Beauville surfaces and groups: a survey

Gareth A. Jones
School of Mathematics
University of Southampton
Southampton SO17 1BJ, U.K.
G.A.Jones@maths.soton.ac.uk

Abstract

This is a survey of recent progress on Beauville surfaces, concentrating almost entirely on the group-theoretic and combinatorial problems associated with them. A Beauville surface $\mathcal{S}$ is a complex surface formed from two orientably regular hypermaps of genus at least 2 (viewed as compact Riemann surfaces and hence as algebraic curves), with the same automorphism group G acting freely on their product. The following questions are discussed: Which groups G (called Beauville groups) have this property? What can be said about the automorphism group and the fundamental group of $\mathcal{S}$? Beauville surfaces are defined (as algebraic varieties) over the field $\overline{\mathbb{Q}}$ of algebraic numbers, so how does the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ act on them?

MSC classification: primary 20B25, secondary 05C10, 11G32, 14J25, 14J50, 51M20.

1 Introduction

The objects now known as Beauville surfaces[1] were introduced by the algebraic geometer Arnaud Beauville in [5, p. 159]. A Beauville surface $\mathcal{S}$ is a complex surface of general type [5, 33], constructed from a pair of orientably regular hypermaps (regular dessins, in Grothendieck's terminology [34]) of genus at least 2, with the same automorphism group G. The basic idea is that $\mathcal{S}$ can be designed to have certain properties by appropriate choices of G and its actions on the hypermaps. Since 2000, the geometric properties of Beauville surfaces, such as their rigidity (discussed in Section 9) have been intensively studied by Bauer, Catanese, Grunewald and others (see [3, 4, 9] for instance). More recently, group-theorists such as Guralnick, Lubotzky, Magaard, Malle and others have been interested

[1]Here, as is customary in algebraic geometry, a 'surface' is an algebraic variety which is 2-dimensional over the field of coefficients; in this case, that field is $\mathbb{C}$ so these surfaces have dimension 4 as real manifolds. Rather confusingly, a complex algebraic curve, 1-dimensional over $\mathbb{C}$, can be regarded as a Riemann surface, where 'surface' now indicates 2-dimensionality over $\mathbb{R}$!

in determining which groups G (known as Beauville groups) can be used in this construction. The rigidity properties of Beauville surfaces have been used by the author [39] to determine the structure of the automorphism group of a Beauville surface, and by González-Diez, Torres-Teigell and the author [31, 30] to extend an example of Serre [55], constructing arbitrarily large orbits of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ consisting of mutually non-homeomorphic algebraic varieties. This survey will describe some of these discoveries, and in addition, will suggest that there are interesting combinatorial questions to be investigated, including connections between Beauville surfaces and polytopes.

The paper is organised as follows. Section 2 explains how Belyĭ's Theorem gives a link between curves and hypermaps, used in Section 3 to give two equivalent definitions of a Beauville surface. These are translated into purely group-theoretic terms in Section 5, after a brief discussion of possible links with polytopes in Section 4. Various classes of Beauville groups are described in Sections 6–8. The fundamental groups and automorphism groups of Beauville surfaces are described in Sections 9 and 10, and the absolute Galois group and its action on Beauville surfaces are discussed in Sections 10 and 11.

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2 Curves and hypermaps

Since Beauville surfaces are constructed from pairs of hypermaps on algebraic curves, this section will briefly summarise the connection between curves and hypermaps.

Compact Riemann surfaces are the same as algebraic curves (smooth, projective, defined over $\mathbb{C}$). This fact, first discovered by Riemann, is now expressed as an equivalence of categories: see [26, 53] for details. It is particularly interesting to know which compact Riemann surfaces are defined (as algebraic varieties) over various subfields of $\mathbb{C}$. Belyĭ's Theorem answers this question for the field $\overline{\mathbb{Q}}$ of algebraic numbers, by showing that the following conditions on a compact Riemann surface (or algebraic curve) $\mathcal{C}$ are equivalent:

(a) $\mathcal{C}$ is defined over $\overline{\mathbb{Q}}$;

(b) there is a meromorphic function $\beta : \mathcal{C} \to \mathbb{P}^1(\mathbb{C})$ branched over at most three points;

(c) $\mathcal{C}$ is uniformised by a subgroup K of finite index in a triangle group Δ;

(d) the complex structure on $\mathcal{C}$ is obtained, in a canonical way, from a hypermap $\mathcal{H}$ on $\mathcal{C}$.

A curve with these properties is called a *Belyĭ curve*. In fact, Belyĭ [6] gave an ingenious proof that (a) implies (b), and a two-line argument, referring to Weil's Rigidity Theorem [64], for the converse; full details (which are rather intricate) were later provided by

Wolfart [67] and Köck [47]. Conditions (c) and (d) are straightforward reinterpretations of (b), due to Grothendieck, Wolfart and others (see [26, 42, 68], for instance). Belyĭ's Theorem has been extended to complex surfaces by González-Diez [28]. He and Girondo have written a very readable account of Belyĭ's Theorem and related matters in [26].

In (b), $\mathbb{P}^1(\mathbb{C})$ is the complex projective line (or Riemann sphere) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; the three ramification points can be assumed, by applying a Möbius transformation, to be $0, 1$ and ∞; a function β with these properties is called a *Belyĭ function*.

In (c), uniformisation means that $\mathcal{C} \cong \mathcal{U}/K$ where $\mathcal{U}$ is one of the three simply connected Riemann surfaces, namely $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or the hyperbolic plane $\mathbb{H}$, and K is a subgroup of a triangle group Δ acting as a group of automorphisms of $\mathcal{U}$. The inclusion $K \to \Delta$ induces a covering $\gamma : \mathcal{U}/K \to \mathcal{U}/\Delta$ corresponding to β:

$$\begin{array}{ccc} \mathcal{U}/K & \cong & \mathcal{C} \\ \gamma \downarrow & & \downarrow \beta \\ \mathcal{U}/\Delta & \cong & \mathbb{P}^1(\mathbb{C}) \end{array}$$

The degree (number of sheets) of this covering is equal to the index of K in Δ. We will be mainly interested in the case where $\mathcal{C}$ has genus at least 2, so that $\mathcal{U} = \mathbb{H}$.

In (d), a hypermap $\mathcal{H}$ on a curve $\mathcal{C}$ can be represented in several ways. Perhaps the most natural way is as a tripartite triangular map $\mathcal{T}$. This consists of a tripartite graph embedded in $\mathcal{C}$ with triangular faces; the three colour classes of vertices represent the hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, and the edges correspond to incidences between them. This map can be constructed as the inverse image under β of the trivial triangulation of $\mathbb{P}^1(\mathbb{C})$; this has vertices at $0, 1$ and ∞, joined by three edges along $\mathbb{R}$, and two triangular faces (the upper and lower half planes), so that its edges and faces lift to $\mathcal{C}$ without branching, which occurs only at the vertices. Thus $\mathcal{T}$ has triangular faces, and its vertices can be 3-coloured as they lie over $0, 1$ or ∞.

