COMPLEX HYPERBOLIC GEOMETRY:
2. Eigenvalues and traces in SU(2,1)

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Dynamical classification of elements in $\mathrm{PU}(2,1)$

$A \in \mathrm{PU}(2,1)$ acts on the $\overline{\mathbf{H}}^2_{\mathbb{C}}$ so has a fixed point. We say

- A is loxodromic if A fixes exactly 2 points of $\partial\mathbf{H}^2_{\mathbb{C}}$.
- A is parabolic if A fixes exactly 1 point of $\partial\mathbf{H}^2_{\mathbb{C}}$.
- A is elliptic if A fixes at least 1 point of $\mathbf{H}^2_{\mathbb{C}}$.

I claim:
A fixed point of $A \in \mathrm{PU}(2,1)$ corresponds to an eigenvector of the associated matrix A in $\mathrm{U}(2,1)$ or $\mathrm{SU}(2,1)$:

Suppose $A \in \mathrm{SU}(2,1)$. We know A acts on $\overline{\mathbf{H}}^2_{\mathbb{C}}$ as $A(z) = \mathbb{P}A\mathbf{z}$ where $z = \mathbb{P}\mathbf{z}$.

Suppose A has eigenvector $\mathbf{v} \in \mathbb{C}^{2,1} - \{\mathbf{0}\}$ with eigenvalue λ, so $A\mathbf{v} = \lambda\mathbf{v}$.
Let $v = \mathbb{P}\mathbf{v}$. Then $A(v) = \mathbb{P}A\mathbf{v} = \mathbb{P}\lambda\mathbf{v} = \mathbb{P}\mathbf{v} = v$.
Thus an eigenvector $\mathbf{v}$ of A corresponds to a fixed point $v = \mathbb{P}\mathbf{v}$ of A.

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Thus an eigenvector $\mathbf{v}$ of A corresponds to a fixed point $v = \mathbb{P}\mathbf{v}$ of A.

The local dynamics around the v is determined by the eigenvalue λ.

Eigenvalues, determinant, trace

Suppose that $A \in \mathrm{SU}(2,1)$, $\lambda \in \mathbb{C}$ and $\mathbf{v} \in \mathbb{C}^{2,1} - \{\mathbf{0}\}$ with $A\mathbf{v} = \lambda\mathbf{v}$.
As $A \in \mathrm{SU}(2,1)$ we know $A^{-1} = H^{-1}A^*H$.

So $H^{-1}A^*H\mathbf{v} = A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$ and so λ^{-1} is an eigenvalue of A^*.
Therefore $\overline{\lambda}^{-1}$ is also an eigenvalue of A.
If $|\lambda| = 1$ then $\overline{\lambda}^{-1} = \lambda$ and this does not say anything!

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Suppose λ, $\mu \in \mathbb{C}$ and $\mathbf{v}$, $\mathbf{w} \in \mathbb{C}^{2,1} - \{\mathbf{0}\}$ with $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \mu\mathbf{w}$.
Then $\langle \mathbf{v}, \mathbf{w} \rangle = \langle A\mathbf{v}, A\mathbf{w} \rangle = \langle \lambda\mathbf{v}, \mu\mathbf{w} \rangle = \lambda\overline{\mu}\langle \mathbf{v}, \mathbf{w} \rangle$.
Hence, either $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ or $\mu = \overline{\lambda}^{-1}$.

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Since $A \in \mathrm{SU}(2,1)$ we have $1 = \det(A)$, the product of the eigenvalues.
Let τ be $\mathrm{tr}(A)$, the sum of the eigenvalues. Then $\overline{\tau} = \mathrm{tr}(A^{-1})$.
This implies that the characteristic polynomial of A is

$$\chi_A(x) = x^3 - \tau x^2 + \overline{\tau}x - 1 = x^3 - \mathrm{tr}(A)x^2 + \mathrm{tr}(A^{-1})x - 1.$$

Expanding/contracting dynamics

Suppose $|\lambda| \neq 1$.
We have $\langle \mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{v}, A\mathbf{v} \rangle = \langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle = |\lambda|^2 \langle \mathbf{v}, \mathbf{v} \rangle$.
Since $|\lambda|^2 \neq 1$ we have $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.
Since $\mathbf{v} \neq \mathbf{0}$ we see that $\mathbf{v} \in V_0$ and so $v = \mathbb{P}\mathbf{v} \in \partial \mathbf{H}^2_{\mathbb{C}}$.

