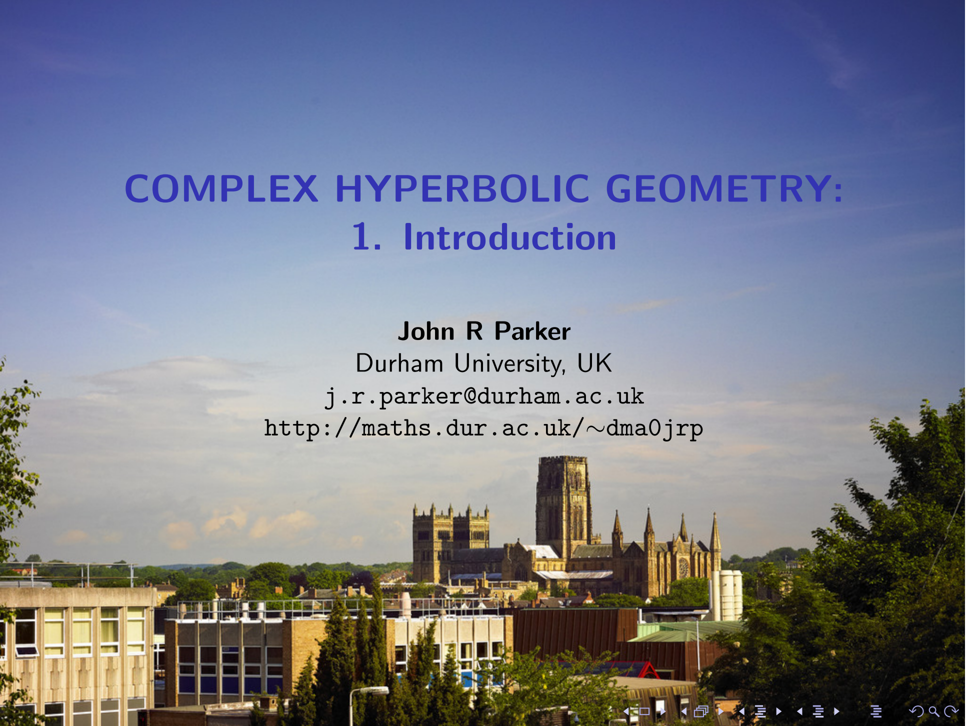

COMPLEX HYPERBOLIC GEOMETRY: 1. Introduction

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Durham

The castle (centre) was begun in 1070.
In 1832 Durham University was founded, and the castle became a college.
The cathedral (on the right) was begun around 1090.
It was recently voted the most beautiful building in the UK.

Hermitian linear algebra

Let $A = (a_{ij})$ be a $k \times l$ complex matrix.
The Hermitian transpose of A is the $l \times k$ complex matrix $A^* = (\bar{a}_{ji})$.
Take transpose, then complex conjugate.

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A $k \times k$ matrix H is Hermitian if $H = H^*$.
Eigenvalues of Hermitian matrices are real.

A non-singular Hermitian matrix has signature (p, q) where $p + q = k$ if it has p positive eigenvalues and q negative eigenvalues.

If H is $k \times k$ Hermitian, define a Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^k$ by
$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z}$.
Note $\langle \lambda \mathbf{z} + \mu \mathbf{z}, \mathbf{v} \rangle = \lambda \langle \mathbf{z}, \mathbf{v} \rangle + \mu \langle \mathbf{w}, \mathbf{v} \rangle$, $\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{w} \rangle}$ and $\langle \mathbf{z}, \mathbf{z} \rangle \in \mathbb{R}$.

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A $k \times k$ matrix A is unitary with respect to H if
$\langle A\mathbf{z}, A\mathbf{w} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle \ \forall\ \mathbf{z}, \mathbf{w} \in \mathbb{C}^k$
Equivalently $A^* H A = H$ so $A^{-1} = H^{-1} A^* H$ if H invertible.

The space $\mathbb{C}^{2,1}$

Define $\mathbb{C}^{2,1}$ to be $\mathbb{C}^3$ with a (non-singular) Hermitian form $\langle \mathbf{z}, \mathbf{w} \rangle$ of signature $(2,1)$ associated to the Hermitian matrix H.

Example: $H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\langle \mathbf{z}, \mathbf{w} \rangle_1 = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3$

Or $H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\langle \mathbf{z}, \mathbf{w} \rangle_2 = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1$.