A more economical and frequently-used representation of $\mathcal{H}$ is as a bipartite map $\mathcal{B}$ on $\mathcal{C}$, called the *Walsh map* of $\mathcal{H}$ [63]. This can be formed from $\mathcal{T}$ by deleting the vertices over ∞ and their incident edges; topologically, no information is lost since one can retrieve $\mathcal{T}$ (up to homeomorphisms fixing the graph) by stellating $\mathcal{B}$, placing a vertex in each face of $\mathcal{B}$, joined by mutually disjoint edges to the incident vertices. Equivalently, $\mathcal{B}$ is the inverse image under β of the trivial bipartite map on $\mathbb{P}^1(\mathbb{C})$; this consists of two vertices at 0 and 1, joined by an edge along the unit interval, and one face. Both $\mathcal{T}$ and $\mathcal{B}$ can also be formed as the quotients by K of Δ-invariant maps of the same type on the universal covering space $\mathcal{U}$ of $\mathcal{C}$ (see [43]).

The most symmetric Belyĭ curves $\mathcal{C}$ (and the only ones we will consider here) are the *quasiplatonic curves*, those which have a Belyĭ function β which is a regular covering, that is, there is a group G of automorphisms of $\mathcal{C}$ inducing the covering $\beta : \mathcal{C} \to \mathcal{C}/G \cong \mathbb{P}^1(\mathbb{C})$. This is equivalent to $\mathcal{C}$ being uniformised by a torsion-free normal subgroup K of finite index in a triangle group Δ, with $\Delta/K \cong G$; then K is a surface group, isomorphic to the fundamental group $\pi_1\mathcal{C}$ of $\mathcal{C}$. This is also equivalent to the hypermap $\mathcal{H}$ in (d) being (orientably) regular, with orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{H} \cong G$; this means that G is a group of orientation- and colour-preserving automorphisms of $\mathcal{T}$, with two orbits (necessarily regular) on the faces of $\mathcal{T}$, or equivalently one regular orbit on the edges of $\mathcal{B}$.

Example 1. Let $\mathcal{C}$ be the Fermat curve

$$\mathcal{F}_n = \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) \mid x^n + y^n + z^n = 0\}$$

of degree n. This is a compact Riemann surface, visibly defined over $\mathbb{Q}$, and hence over $\overline{\mathbb{Q}}$. The meromorphic function

$$\beta : [x, y, z] \mapsto -\left(\frac{x}{z}\right)^n$$

is an n^2-sheeted covering $\mathcal{C} \to \mathbb{P}^1(\mathbb{C})$, branched where $xyz = 0$, that is, over $0, 1$ and ∞, each of which lifts to n points on $\mathcal{C}$. The triangulation $\mathcal{T}$ therefore has $3n$ vertices, $3n^2$ edges and $2n^2$ faces, so $\mathcal{C}$ has Euler characteristic $n(3-n)$ and hence genus $(n-1)(n-2)/2$; the underlying graph of $\mathcal{T}$ is, in fact, the complete tripartite graph $K_{n,n,n}$, and this is a minimum genus embedding of that graph. Similarly, $\mathcal{B}$ is an embedding of the complete bipartite graph $K_{n,n}$: see Fig. 1 for the case $n = 3$, with opposite sides of the outer hexagon identified to form a torus.

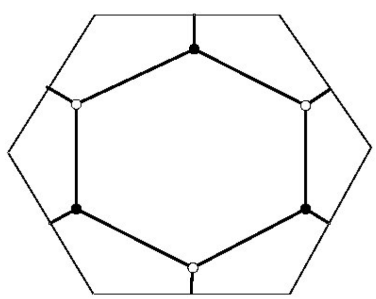

Figure 1: $K_{3,3}$ embedded in the Fermat curve $\mathcal{F}_3$

As a Riemann surface, $\mathcal{C}$ is uniformised by the commutator subgroup $K = \Delta'$ of the triangle group $\Delta = \Delta(n, n, n)$, acting on $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or $\mathbb{H}$ as $n < 3$, $n = 3$ or $n > 3$. This is a normal subgroup of Δ, with $\Delta/K \cong G := \mathbb{Z}_n \oplus \mathbb{Z}_n$ acting as a group of automorphisms

$$(j, k) : [x, y, z] \mapsto [\zeta_n^j x, \zeta_n^k y, z] \quad (j, k \in \mathbb{Z}_n)$$

of $\mathcal{C}$, where $\zeta_n = \exp(2\pi i/n)$, and inducing the regular covering β. Thus $\mathcal{C}$ is a quasiplatonic curve, and $\mathcal{H}$ is an orientably regular hypermap with $\mathrm{Aut}^+\mathcal{H} \cong G$. (In fact, if $n > 3$ then the full automorphism group of $\mathcal{C}$ is a semidirect product of G by S_3, permuting the coordinates x, y and z; this corresponds to K being normal in the maximal triangle group $\Delta(2, 3, 2n)$, which contains $\Delta(n, n, n)$ as a normal subgroup with quotient group S_3.)

3 Definition of a Beauville surface

We say that $\mathcal{S}$ is a *Beauville surface* (of unmixed type) if

1. $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ where each $\mathcal{C}_i$ is a complex projective algebraic curve of genus $g_i > 1$, and G is a finite group acting faithfully as a group of automorphisms of each $\mathcal{C}_i$, so that it acts freely (i.e. without fixed points) on $\mathcal{C}_1 \times \mathcal{C}_2$;

2. for each i, $\mathcal{C}_i/G$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$, with the induced projection $\beta_i : \mathcal{C}_i \to \mathbb{P}^1(\mathbb{C})$ branched over three points.

Here we will ignore the technically more difficult case of Beauville surfaces of mixed type [4, §7], where half the elements of G transpose two isomorphic factors $\mathcal{C}_i$. The product $\mathcal{C}_1 \times \mathcal{C}_2$ is a complex manifold (in fact, an algebraic variety) of dimension 2, and hence so is the quotient $\mathcal{S}$ since G acts freely on $\mathcal{C}_1 \times \mathcal{C}_2$. In combinatorial terms, the above conditions can be restated as follows:

1. $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ where each $\mathcal{C}_i$ is a quasiplatonic curve of genus $g_i > 1$, carrying an orientably regular hypermap $\mathcal{H}_i$ with $\mathrm{Aut}^+\mathcal{H}_i \cong G$;

2. the induced action of G on $\mathcal{H}_1 \times \mathcal{H}_2$ is fixed-point-free.

Thus a Beauville surface is formed from a pair of orientably regular hypermaps of hyperbolic type, with the same automorphism group acting freely on their product.