Moreover, $\overline{\lambda}^{-1} \neq \lambda$ and there exists $\mathbf{w} \in \mathbb{C}^{2,1} - \{\mathbf{0}\}$ with $A\mathbf{w} = \overline{\lambda}^{-1}\mathbf{w}$.
As above $\langle \mathbf{w}, \mathbf{w} \rangle = |\lambda|^{-2} \langle \mathbf{w}, \mathbf{w} \rangle$. So $\mathbf{w} \in V_0$.
Hence A has a second fixed point $w = \mathbb{P}\mathbf{w} \in \partial \mathbf{H}^2_{\mathbb{C}}$.

Then v and w are attracting/repelling fixed points of A in $\partial \mathbf{H}^2_{\mathbb{C}}$.
It is not hard to show A has no other fixed points in $\overline{\mathbf{H}}^2_{\mathbb{C}}$.

Thus A has an eigenvalue λ with $|\lambda| \neq 1$ if and only if A is loxodromic.

Since λ and $\overline{\lambda}^{-1}$ are two eigenvalues, the third must be $\overline{\lambda}\lambda^{-1}$.
Therefore $\mathrm{tr}(A) = \lambda + \overline{\lambda}^{-1} + \overline{\lambda}\lambda^{-1}$.

Let a be the geodesic with endpoints v and w, called the axis of A.
A translates along a by distance ℓ and rotates around a an angle ϕ.

Then $\lambda = e^{\pm\ell/2} e^{-i\phi/3}$ and $\mathrm{tr}(A) = 2\cosh(\ell/2) e^{-i\phi/3} + e^{2i\phi/3}$.

Non-diagonalizable elements of $\mathrm{SU}(2,1)$

Suppose that $A \in \mathrm{SU}(2,1)$ has a repeated eigenvalue $\lambda \in \mathbb{C}$.
By the previous slide, we have $|\lambda| = 1$, that is $\lambda = e^{i\phi}$.

Suppose A is non-diagonalizable.
Then there exist $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{C}^{2,1} - \{\mathbf{0}\}$ with $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \lambda\mathbf{w} + \mathbf{v}$.
Then $\langle \mathbf{v}, \mathbf{w} \rangle = \langle A\mathbf{v}, A\mathbf{w} \rangle = \langle \lambda\mathbf{v}, \lambda\mathbf{w} \rangle + \langle \lambda\mathbf{v}, \mathbf{v} \rangle = |\lambda|^2 \langle \mathbf{v}, \mathbf{w} \rangle + \lambda \langle \mathbf{v}, \mathbf{v} \rangle$.
As $|\lambda| = 1$ we see that $\mathbf{v} \in V_0$ and $v = \mathbb{P}\mathbf{v} \in \partial\mathbb{H}^2_{\mathbb{C}}$.
It is not hard to show v is the only fixed point of A in $\overline{\mathbf{H}}^2_{\mathbb{C}}$.
Thus A is non-diagonalizable if and only if A is parabolic.

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It is not hard to show v is the only fixed point of A in $\overline{\mathbf{H}}^2_{\mathbb{C}}$.
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There are three possibilities

- λ has multiplicity 2.
- λ has multiplicity 3 and A has minimal polynomial $(t - \lambda)^2$.
- λ has multiplicity 3 and A has minimal polynomial $(t - \lambda)^3$.

In the first case $\lambda = e^{i\phi}$ and the third eigenvalue is $e^{-2i\phi} \neq e^{i\phi}$.
Then A is screw-parabolic.

In the second and third case $\lambda^3 = 1$.
Then (projectively) A is conjugate to a Heisenberg translation.

Diagonalizable maps with unit eigenvalues

The one remaining case to consider is:

Diagonalizable $A \in \mathrm{SU}(2,1)$ with all eigenvalues λ of A have $|\lambda| = 1$.

As A diagonalizable, there exists a basis of eigenvectors for $\mathrm{SU}(2,1)$.

Since eigenvectors with distinct eigenvalues are Hermitian orthogonal:

There exists an eigenvector $\mathbf{v}$ of A in V_- corresponding to a fixed point $v = \mathbb{P}\mathbf{v} \in \mathbf{H}^2$.

