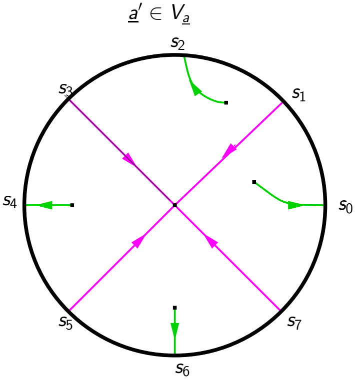

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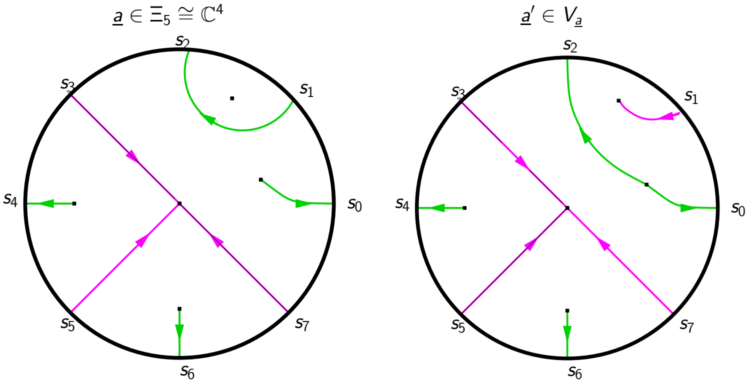

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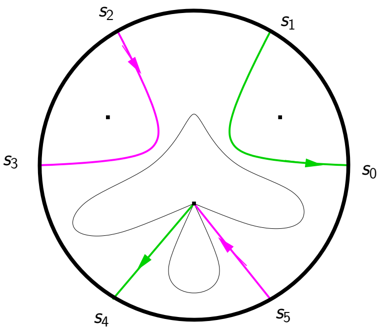

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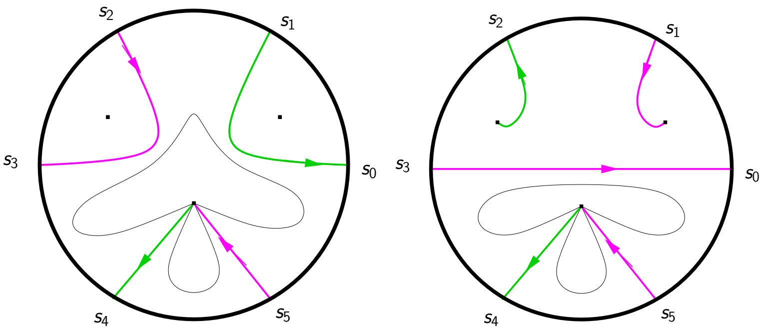

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Proof idea.

Show that for every $\mathcal{C} \cap \tilde{V}_{\underline{a}} \neq \emptyset$, there exists a sequence of classes $\mathcal{C}_i, \ i = 1, \dots, k-1$ satisfying $\mathcal{C}_i \cap \tilde{V}_{\underline{a}} \neq \emptyset$, such that

$$\partial\mathcal{C} \supset \mathcal{C}_1, \ \partial\mathcal{C}_1 \supset \mathcal{C}_2, \dots, \partial\mathcal{C}_{k-1} \supset \mathcal{C}_{\underline{a}},$$

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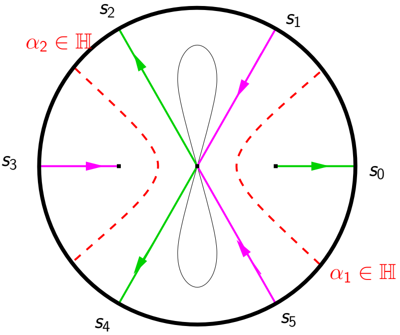
Remark

Proof relies on $\mathcal{C}_1 \cap \partial\mathcal{C}_2 \neq \emptyset \Rightarrow \mathcal{C}_1 \subset \partial\mathcal{C}_2$ *for* $\mathcal{C} \cap \tilde{V}_{\underline{a}} \neq \emptyset$.