4 Combinatorial structures

If $\mathcal{S}$ is a Beauville surface $(\mathcal{C}_1 \times \mathcal{C}_2)/G$, then each curve $\mathcal{C}_i$ carries an orientably regular hypermap (or regular dessin) $\mathcal{H}_i$ with $\mathrm{Aut}^+\mathcal{H}_i \cong G$. As explained in Section 2, these hypermaps can be represented combinatorially in several ways, as triangulations $\mathcal{T}_i$ or bipartite maps $\mathcal{B}_i$, for instance. These combinatorial structures on the curves $\mathcal{C}_i$ induce further combinatorial structures on $\mathcal{C}_1 \times \mathcal{C}_2$, and hence, by means of the smooth covering $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$, on the Beauville surface $\mathcal{S}$.

For example $\mathcal{T}_1 \times \mathcal{T}_2$ can be regarded as a 4-dimensional CW-complex structure on $\mathcal{C}_1 \times \mathcal{C}_2$: each 2-cell is either a triangle (the product of a vertex on one curve and a face on the other) or a square (the product of two edges), each 3-cell is a triangular prism (the product of a triangular face and an edge), and each 4-cell is the product of two triangles (a 3,3 - duoprism). In addition, the 3-colourings of the vertices of the triangulations $\mathcal{T}_i$ induce a 9-colouring of the vertices of $\mathcal{T}_1 \times \mathcal{T}_2$. This structure, including its vertex-colouring, is invariant under the natural action of $G \times G$ on $\mathcal{C}_1 \times \mathcal{C}_2$. The free action of the diagonal subgroup means that the quotient surface $\mathcal{S}$ inherits the structure of a 4-dimensional CW-complex $(\mathcal{T}_1 \times \mathcal{T}_2)/G$, with the number of k-cells divided by $|G|$ for each dimension $k = 0, \ldots, 4$. This structure on $\mathcal{S}$ is preserved by the automorphisms of $\mathcal{S}$, which are described in Section 10. Similarly, the bipartite maps $\mathcal{B}_i$ induce a CW-complex $(\mathcal{B}_1\times\mathcal{B}_2)/G$ on $\mathcal{S}$, with each k-cell a union of k-cells of $(\mathcal{T}_1\times\mathcal{T}_2)/G$.

Although Beauville surfaces have been studied quite extensively from the points of view of algebraic geometry and group theory, this aspect of the theory seems not to have been investigated so far. It should be noted that although the curves $\mathcal{C}_i$ carry regular dessins, these maps need not be regular when viewed as 3-polytopes: they could be chiral, with automorphism groups having two orbits on flags: this happens for the Beauville surfaces in Example 3 when $f < e$ (see Section 7 and [41]), and also for those based on Ree groups and Suzuki groups in [22] (see Section 8). Moreover, although $\mathcal{C}_1\times\mathcal{C}_2$ will have many automorphisms, as a surface or a polytope, taking a quotient by G may destroy most, and possibly all, of this symmetry: see Section 10, where automorphisms are discussed.

5 Beauville groups

We call a finite group G a *Beauville group* if there is a Beauville surface $\mathcal{S} = (\mathcal{C}_1\times\mathcal{C}_2)/G$. Here we translate that definition into purely group-theoretic terms.

A group G is a quotient of a triangle group

$$\Delta_i = \Delta(l_i, m_i, n_i) = \langle A_i, B_i, C_i \mid A_i^{l_i} = B_i^{m_i} = C_i^{n_i} = A_iB_iC_i = 1\rangle$$

if and only if it has a presentation

$$G = \langle a_i, b_i, c_i \mid a_i^{l_i} = b_i^{m_i} = c_i^{n_i} = a_ib_ic_i = 1, \dots\rangle, \tag{1}$$

with each a_i, b_i, c_i the image of A_i, B_i or C_i. The torsion elements of Δ_i are the conjugates of the powers of the generators A_i, B_i and C_i, so the kernel K_i of the natural epimorphism $\Delta_i \to G$ is torsion-free if and only if the generators a_i, b_i and c_i have orders

$$|a_i| = l_i,\ |b_i| = m_i,\ |c_i| = n_i. \tag{2}$$

The triangle group Δ_i acts on $\mathbb{H}$ if and only if

$$\frac{1}{l_i} + \frac{1}{m_i} + \frac{1}{n_i} < 1, \tag{3}$$

in which case there is an induced action of G on the Riemann surface $\mathbb{H}/K_i$, which is compact (and thus an algebraic curve $\mathcal{C}_i$) if and only if G is finite. The elements of G with fixed points in $\mathcal{C}_i$ are the conjugates of the powers of the generators a_i, b_i and c_i, forming a subset

$$\Sigma_i = \Sigma_i(G) = \bigcup_{g\in G}(\langle a_i\rangle \cup \langle b_i\rangle \cup \langle c_i\rangle)^g$$

of G. Then G acts freely on the product $\mathcal{C}_1\times\mathcal{C}_2$ of two such curves $\mathcal{C}_i$ $(i = 1, 2)$ if and only if no non-identity element of G has fixed points on both curves, that is,

$$\Sigma_1 \cap \Sigma_2 = \{1\}. \tag{4}$$

Thus conditions (1), (2), (3) and (4) are necessary and sufficient for a finite group G to be a Beauville group. When these conditions are satisfied, we call the pair of generating triples (a_i, b_i, c_i) a *Beauville structure* of *type* $(l_1, m_1, n_1; l_2, m_2, n_2)$ on G. Such a structure on G uniquely determines the curves $\mathcal{C}_i$, and hence the Beauville surface $\mathcal{S}$. This equivalence between surfaces and structures means that one can study many aspects of Beauville surfaces entirely within the theories of finite groups or of regular hypermaps.

6 Beauville's example

The original examples of Beauville surfaces are constructed as follows:

Example 2. Let $\mathcal{C}_1 = \mathcal{C}_2$ be the Fermat curve $\mathcal{F}_n$ of degree n, described in Example 1. There is a faithful action $\rho_1 : G \to \mathrm{Aut}\,\mathcal{F}_n$ of the group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ on $\mathcal{F}_n$, given by

$$(j, k) : [x, y, z] \mapsto [\zeta_n^j x, \zeta_n^k y, z]$$

for all $j, k \in \mathbb{Z}_n$. In this action of G, the elements with fixed points are the multiples of the generating triple $a_1 = (1, 0)$ fixing points $[0, y, z] \in \mathcal{F}_n$, $b_1 = (0, 1)$ fixing points $[x, 0, z] \in \mathcal{F}_n$, and $c_1 = (-1, -1)$ fixing points $[x, y, 0] \in \mathcal{F}_n$. Thus

$$\Sigma_1 = \{(j, k) \in G \mid j = 0, k = 0 \text{ or } j = k\}.$$

We need a second action of G on this curve. If α is an automorphism of G then (composing from right to left) $\rho_2 := \rho_1 \circ \alpha^{-1} : G \to \mathrm{Aut}\,\mathcal{F}_n$ is a faithful action of G on $\mathcal{F}_n$ with $\Sigma_2 = \alpha(\Sigma_1)$. If we define $\alpha : (j, k) \mapsto (4j + 2k, j + k)$, then simple number theory shows that this is an automorphism of G with $\Sigma_1 \cap \Sigma_2 = \{(0, 0)\}$ if and only if n is coprime to 6.