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The possibilities are

- A has three distinct eigenvalues. There are three conjugacy classes depending on which eigenvector lies in V_-. A is regular elliptic.
- A has an eigenvector λ of multiplicity 2. A is a complex reflection.
  - If the 2-dimensional eigenspace V_λ intersects V_-.
    A is complex reflection in the complex line $\mathbb{P}V_\lambda$.
  - If the 2-dimensional eigenspace V_λ lies in V_+.
    A is complex reflection in a point (comes from $\mathbb{P}V_{\lambda^{-2}}$).
- A has an eigenvector of multiplicity 3. A is projectively the identity.

Goldman's discriminant function

Let $A \in \mathrm{SU}(2,1)$ and write $\tau = \mathrm{tr}(A)$. Characteristic polynomial is $\chi_A(x) = x^3 - \tau x^2 + \overline{\tau} x - 1$.
Define Goldman's discriminant function $f(\tau)$ by
$f(\tau) = |\tau|^4 - 4\tau^3 - 4\overline{\tau}^3 + 18|\tau|^2 - 27$.
Then $f(\tau) = 0$ if and only if $\chi_A(x)$ and $\chi'_A(x)$ have a common root, so if and only if A has a repeated eigenvalue.

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We can use $f(\tau)$ to characterise isometries:

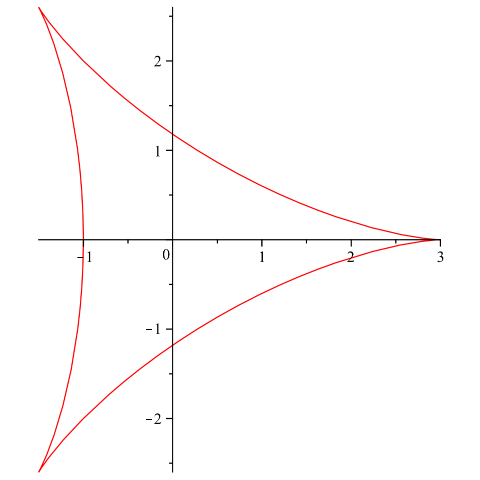
- $f(\tau) > 0$ if and only if A is loxodromic
- $f(\tau) = 0$ if and only if A is parabolic or a complex reflection
- $f(\tau) < 0$ if and only if A is regular elliptic

The curve $f(\tau) = 0$ is a classical curve called a deltoid.

- τ outside deltoid if and only if $f(\tau) > 0$
- τ on deltoid if and only if $f(\tau) = 0$
- τ inside deltoid if and only if $f(\tau) < 0$

The deltoid $f(\tau) = 0$

$$f(\tau) = |\tau|^4 - 4\tau^3 - 4\bar{\tau}^3 + 18|\tau|^2 - 27 = 0.$$



Two generator subgroups of $\mathrm{SU}(2,1)$

We want to parametrise subgroups $\langle A, B\rangle < \mathrm{SU}(2,1)$ up to conjugation.
Traces give a convenient way to do this.

Theorem. Wen
Let $A, B \in \mathrm{SL}(3,\mathbb{C})$. Suppose $\Gamma = \langle A, B\rangle$ is Zariski dense.
Then Γ is determined up to conjugation by the nine traces
$\mathrm{tr}(A)$, $\mathrm{tr}(A^{-1})$, $\mathrm{tr}(B)$, $\mathrm{tr}(B^{-1})$, $\mathrm{tr}(AB)$, $\mathrm{tr}(B^{-1}A^{-1})$,
$\mathrm{tr}(A^{-1}B)$, $\mathrm{tr}(B^{-1}A)$, $\mathrm{tr}[A,B] = \mathrm{tr}(ABA^{-1}B^{-1})$.

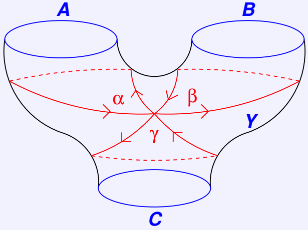
Note that $|\mathrm{tr}[A,B]|$ and $\mathrm{Re}\left(\mathrm{tr}[A,B]\right)$ are determined by the other traces.
We only need to know the sign of $\mathrm{Im}\left(\mathrm{tr}[A,B]\right)$.
This is consistent with the fact that $\mathrm{SL}(3,\mathbb{C})$ has *complex* dimension 8.

Corollary. Lawton, Will
Let $A, B \in \mathrm{SU}(2,1)$. Suppose $\Gamma = \langle A, B\rangle$ is Zariski dense.
Then Γ is determined up to conjugation by the five traces
$\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(AB)$, $\mathrm{tr}(A^{-1}B)$, $\mathrm{tr}[A,B]$.

