COMPLEX HYPERBOLIC GEOMETRY:
3. Representations of surface groups

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Hyperbolic surfaces

Let Σ be an orientable surface of finite type:
genus g with p boundary components
Suppose Euler characteristic $\chi(\Sigma) = 2 - 2g - p < 0$.
Let $\pi_1 = \pi_1(\Sigma)$ be the fundamental group of Σ.

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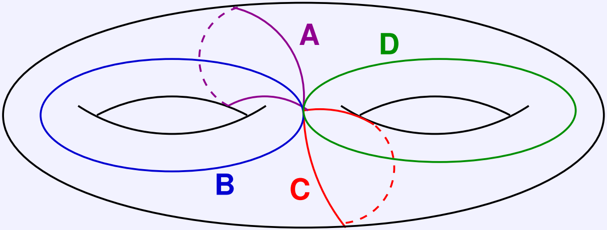


A surface with $g = 2$, $p = 0$. Generators of π_1 are shown.

Let $\widetilde{\Sigma}$ be the universal cover of Σ.
This is topologically a disc and π_1 acts on $\widetilde{\Sigma}$ by deck transformations:
So if $\pi : \widetilde{\Sigma} \longrightarrow \Sigma$ is the projection, $p_1, p_2 \in \widetilde{\Sigma}$ with $\pi(p_1) = \pi(p_2)$
if and only if $p_2 = \alpha(p_1)$ for some $\alpha \in \pi_1(\Sigma)$.

From topology to geometry...and algebra

We want to take this topological picture and impose some geometry, and then analyse using algebra.

Let G be a matrix group corresponding to the isometry group of the hyperbolic plane.
G is one of $\mathrm{SU}(1,1)$, $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SO}(2,1)$
A representation of π_1 is a homomorphism $\rho : \pi_1 \longrightarrow G$
We are only interested in conjugacy classes of representations.
We write $\Gamma = \rho(\phi_1)$, a subgroup of G.

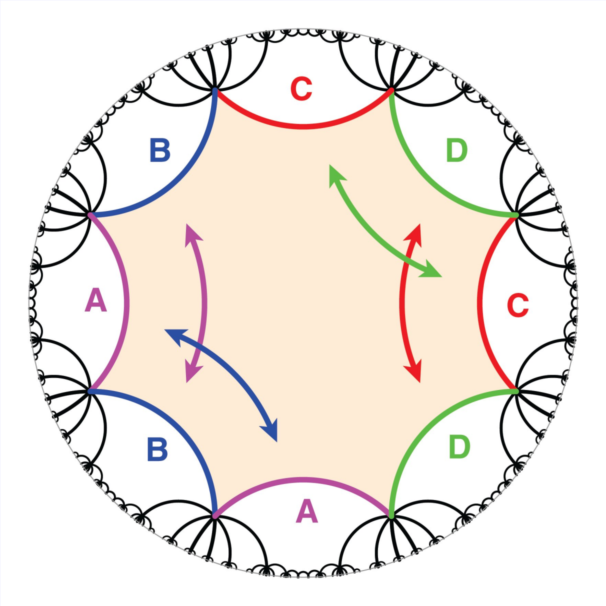
A map $f : \widetilde{\Sigma} \longrightarrow$ hyperbolic plane is equivariant with respect to the representation ρ if $f(\alpha p) = \rho(\alpha) f(p)$ for all $p \in \widetilde{\Sigma}$, $\alpha \in \pi_1$.

This gives correspondence between topology and geometry and algebra:

- deck transformations, hyperbolic isometries, matrices
- $f : \widetilde{\Sigma} \longrightarrow$ hyperbolic plane.
- $\rho : \pi_1 \longrightarrow \Gamma \subset G$, action by isometries.
- loop homotopic to boundary, cusp of Σ, parabolic matrix.

Here parabolic means non-diagonalisible.

Geometric action of $\pi_1(\Sigma)$ on the hyperbolic plane



From: WM Goldman *What is a Projective Structure*, Notices AMS **54** (2007) 30–33

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A representation $\rho : \pi_1 \longrightarrow \Gamma < G$ is Fuchsian if ρ is

- discrete, so action is properly discontinuous;
- faithful, so Γ isomorphic to π_1
- type-preserving see below
- geometrically finite, so finite sided fundamental parallelogram.