In fact, Beauville set the case $n = 5$ as an exercise in [5], and then invited the reader to generalise this construction. In 2000 Catanese [9] showed that these are the only abelian examples:

Theorem 6.1 (Catanese) *The only abelian Beauville groups are the groups $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ where $n > 1$ and n is coprime to 6.*

The proof depends on simple applications of the structure theorems for finite abelian groups. This result raises the question of how many Beauville surfaces are associated with the group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$. Bauer, Catanese and Grunewald gave asymptotic estimates in [3], and Garion and Penegini gave upper and lower bounds in [23]. The following argument, due to González-Diez, Torres-Teigell and the author [29], gives an exact formula.

Without loss of generality, one can assume that the first generating triple (a_1, b_1, c_1) is as above. The second triple differs from it by an automorphism of G, i.e. a matrix $A \in GL_2(\mathbb{Z}_n)$. It is shown in both [23] and [29] that the set $\mathfrak{F}_n$ of matrices A inducing automorphisms of G satisfying $\Sigma_1 \cap \Sigma_2 = \{(0, 0)\}$ has cardinality

$$|\mathfrak{F}_n| = n^4 \prod_{p|n} \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)\left(1 - \frac{3}{p}\right)\left(1 - \frac{4}{p}\right), \qquad (5)$$

where p ranges over the distinct primes dividing n. (Notice that this expression is 0 unless n is coprime to 6.) One can prove this by using basic linear algebra in the case where n is prime, then lifting to powers of that prime by Hensel's Lemma, and finally using the Chinese Remainder Theorem for general integers n.

Now two matrices $A, A' \in GL_2(\mathbb{Z}_n)$ give isomorphic Beauville surfaces if and only if $A' = PA^{\pm 1}Q$ where P and Q are elements of a certain subgroup of $GL_2(\mathbf{Z}_n)$ isomorphic to S_3, permuting the standard triple $\{a_1, b_1, c_1\}$. We thus have an action on $\mathfrak{F}_n$ by the wreath product $W = S_3 \wr S_2$, a semidirect product of $S_3 \times S_3$ by S_2: here the two direct factors S_3 correspond to the matrices P and Q, permuting the three vertex colours on each curve $\mathcal{C}_i$, and the complement S_2 corresponds to inverting A and transposing the curves. The number of non-isomorphic Beauville surfaces obtained is equal to the number of orbits of W on $\mathfrak{F}_n$, and this can be found by applying the Cauchy-Frobenius Counting Lemma (otherwise known as Burnside's Lemma). This states that the number of orbits of a finite group on a finite set is equal to the average number of points fixed by the elements of the group. In our case, inspection shows that most of the elements of W act without fixed points on $\mathfrak{F}_n$, giving the following result (see [29] for details):

Theorem 6.2 *Let $n = p_1^{e_1} \cdot \ldots \cdot p_s^{e_s}$ be a natural number coprime to 6, where $p_1, \ldots, p_k$ are distinct primes. Then the number of isomorphism classes of Beauville surfaces with Beauville group $\mathbb{Z}_n \oplus \mathbb{Z}_n$ is*

$$\Theta(n) = \frac{1}{72}\left(\Theta_1(n) + 4\prod_{i=1}^{s}\Theta_2(p_i^{e_i}) + 6\prod_{i=1}^{s}\Theta_3(p_i^{e_i}) + 12\prod_{i=1}^{s}\Theta_4(p_i^{e_i})\right), \tag{6}$$

where $\Theta_1(n) = |\mathfrak{F}_n|$,

$$\Theta_2(p^e) := \begin{cases} p^{2e}\left(1-\frac{1}{p}\right)\left(1-\frac{4}{p}\right) & \textit{if } p \equiv 1 \bmod (3), \\ p^{2e}\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right) & \textit{if } p \equiv 2 \bmod (3), \end{cases}$$

$$\Theta_3(p^e) := p^{2e}(1-3/p)(1-5/p),$$

and

$$\Theta_4(p^e) := \begin{cases} 2 & \textit{if } p \equiv 1 \bmod (3), \\ 0 & \textit{if } p \equiv 2 \bmod (3). \end{cases}$$

Here 72 is the order of W, while $\Theta_1(n)$ is the number $|\mathfrak{F}_n|$ of fixed points of its identity element, given by (5), and the terms in (6) involving Θ_2, Θ_3 and Θ_4 are the contributions to the average from conjugacy classes in W containing four, six and twelve elements of orders 3, 2 and 6.

For large n the sum in (6) is dominated by $\Theta_1(n)$, so we have

$$\Theta(n) \sim \frac{1}{72}\Theta_1(n) = \frac{n^4}{72}\prod_{p|n}\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\left(1-\frac{3}{p}\right)\left(1-\frac{4}{p}\right)$$

as $n \to \infty$ with n coprime to 6. (Note that, despite appearances, $\Theta(n)/n^4$ is not bounded away from 0: if we take n to be the product of the first k primes $p > 3$, then

$$\prod_{p|n}\left(1-\frac{1}{p}\right) \to 0$$

as $k \to \infty$ (see [40, Exercise 9.3]), and hence $\Theta(n)/n^4 \to 0$ for such integers n.)

7 Beauville p-groups

It is natural to try to extend the classification of Beauville groups from abelian groups to wider classes, such as nilpotent groups. A finite group is nilpotent if and only if it is a direct product of its Sylow subgroups, and a direct product of Beauville groups of mutually coprime orders is clearly a Beauville group, so the main objective in such an extension is to study Beauville structures on p-groups for the various primes p. Barker, Boston, Peyerimhoff and Vdovina [2] have obtained Beauville 2-groups as quotients of the fundamental group of a certain simplicial complex, while Barker, Boston and Fairbairn [1] have constructed many examples for all p. For instance, they show that in addition to the abelian p-groups $C_{p^e} \times C_{p^e}$ with $p \geq 5$, given by Example 2, there is at least one nonabelian Beauville group of every prime-power order p^k provided $p \geq 7$ and $k \geq 3$. (For primes $p < 7$, the smallest nonabelian Beauville p-groups have orders 2^7, 3^5 and 5^4.)