Local Combinatorics Depends on Analytic Data

Combinatorics in $V_{\underline{a}}$ for $\xi_{\underline{a}}$ with multiple points

$\mathcal{C}_{\underline{a}}, \quad \underline{a} \in \Xi_4 \cong \mathbb{C}^3$



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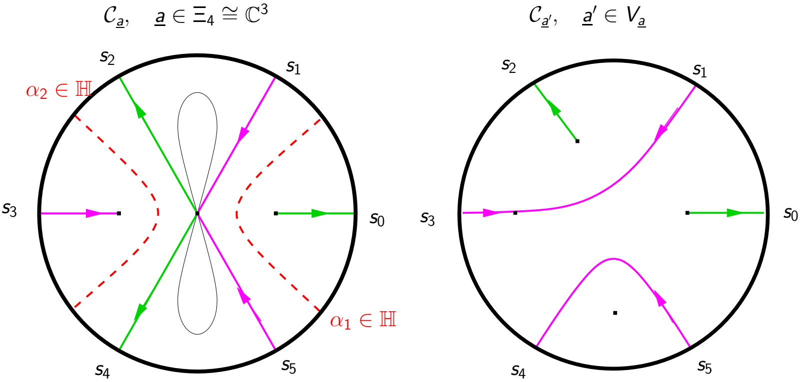

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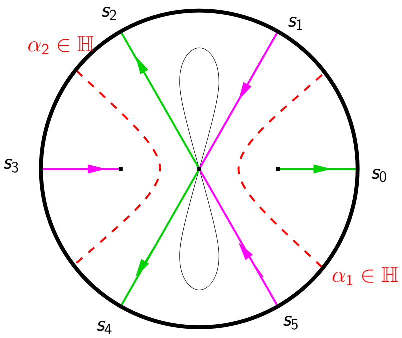

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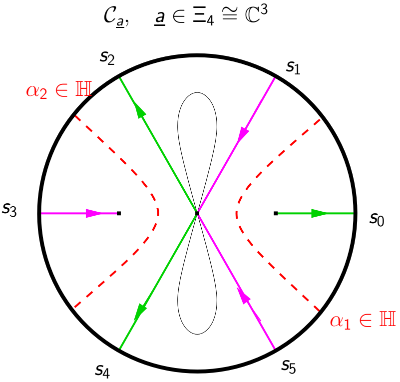

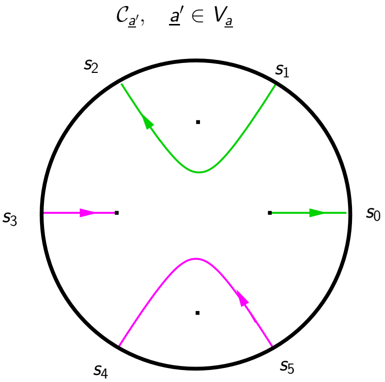

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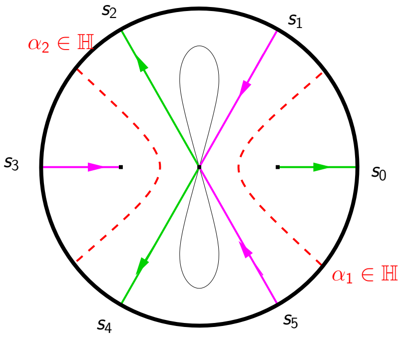

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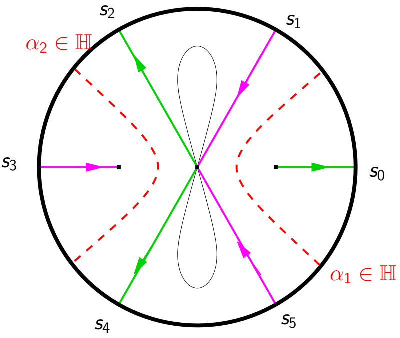