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Let $V_- = \left\{\mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle < 0\right\}$, $V_0 = \left\{\mathbf{z} \in \mathbb{C}^{2,1} - \{0\} | \langle \mathbf{z}, \mathbf{z} \rangle = 0\right\}$,

Define $\mathbb{P} : \mathbb{C}^{2,1} - \{\mathbf{0}\} \longrightarrow \mathbb{CP}^2$ to be the canonical projection

$$\mathbb{P} : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \longmapsto [z_1 : z_2 : z_3].$$

Complex hyperbolic space

Let $\mathbb{C}^{2,1}$, V_-, V_0, $\mathbb{P}$ be as above. Then
complex hyperbolic 2-space is $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_-$
Its boundary is $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_0$.

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What does this look like more concretely?

Consider $\mathbb{C}^{2,1}$ with the first Hermitian form H_1.
So $\langle \mathbf{z}, \mathbf{z} \rangle_1 = |z_1|^2 + |z_2|^2 - |z_3|^2$.
If $\mathbf{z} \in V_-$ then $|z_1|^2 + |z_2|^2 - |z_3|^2 < 0$ so $z_3 \neq 0$.
In $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_-$ we may take $z_3 = 1$, so points satisfy $|z_1|^2 + |z_2|^2 < 1$.
Thus $\mathbf{H}^2_{\mathbb{C}}$ is the unit ball in $\mathbb{C}^2$.
Likewise $\partial\mathbf{H}^2_{\mathbb{C}}$ is the unit sphere in $\mathbb{C}^2$.

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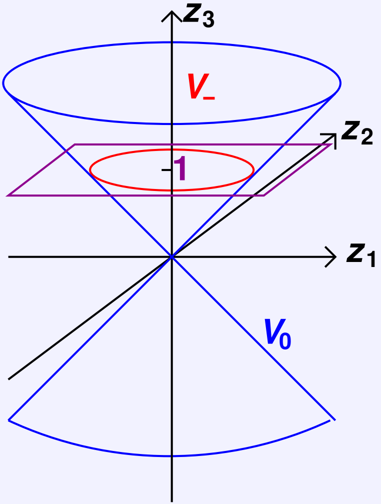
Likewise $\partial\mathbf{H}^2_{\mathbb{C}}$ is the unit sphere in $\mathbb{C}^2$.

Consider $\mathbb{C}^{2,1}$ with the second Hermitian form H_2.
So $\langle \mathbf{z}, \mathbf{z} \rangle_2 = z_1\overline{z}_3 + z_3\overline{z}_1 + |z_2|^2$.
If $\mathbf{z} \in V_-$ then $z_1\overline{z}_3 + z_3\overline{z}_1 + |z_2|^2 < 0$ so $z_3 \neq 0$.
In $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_-$ we may take $z_3 = 1$, so points satisfy $2\mathrm{Re}\,(z_1) + |z_2|^2 < 0$.
This is a paraboloid in $\mathbb{C}^2$ called the Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$.

The light cone for H_1



The set V_0 for the first Hermitian form H_1, sometimes called the light cone.

The (complex) hyperplane $z_3 = 1$ intersects V_- in a ball which is $\mathbf{H}^2_{\mathbb{C}}$.

The Bergman metric, distance function and volume form

We define a metric, called the Bergman metric, on $\mathbf{H}^2 = \mathbb{P}V_-$ by

$$ds^2 = \frac{-4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} \det \begin{pmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, \mathbf{z} \rangle \\ \langle \mathbf{z}, d\mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{pmatrix}.$$

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It gives a distance function ρ.
Let z, w be points in $\mathbf{H}^2_{\mathbb{C}}$ corresponding to $\mathbf{z}$, $\mathbf{w}$ in $\mathbb{C}^{2,1}$. Then

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}.$$

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This gives a volume form and Kähler form ω on $\mathbf{H}^2_{\mathbb{C}}$.

Lift $z = (z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} \subset \mathbb{C}^2$ to $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} \in \mathbb{C}^{2,1}$. For forms H_1 or H_2

$$d\mathrm{Vol} = \frac{16}{-\langle \mathbf{z}, \mathbf{z} \rangle^3} \, d\mathrm{vol}, \qquad \omega = 4i\partial\overline{\partial} \log \langle \mathbf{z}, \mathbf{z} \rangle$$

where $d\,\mathrm{vol}$ is the volume element
$(1/2i)^2 d\,z_1 \wedge d\,\overline{z}_1 \wedge d\,z_2 \wedge d\,\overline{z}_2 = d\,x_1 d\,y_1 d\,x_2 d\,y_2$.