Example 3. For each prime $p \geq 5$, let

$$G = G(e,f) = \langle x, y \mid x^{p^e} = y^{p^e} = 1,\ y^x = y^{1+p^f} \rangle$$

where $1 \leq f \leq e$. Thus G is a semidirect product of two cyclic groups $\langle x \rangle$ and $\langle y \rangle$ of order p^e, so G has order p^{2e}; it is abelian if and only if $f = e$. The Frattini subgroup of G is the normal subgroup $\Phi = \langle x^p, y^p \rangle$, with $G/\Phi \cong C_p \times C_p$. The Beauville structures of type $(p,p,p;p,p,p)$ on $C_p \times C_p$ constructed in Example 2 lift back to Beauville structures of type $(p^e,p^e,p^e;p^e,p^e,p^e)$ on G. These groups appeared in connection with the classification of orientably regular embeddings of complete bipartite graphs in [38, 41], and their connections with dessins were studied in [45].

This example deals with even powers of primes $p \geq 5$. Barker, Boston and Fairbairn [1] give a similar construction for odd powers.

Example 4. Let G be a 2-generator finite group of prime exponent $p \geq 5$. As in Example 3, any Beauville structure on the quotient group $G/\Phi \cong C_p \times C_p$ lifts to a Beauville structure on G, this time of type $(p,p,p;p,p,p)$. By Kostrikin's solution [48] of the restricted Burnside problem for prime exponents, for each p there is a largest such 2-generator finite group G, denoted by $R(2,p)$, and all others are quotients of it. These groups $R(2,p)$ are in fact very large: for instance, Havas, Wall and Wamsley [36] have shown that $|R(2,5)| = 5^{34}$, while O'Brien and Vaughan-Lee [52] have shown that $|R(2,7)| = 7^{20416}$. For a detailed survey of the restricted Burnside problem, see [61].

Barker, Boston and Fairbairn show in [1] that the proportion of 2-generator groups of order p^5 which are Beauville groups tends to 1 as $p \to \infty$, but that this is not the case for groups of order p^6. The question raised by Fuertes, González-Diez and Jaikin-Zapirain in [21], namely whether, in any sense, most 2-generator p-groups are Beauville groups, remains open.

8 Simple Beauville groups

It is easy to see that the alternating group A_5 is not a Beauville group. For instance, its non-identity elements have orders $2, 3$ or 5. If $l, m, n \in \{2, 3\}$ then the triangle group $\Delta(l, m, n)$ is solvable, whereas A_5 is not, so any generating triple for A_5 must contain an element of order 5. Since the Sylow 5-subgroups of A_5 are cyclic, any two elements of order 5 are conjugate to powers of each other, so no two generating triples can satisfy the Beauville condition (4).

In 2005, Bauer, Catanese and Grunewald [3] made the following conjecture:

Every non-abelian finite simple group except A_5 is a Beauville group.

As evidence for this, they showed that A_n is a Beauville group for all sufficiently large n, as are the groups $PSL_2(p)$ for all primes $p > 5$ (note that $PSL_2(5) \cong A_5$) and the Suzuki groups $Sz(2^e)$ for all odd primes e. Fuertes and González-Diez [19] showed that A_n is a Beauville group for all $n \geq 6$. In [22], Fuertes and the author showed that various other simple groups are Beauville groups, namely $PSL_2(q)$ for all prime powers $q > 5$, and the Suzuki groups $Sz(2^e)$ and the Ree groups $R(3^e)$ for all odd $e \geq 3$. They also showed that certain quasisimple groups (perfect central extensions of simple groups) are Beauville groups, namely the groups $SL_2(q)$ for $q > 5$, again extending a result for prime q in [3].

Around the same time, Garion and Penegini [23] obtained the above result for $PSL_2(q)$, using results of Macbeath [50] on generating triples for this group. They also used probabilistic methods to show that $Sz(2^e)$ and $R(3^e)$ are Beauville groups for all sufficiently large odd e, with similar results for several other families of simple groups, including $PSL_3(q)$ and the unitary groups $U_3(q)$.

Soon, specialists in the study of finite simple groups became interested in this problem: the classification of such groups, announced around 30 years ago but not completely proved until 2004, allows conjectures such as this to be obtained by inspection. Several major advances were announced in 2010. Firstly, Garion, Larsen, and Lubotzky [24] used probabilistic methods to show that the conjecture is true with at most finitely many exceptions. Soon afterwards, Guralnick and Malle [35] gave a complete proof of the conjecture, while Fairbairn, Magaard and Parker [18] extended it further to all finite quasisimple groups except A_5 and its central cover $SL_2(5)$. In all three cases, the proofs require deep knowledge of the structure of finite simple groups, especially those of Lie type; see [8, 66] for detailed accounts of these groups, and [13] for a concise (but hardly pocket-sized) summary.

9 Fundamental groups and rigidity

Just as each Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ is constructed from a group G, it gives rise to two more groups: as a connected topological space it has a fundamental group $\pi_1\mathcal{S}$, and as an algebraic variety it has an automorphism group $\mathrm{Aut}\,\mathcal{S}$.

The fundamental group of $\mathcal{S}$ is easily described. We have a pair of triangle groups Δ_i, each with a normal subgroup $K_i \cong \pi_1\mathcal{C}_i$ such that $\mathcal{C}_i \cong \mathbb{H}/K_i$ and $\Delta_i/K_i \cong G$. Each Δ_i acts on $\mathbb{H}$, so there is an induced action of $\Delta_1 \times \Delta_2$ on the simply connected space $\mathbb{H} \times \mathbb{H}$. Let Π denote the inverse image of the diagonal subgroup in the natural epimorphism $\Delta_1 \times \Delta_2 \to G \times G$, that is, the subgroup of $\Delta_1 \times \Delta_2$ consisting of those pairs which map onto the same element of G. Beauville condition (4) implies that Π acts freely on $\mathbb{H} \times \mathbb{H}$, with $(\mathbb{H} \times \mathbb{H})/\Pi \cong \mathcal{S}$, so $\pi_1\mathcal{S}$ can be identified with Π. Thus $\pi_1\mathcal{S}$ has a normal subgroup $K_1 \times K_2 \cong \pi_1\mathcal{C}_1 \times \pi_1\mathcal{C}_2 \cong \pi_1(\mathcal{C}_1 \times \mathcal{C}_2)$, with quotient group G, corresponding to the regular covering $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{S}$ with covering group G. It also has a normal subgroup K_1, with quotient group Δ_2, corresponding to the regular covering $\mathcal{C}_1 \times \mathbb{H} \to \mathcal{S}$ with covering group Δ_2, and similarly for the normal subgroup K_2.