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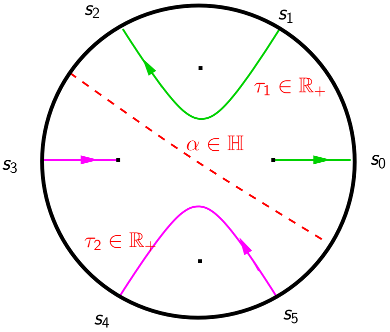

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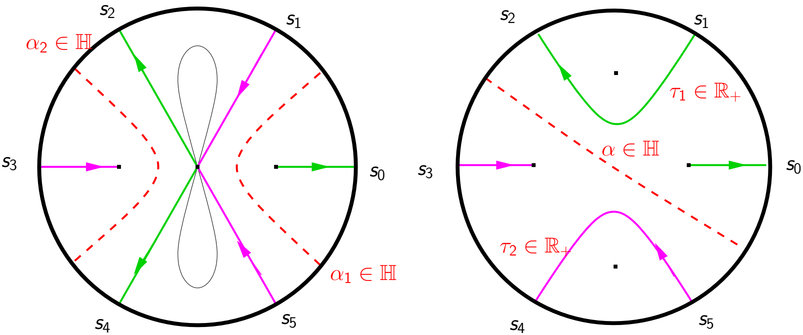

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Problem

Given $\underline{a} \in \mathcal{C}_{\underline{a}}$ and $\mathcal{C}$ such that $\underline{a} \in \partial\mathcal{C}$, prove that there exists a sequence of combinatorial classes $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$ and respective subsets $C_1, \ldots, C_{k-1}$ of these such that

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Conjecture

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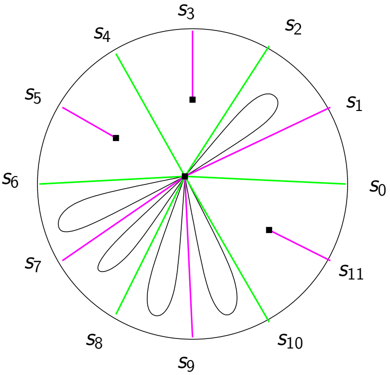

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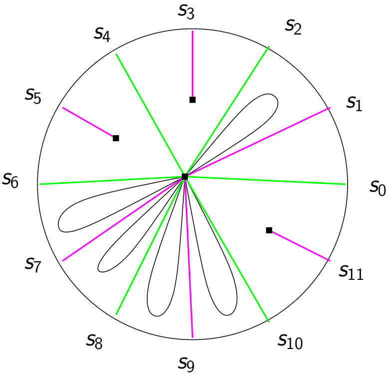

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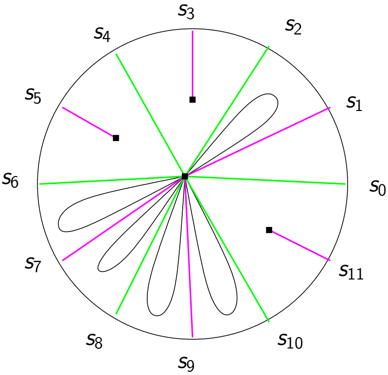

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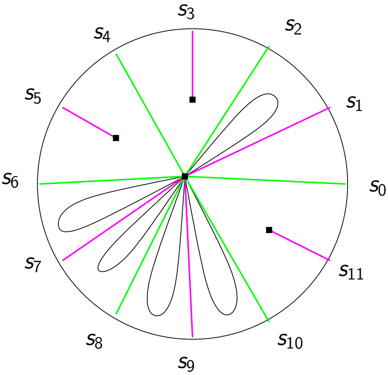

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Thank You!