Again we only need to know the sign of $\mathrm{Im}\left(\mathrm{tr}[A,B]\right)$.
This is consistent with the fact that $\mathrm{SU}(2,1)$ has *complex* dimension 4.

The three holed sphere

Let Y be a three holed sphere (sometimes called pair of pants).
If the three boundary curves are denoted α, β, γ then
The fundamental group of Y is $\pi_1(Y) = \langle [\alpha], [\beta], [\gamma] : [\alpha\beta\gamma] = id \rangle$.



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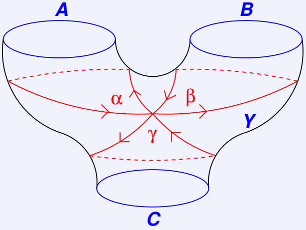

We want to study representations (conjugacy class of homomorphisms)
$\rho : \pi_1(Y) \longrightarrow \Gamma_Y < \mathrm{SU}(2,1)$
Let $\rho([\alpha]) = A$, $\rho([\beta]) = B$, $\rho([\gamma]) = C$
Then $\rho(\pi_1(Y)) = \Gamma_Y$ is a subgroup of $\mathrm{SU}(2,1)$
generated by A, B, C with $ABC = I$.
In other words, $C = (AB)^{-1} = B^{-1}A^{-1}$.

Representations of Γ_Y

Let $\rho(\pi_1(Y)) = \Gamma_Y < \mathrm{SU}(2,1)$. By Lawton-Will Γ_Y determined by $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(C)$, $\mathrm{tr}(A^{-1}B)$, $\mathrm{tr}[A,B]$.

How do we make the last two symmetric in A, B, C?

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Using $C = (AB)^{-1} = B^{-1}A^{-1}$
$[B,C] = BCB^{-1}C^{-1} = BB^{-1}A^{-1}B^{-1}AB = C[A,B]C^{-1}$
$[C,A] = CAC^{-1}A^{-1} = B^{-1}A^{-1}AABA^{-1} = B^{-1}[A,B]B$
So $\mathrm{tr}[A,B] = \mathrm{tr}[B,C] = \mathrm{tr}[C,A]$.

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I claim
$\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B) = \mathrm{tr}(B^{-1}C) - \mathrm{tr}(B^{-1})\mathrm{tr}(C)$
$= \mathrm{tr}(C^{-1}A) - \mathrm{tr}(C^{-1})\mathrm{tr}(A)$.

By the Cayley-Hamilton Theorem:
$\chi_A(A) = A^3 - \mathrm{tr}(A)A^2 + \mathrm{tr}(A^{-1})A - I = O$.
Multiply on the right by $A^{-1}B$ and rearrange to get
$AC^{-1} - \mathrm{tr}(A)C^{-1} = A^2B - \mathrm{tr}(A)AB = A^{-1}B - \mathrm{tr}(A^{-1})B$.
Take traces to get
$\mathrm{tr}(C^{-1}A) - \mathrm{tr}(C^{-1})\mathrm{tr}(A) = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$.

Geometric structure on Y

We suppose that $A = \rho([\alpha])$, $B = \rho([\beta])$, $C = \rho([\gamma])$ are all loxodromic.
Let a_α, a_β, a_γ be the axes of A, B, C (geodesic joining fixed points).
Let c_α, c_β, c_γ be geodesics in homotopy classes $[\alpha]$, $[\beta]$, $[\gamma]$.
Then $c_\alpha = a_\alpha/\langle A\rangle$, $c_\beta = a_\beta/\langle B\rangle$, $c_\gamma = a_\gamma/\langle C\rangle$.

Let ℓ_α, ℓ_β, ℓ_γ be the (Bergman) lengths of c_α, c_β, c_γ.
Let ϕ_α, ϕ_β, ϕ_γ be the holonomy angles around c_α, c_β, c_γ.
Then
$\mathrm{tr}(A) = 2\cosh(\ell_\alpha/2)e^{-i\phi_\alpha/3} + e^{2i\phi_\alpha/3}$,
$\mathrm{tr}(B) = 2\cosh(\ell_\beta/2)e^{-i\phi_\beta/3} + e^{2i\phi_\beta/3}$,
$\mathrm{tr}(C) = 2\cosh(\ell_\gamma/2)e^{-i\phi_\gamma/3} + e^{2i\phi_\gamma/3}$.

The invariants $\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$ and $\mathrm{tr}[A, B]$ are harder to interpret geometrically.