This leads to a property of Beauville surfaces called rigidity [9], meaning essentially that the topology determines the geometric structure. In a hyperbolic triangle group, the centraliser of each non-identity element is cyclic. Thus the centraliser in $\Pi = \pi_1\mathcal{S}$ of any element of K_i contains a surface group (namely K_{3-i}), and is therefore nonabelian, whereas any other element of Π has an abelian centraliser. It follows that if $\mathcal{S}' = (\mathcal{C}'_1 \times \mathcal{C}'_2)/G'$ is another Beauville surface, then any isomorphism $\pi_1\mathcal{S} \to \pi_1\mathcal{S}'$ induces isomorphisms $\Delta_i \to \Delta'_i$ between the corresponding triangle groups (possibly after transposing factors), and an isomorphism $G \to G'$ of their Beauville groups. Now any isomorphism of cocompact hyperbolic triangle groups is induced by an isometry of $\mathbb{H}$, since the corresponding triangles are isometric. It follows that homeomorphic Beauville surfaces are in fact isometric, and that $\mathcal{S}$ is uniquely determined, up to complex conjugation of either or both of the curves $\mathcal{C}_i$, by its fundamental group. Such rigidity properties help to explain why Beauville surfaces are so interesting to algebraic geometers. (The above argument, taken from a more detailed proof given by González-Diez and Torres-Teigell in [31], is a group-theoretic analogue of the arguments based on algebraic geometry given by Catanese in [9] and by Bauer, Catanese and Grunewald in [4].)

10 Automorphism groups of Beauville surfaces

This section summarises results of the author on automorphism groups of Beauville surfaces in [39]; some of these results have been obtained independently by Fuertes and González-Diez in [20], and have been extended to mixed Beauville surfaces by González-Diez and Torres-Teigell in [32].

The rigidity results outlined in the preceding section show that any automorphism of a Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ lifts to an automorphism of $\mathcal{C}_1 \times \mathcal{C}_2$, and this either preserves or transposes the curves $\mathcal{C}_i$; such automorphisms of $\mathcal{S}$ are called *direct* or *indirect* respectively. First we consider the group $\mathrm{Aut}^0\mathcal{S}$ of direct automorphisms of $\mathcal{S}$, a subgroup of index at most 2 in $\mathrm{Aut}\,\mathcal{S}$.

Let $A_i := \mathrm{Aut}\,\mathcal{C}_i$. There is a natural action of $A_1 \times A_2$ on $\mathcal{C}_1 \times \mathcal{C}_2$, and we can regard $\mathcal{S}$ as the quotient of $\mathcal{C}_1 \times \mathcal{C}_2$ by the diagonal subgroup D of the subgroup $G \times G$ of $A_1 \times A_2$. A simple calculation shows that an element $(\alpha_1, \alpha_2) \in A_1 \times A_2$, acting on $\mathcal{C}_1 \times \mathcal{C}_2$, permutes the orbits of D, and hence induces an automorphism of $\mathcal{S}$, if and only if

1. each α_i is in the normaliser $N_i := N_{A_i}(G)$ of G in A_i, and

2. α_1 and α_1, acting by conjugation, induce the same automorphism of G.

Such elements (α_1, α_2) form a subgroup N of $N_1 \times N_2$, the inverse image of the diagonal subgroup of $\mathrm{Aut}\,G \times \mathrm{Aut}\,G$ under the natural homomorphism $N_1 \times N_2 \to \mathrm{Aut}\,G \times \mathrm{Aut}\,G$. The kernel of this action of N is D, so the group $A^0 = \mathrm{Aut}^0\mathcal{S}$ of direct automorphisms of $\mathcal{S}$ is isomorphic to N/D.

In particular, if each $\alpha_i \in G$ then condition (1) is satisfied, and (2) is satisfied if and only if $\alpha_1\alpha_2^{-1}$ is in the centre $Z := Z(G)$ of G. Thus N contains a normal subgroup

$$M = N \cap (G \times G) = \{(\alpha_1, \alpha_2) \in G \times G \mid \alpha_1\alpha_2^{-1} \in Z\} \cong D \times Z,$$

inducing on $\mathcal{S}$ a normal subgroup $I := \mathrm{Inn}\,\mathcal{S} \cong M/D \cong Z$ of A^0; the elements of I are called the *inner automorphisms* of $\mathcal{S}$, induced by compatible pairs of elements of G acting on the curves $\mathcal{C}_i$. Since I is isomorphic to the centre of G, it is finite and abelian. The quotient group $A^0/I \cong N/M$ is called the *direct outer automorphism group* $\mathrm{Out}^0\mathcal{S}$ of $\mathcal{S}$.

In many cases $G = N_i$ for each i (for instance if $G = A_i$), so that $M = N$ and hence $A^0 = I \cong Z$. If $G < N_i$ for some i, then Δ_i is a proper normal subgroup of a Fuchsian group $\tilde{\Delta}_i$, with $\tilde{\Delta}_i/K_i \cong N_i$. Singerman [57] has shown that any Fuchsian group containing a triangle group must also be a triangle group, and that any proper normal inclusion between them must be (up to permutations of the periods) of one of the forms

$$\text{(a) } \Delta(s, s, t) \lhd \Delta(2, s, 2t), \quad \text{(b) } \Delta(t, t, t) \lhd \Delta(3, 3, t), \quad \text{(c) } \Delta(t, t, t) \lhd \Delta(2, 3, 2t),$$

with the quotient group isomorphic to C_2, C_3 or S_3 respectively. In all three cases, at least two of the three periods of Δ_i are equal, so we have:

Proposition 10.1 *If a Beauville structure on a group G has type $(l_1, m_1, n_1; l_2, m_2, n_2)$, and for each i the periods l_i, m_i and n_i are mutually distinct, then the direct automorphism group $\mathrm{Aut}^0\mathcal{S}$ of the corresponding Beauville surface $\mathcal{S}$ is isomorphic to the centre of G.* □

If there are repetitions among either or both of the triples l_i, m_i, n_i, then $\mathcal{S}$ may have direct outer automorphisms, arising from proper normal inclusions $\Delta_i \lhd \tilde{\Delta}_i$. In this case Singerman's results, stated above, allow us to deduce the following:

Proposition 10.2 *The direct automorphism group $\mathrm{Aut}^0\mathcal{S}$ of a Beauville surface $\mathcal{S}$ has a normal subgroup $\mathrm{Inn}\,\mathcal{S} \cong Z(G)$ with $\mathrm{Aut}^0\mathcal{S}/\mathrm{Inn}\,\mathcal{S}$ isomorphic to a subgroup of $S_3 \times S_3$. In particular, $\mathrm{Aut}^0\mathcal{S}$ is a finite solvable group, of derived length at most 3.* □

The direct factors S_3 can be regarded as permuting the fibres of β_i over 0, 1 and ∞.

Example 4. Let $\mathcal{S} = (\mathcal{F}_n \times \mathcal{F}_n)/G$ as in Example 3, with the Beauville group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$. Since G is abelian we have $\mathrm{Inn}\,\mathcal{S} \cong G$. Then $\mathrm{Out}^0\mathcal{S} \cong C_3$ or C_1 as the automorphism of G induced by the 3-cycle (a_1, b_1, c_1) is or is not the same as that induced by (a_2, b_2, c_2). Thus $\mathrm{Aut}^0\mathcal{S}$ is isomorphic to an extension of G by C_3, or to G, depending on the choice of the matrix $A \in GL_2(\mathbb{Z}_n) = \mathrm{Aut}\,G$ linking the two representations ρ_i of G on $\mathcal{F}_n$.