Type-preserving means for each homotopy class α:
If α is non-trivial, non-peripheral loops then $\rho(\alpha)$ loxodromic.
If α is non-trivial, peripheral loops then $\rho(\alpha)$ parabolic.

Note in this case type-preserving implies discrete and geometrically finite.

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Space of all Fuchsian representations, up to conjugacy, is Fricke space.
It is one realisation of Teichmüller space.
It is homeomorphic to a ball of dimension $6g - 6 + 2p$.

Representations of π_1 in $\mathrm{SU}(2,1)$

We now consider representations $\rho : \pi_1 \longrightarrow \Gamma < \mathrm{SU}(2,1)$

Example 1. *A representation where* $\rho(\pi_1) = \Gamma < \mathrm{SU}(1,1)$.
It can be embedded as the block diagonal representation.
$\{1\} \times \Gamma < \{1\} \times \mathrm{SU}(1,1) < \mathrm{SU}(2,1)$.
ρ *preserves a complex line.*

Example 2. *A representation where* $\rho(\pi_1) = \Gamma < \mathrm{SO}(2,1)$.
Real matrices are special cases of complex matrices, so
$\Gamma < \mathrm{SO}(2,1) < \mathrm{SU}(2,1)$.
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If the original representation in Example 1 is Fuchsian, we say
$\rho : \pi_1 \longrightarrow \Gamma < \{1\} \times \mathrm{SU}(1,1) < \mathrm{SU}(2,1)$ is $\mathbb{C}$-Fuchsian.
If the original representation in Example 2 is Fuchsian, we say
$\rho : \pi_1 \longrightarrow \Gamma < \mathrm{SO}(2,1) < \mathrm{SU}(2,1)$ is $\mathbb{R}$-Fuchsian.

We want to study more general representations of π_1 in $\mathrm{SU}(2,1)$.

The Toledo invariant

Let $\rho : \pi_1 \longrightarrow \Gamma < \mathrm{SU}(2,1)$ be a representation.
Let $f : \widetilde{\Sigma} \longrightarrow \mathbf{H}^2_{\mathbb{C}}$ be a smooth, ρ-equivariant map. So $f(\alpha p) = \rho(\alpha) f(p)$.

The Toledo invariant is $\tau = \tau(\rho) = \dfrac{1}{2\pi} \displaystyle\int_{\Sigma} f^* \omega$.

Here ω is the Kähler form $\omega = 4i\partial\overline{\partial} \log\langle \mathbf{z}, \mathbf{z}\rangle$.
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In fact τ only depends on ρ and is independent of f.

τ has the following properties

- τ varies continuously with ρ
- $\chi < \tau < -\chi$ where $\chi = 2 - 2g - p$ is the Euler characteristic
- ρ is $\mathbb{C}$-Fuchsian if and only if $\tau = \pm\chi$
- if ρ is $\mathbb{R}$-Fuchsian then $\tau = 0$ (converse of this not true)
- if $p = 0$ (so Σ closed) then $\tau \in 2\mathbb{Z}$
- if $p = 0$ then τ indexes components of representation variety (Xia)

Think of τ as generalisation of Cartan's angular invariant.

Complex hyperbolic quasi-Fuchsian representations

We are not interested in all representations.
A representation $\rho : \pi \longrightarrow \Gamma < \mathrm{SU}(2,1)$ is
complex hyperbolic quasi-Fuchsian (CHQF) if it is

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Every $\mathbb{R}$-Fuchsian and $\mathbb{C}$-Fuchsian representation is also CHQF.

When $p = 0$ then every component of the representation variety contains a CHQF representation Goldman, Kapovich, Leeb

When $p \neq 0$ there is a one parameter family of CHQF representations interpolating between $\mathbb{R}$-Fuchsian and $\mathbb{C}$-Fuchsian Gusevskii, Parker

Together, these results means that for every allowable value of τ there is a CHQF representation with this value of τ

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There exist exotic CHQF representations with limit set a wild knot Dutenhefner, Gusevskii