Any indirect automorphism of $\mathcal{S}$ is induced by an automorphism of $\mathcal{C}_1 \times \mathcal{C}_2$ of the form

$$(p_1, p_2) \mapsto (p_2\phi_2, p_1\phi_1),$$

where $\phi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\phi_2 : \mathcal{C}_2 \to \mathcal{C}_1$ are isomorphisms of curves. It is not hard to prove:

Proposition 10.3 *A Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ has an indirect automorphism if and only if $\mathcal{C}_1 \cong \mathcal{C}_2$ and G has an automorphism ζ transposing the equivalence classes of its representations on $\mathcal{C}_1$ and $\mathcal{C}_2$.* □

Here two representations are defined to be equivalent if each is obtained from the other by composition with an isomorphism of curves.

Corollary 10.4 *If a Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ has an indirect automorphism, then the corresponding Beauville structure on G must consist of two triples of equivalent types.* □

Here two types are defined to be equivalent if each is a permutation of the other. The analogue of Proposition 10.3 is the following:

Proposition 10.5 *The automorphism group $\mathrm{Aut}\,\mathcal{S}$ of a Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ has a normal subgroup $\mathrm{Inn}\,\mathcal{S} \cong Z(G)$ with $\mathrm{Aut}\,\mathcal{S}/\mathrm{Inn}\,\mathcal{S}$ isomorphic to a subgroup of $S_3 \wr S_2$. In particular, $\mathrm{Aut}\,\mathcal{S}$ is a finite solvable group, of derived length at most 4.* □

By the above results, many Beauville surfaces (for instance, most of those with simple Beauville groups) have only the identity automorphism.

There are no restrictions on the centre of a Beauville group, and hence on $\mathrm{Inn}\,\mathcal{S}$, other than the obvious ones that it should be finite and abelian:

Theorem 10.6 *Given any finite abelian group H, there is a Beauville group G with centre $Z(G) \cong H$.* □

It immediately follows that there is a Beauville surface $\mathcal{S}$ with $\mathrm{Inn}\,\mathcal{S} \cong H$; this remains true, even even if one requires $\mathrm{Out}\,\mathcal{S}$ to be as large (isomorphic to $S_3 \wr S_2$) or as small (the trivial group) as possible. A key ingredient of the proof adapts a method used by Conder [11] for constructing Hurwitz groups with large centres: we represent H as a direct product of cyclic groups C_{m_i}, each isomorphic to the centre of some group $SL_{n_i}(q_i)$, where $m_i = \gcd(n_i, q_i - 1)$, so that the direct product G of these groups $SL_{n_i}(q_i)$ has centre $Z(G) \cong H$. Results of Lucchini [49] on generators of special linear groups allow one to choose the groups $SL_{n_i}(q_i)$, and hence also their product G, to be quotients of $\Delta(2, 3, p)$ and hence of $\Delta(p, p, p)$, for two different primes $p = p_1, p_2$, thus giving a Beauville structure of type $(p_1, p_1, p_1; p_2, p_2, p_2)$ on G. Modifications of this construction provide some control over the outer automorphism group of the resulting Beauville surface $\mathcal{S}$. For details, see [39].

11 The absolute Galois group

Belyĭ's Theorem [6] implies that the curves $\mathcal{C}_i$ used in constructing a Beauville surface $\mathcal{S}$ are defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, and it follows that $\mathcal{S}$ is also defined over this field. The *absolute Galois group* is the automorphism group

$$\Gamma = \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$$

of this field. Since $\overline{\mathbb{Q}}$ is the direct limit (i.e. union) of the Galois (finite normal) extensions K of $\mathbb{Q}$, it follows that Γ is the inverse limit

$$\Gamma = \lim_{\leftarrow} \mathrm{Gal}\, K/\mathbb{Q}$$

of the Galois groups of these fields; the homomorphisms in this inverse system are the restriction mappings

$$\mathrm{Gal}\, L/\mathbb{Q} \to \mathrm{Gal}\, K/\mathbb{Q}$$

induced by inclusions $K \subseteq L$ between such fields. Since these are all epimorphisms between finite groups, Γ is in fact a profinite group, that is, a projective limit of finite groups: it can be identified with the subgroup of the cartesian product Π of all such groups $\mathrm{Gal}\, K/\mathbb{Q}$ consisting of the elements whose coordinates are compatible with the restriction mappings.

Giving the finite groups $\mathrm{Gal}\, K/\mathbb{Q}$ the discrete topology makes Π a topological group, compact by Tychonoff's Theorem, so Γ, as a closed subgroup of Π, is also a compact topological group (in fact, homeomorphic to a Cantor set). The Galois correspondence is then between the subfields of $\overline{\mathbb{Q}}$ and the closed subgroups of Γ. Understanding Γ is therefore critical to an understanding of algebraic number theory. There are many important open problems associated with this group. For instance the Inverse Galois Problem, Hilbert's question whether every finite group is isomorphic to a Galois group over $\mathbb{Q}$, is equivalent to asking whether every finite group is the quotient of Γ by some closed normal subgroup. Books by Malle and Matzat [51], Serre [56] and Völklein [62] describe progress on this.

In the mid-1980s, Grothendieck [34] proposed that one should study Γ through its actions on various geometric and combinatorial objects, the simplest of which are oriented hypermaps, or *dessins d'enfants* (children's drawings) as he called them, viewed as unbranched finite coverings of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. By Belyĭ's Theorem these are defined over $\overline{\mathbb{Q}}$, and there is a natural action of Γ on them, through its action on the coefficients of the polynomials and rational functions defining them. This action preserves the obvious numerical parameters of a dessin, such as the numbers and valencies of its vertices and faces, and hence its genus [44]. However, using elementary properties of the modular j-function it is easy to show that Γ acts faithfully on dessins of genus 1 (those on elliptic curves). Less obviously, Schneps [54] has shown that it acts faithfully on plane trees, while Girondo and González-Diez [25] have shown that is faithful on dessins of each genus $g \geq 2$. It is an open problem whether Γ acts faithfully on regular dessins, i.e. orientably regular hypermaps.

Example 5. Hurwitz [37] showed that if $\mathcal{C}$ is a compact Riemann surface (or algebraic curve) of genus $g \geq 2$ then $\mathrm{Aut}\,\mathcal{C}$ has order at most $84(g-1)$. The finite groups G attaining this bound, namely the nontrivial finite quotients of the triangle group $\Delta = \Delta(2, 3, 7)$, are called *Hurwitz groups*. Macbeath [50] classified those groups $PSL_2(q)$ which are Hurwitz groups, and these include the groups $G = PSL_2(p)$ for all primes $p \equiv \pm 1 \bmod (7)$. For such groups G there are, in fact, three normal subgroups N of Δ with $\Delta/N \cong G$, corresponding to choosing elements from the three conjugacy classes of elements of order 7 as members of generating triples for G. We thus obtain three non-isomorphic Riemann surfaces $\mathcal{C} = \mathbb{H}/N$, of genus

$$g = 1 + \frac{p(p^2 - 1)}{168}$$

and with automorphism group $PSL_2(p)$, attaining Hurwitz's bound. Streit [58] showed that, as algebraic curves, these are defined over the cubic field $K = \mathbb{Q}(\zeta_7) \cap \mathbb{R}$, and are conjugate under the Galois group $\mathrm{Gal}\, K/\mathbb{Q} \cong C_3$ of that field. The normal inclusions of the subgroups N in Δ equip each $\mathcal{C}$ with a regular dessin, specifically an orientably regular 7-valent triangular map, inherited from the corresponding Δ-invariant tessellation of $\mathbb{H}$. These three algebraically conjugate maps are mutually non-isomorphic, and in fact so are their embedded graphs [46].

Example 6. In [58], Streit generalised the above example, replacing the integer 7 with an arbitrary integer $n \geq 7$. For any prime $p \equiv \pm 1 \bmod (2n)$ there are $\phi(n)/2$ conjugacy classes of elements of order n in the group $G = PSL_2(p)$, giving rise to $\phi(n)/2$ normal subgroups N of the triangle group $\Delta = \Delta(2, 3, n)$ with $\Delta/N \cong G$. These in turn correspond to the same number of non-isomorphic curves $\mathcal{C} = \mathbb{H}/N$, all with automorphism group G and carrying orientably regular n-valent triangular maps. These curves are defined over the field $\mathbb{Q}(\zeta_n) \cap \mathbb{R}$, and are equivalent under the Galois group of that field, isomorphic to $\mathbb{Z}_n^*/\{\pm 1\}$. As before, these maps are mutually non-isomorphic.

12 Conjugate but non-homeomorphic varieties

The examples in the preceding section show how the action of Γ can change analytic and combinatorial structures defined over $\overline{\mathbb{Q}}$, but what about topology? The genus of an algebraic curve can be defined purely algebraically (using the Riemann-Roch Theorem, for example), so it is invariant under Γ; thus Galois conjugate curves are homeomorphic to each other. However, in 1964 Serre [55] showed that in each dimension greater than 1 there are pairs of algebraic varieties, defined over $\overline{\mathbb{Q}}$, which are conjugate under Γ but not homeomorphic to each other. Subsequently, further examples of such pairs have been constructed. In fact, there exist arbitrarily large Galois orbits consisting of mutually non-homeomorphic Beauville surfaces.

Example 7. The first such examples were given by González-Diez and Torres-Teigell [31], using Beauville structures of type $(2, 3, n; p, p, p)$ on the group $G = PSL_2(p)$, for integers $n \geq 7$ and primes $p \equiv \pm 1 \bmod (2n)$. As in Example 6, generating triples of type $(2, 3, n)$ in G give rise to a Galois orbit of $\phi(n)/2$ non-isomorphic curves $\mathcal{C}_1$. By using triples of type (p, p, p) for $\mathcal{C}_2$ they obtained an orbit of at least $\phi(n)/2$ mutually non-isomorphic Beauville surfaces. By rigidity, these have non-isomorphic fundamental groups, so they are mutually non-homeomorphic. For fixed n, Dirichlet's Theorem gives infinitely many suitable primes p, and elementary properties of Euler's function show that the size of these orbits of Γ tends to infinity as n increases.

Example 8. The authors of [31] were unable to determine the exact size of the orbits in Example 7 because of the technical difficulty of finding how the outer automprphism of $PSL_2(p)$, induced by conjugation in $PGL_2(p)$, acts on the associated Beauville surfaces. In [30], they and the present author avoided this problem by using a similar construction based on the Beauville group $G = PGL_2(p)$, which has only inner automorphisms.

If p is an odd prime then the non-identity elements of $PGL_2(p)$ are of three types: elliptic elements, of order dividing $p+1$, with no fixed points on the projective line $\mathbb{P}^1(p)$; parabolic elements, of order p, with one fixed point; and hyperbolic elements, of order dividing $p-1$, with two fixed points. An element of one type cannot be conjugate to a power of an element of another type. For any prime $p \equiv 19 \bmod (24)$ one can find generating triples for G of types $(2, 3, p-1)$, consisting of hyperbolic elements, and $(2, 4, p+1)$, consisting of elliptic elements; any such pair of triples forms a Beauville structure on G.

There are $\phi(p \pm 1)/2$ conjugacy classes of elements of order $p \pm 1$ in G. This is therefore the number of normal subgroups of the triangle group $\Delta_1 = \Delta(2, 3, p-1)$ or $\Delta_2 = \Delta(2, 4, p+1)$ with quotient group G, and hence also the number of non-isomorphic algebraic curves $\mathcal{C}_1$ or $\mathcal{C}_2$ uniformised by such subgroups. These curves all have automorphism group G since Singerman's results [57] show that Δ_1 and Δ_2 are maximal Fuchsian groups. These two families of curves $\mathcal{C}_i$ are defined over the field $K_i = \mathbb{Q}(\zeta_{p\pm1}) \cap \mathbb{R}$, and the members of each family are conjugate under the Galois group of that field. We thus obtain $\phi(p-1)\phi(p+1)/4$ Beauville surfaces $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$, defined over the field $K = K_1K_2$; they are conjugate under $\mathrm{Gal}\, K/\mathbb{Q}$ and hence under Γ. By rigidity, these surfaces have mutually non-isomorphic fundamental groups, so they are mutually non-homeomorphic. As before, the size of this orbit of Γ tends to infinity as p increases.

In both of these examples, although the topological fundamental groups $\pi_1\mathcal{S}$ of the surfaces $\mathcal{S}$ in a given orbit are mutually non-isomorphic, the algebraic fundamental groups $\pi_1^{\mathrm{alg}}\mathcal{S}$, the profinite completions $\widehat{\pi_1\mathcal{S}}$ of the topological fundamental groups, are mutually isomorphic. This is because the finite quotients of the groups $\pi_1\mathcal{S}$ correspond to the finite regular unbranched coverings of $\mathcal{S}$, and these, being algebraically defined, are invariant under Γ (see [55]). By contrast with the groups $\pi_1\mathcal{S}$, Conder [12] has recently shown that triangle groups are determined, up to isomorphism, by their finite quotient groups.

In Examples 5–8, together with other similar examples in [14, 45, 46, 59, 60], for instance, the curves and surfaces in an orbit of Γ are all defined over some subfield of a cyclotomic field. The group of transformations induced by Γ on such an orbit is therefore abelian, so the commutator subgroup Γ' is contained in the kernel of the action. It would be interesting to have some nonabelian examples, which reveal more of the structure of Γ. In particular, it would be interesting to know whether Γ acts faithfully on the set of all Beauville surfaces (as it does on all dessins). If so, then rigidity would imply that it acts faithfully on regular dessins (see Section 11).

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