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The Bergman metric, distance function, volume form and Kähler form are only defined in terms of the Hermitian form.

The group $\mathrm{PU}(H)$

Let H be the Hermitian form of signature $(2,1)$ on $\mathbb{C}^{2,1}$.
Let the unitary group of H be $\mathrm{U}(H) = \{A \in \mathrm{GL}(3,\mathbb{C}) \mid A^*HA = H\}$
Equivalently $\langle A\mathbf{z}, A\mathbf{w}\rangle = \langle \mathbf{z}, \mathbf{w}\rangle$ for all $\mathbf{z}$, $\mathbf{w}$ in $\mathbb{C}^{2,1}$.
If signature $(2,1)$ specified, not particular form then we write $\mathrm{PU}(2,1)$.

Then A acts on $\mathbf{H}^2_{\mathbb{C}}$ as follows. If $z \in \mathbf{H}^2_{\mathbb{C}}$ corresponds to $\mathbf{z} \in \mathbb{C}^{2,1}$ then:
$A(z) = \mathbb{P}A\mathbf{z}$.

Concretely: if $z = (z_1, z_2)$, $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$. Then

$$A(z) = \mathbb{P}\begin{pmatrix} az_1 + bz_2 + c \\ dz_2 + ez_2 + f \\ gz_1 + hz_2 + j \end{pmatrix} = \left(\frac{az_1 + bz_2 + c}{gz_1 + hz_2 + j}, \frac{dz_1 + ez_2 + f}{gz_1 + hz_2 + j}\right).$$

Note that the action of A is the same as the action of λA for any scalar λ.
So we define $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\{\lambda I\}$.

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We also define $\mathrm{SU}(2,1) = \{A \in \mathrm{U}(2,1) \mid \det(A) = 1\}$.
Then $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{I, \omega I, \overline{\omega} I\}$ where $\omega = e^{2i\pi/3}$.
So $\mathrm{SU}(2,1)$ is a triple cover of $\mathrm{SU}(2,1)$.

Complex hyperbolic isometries

Since ds^2, $d\mathrm{Vol}$ and ρ are all only defined in terms of the Hermitian form, we see that if $A \in \mathrm{PU}(2,1)$ then A preserves all three of them. So $\mathrm{PU}(2,1) \subset \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$ the group of complex hyperbolic isometries.

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Theorem.
Every isometry of $\mathbf{H}^2_{\mathbb{C}}$ is either holomorphic or anti-holomorphic.
Every holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ is given by some $A \in \mathrm{PU}(2,1)$.
Every anti-holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ is given by complex conjugation followed by some $A \in \mathrm{PU}(2,1)$.

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$\mathrm{PU}(2,1)$ acts transitively on $\mathbf{H}^2_{\mathbb{C}}$
If z, $z' \in \mathbf{H}^2_{\mathbb{C}}$ there is $A \in \mathrm{PU}(2,1)$ with $z' = A(z)$.
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If z, z', w, $w' \in \mathbf{H}^2_{\mathbb{C}}$ with $\rho(z', w') = \rho(z, w)$
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$\mathrm{PU}(2,1)$ acts transitively on pairs of points in $\partial\mathbf{H}^2_{\mathbb{C}}$
If z, z', w, $w' \in \partial\mathbf{H}^2_{\mathbb{C}}$ there is $A \in \mathrm{PU}(2,1)$ with $z' = A(z)$, $w' = A(z)$.

Subspaces of $\mathbf{H}^2_{\mathbb{C}}$ – complex lines

The ball model and Siegel domain models of $\mathbf{H}^2_{\mathbb{C}}$ are embedded in $\mathbb{C}^2$.
Let L be a complex line in $\mathbb{C}^2$.
So L is the image under $\mathbb{P}$ of a complex hyperplane in $\mathbb{C}^{2,1}$ through $\mathbf{0}$.

If $L \cap \mathbf{H}^2_{\mathbb{C}}$ is nonempty, then the intersection is a disc (or halfplane).
We say this is a complex line in $\mathbf{H}^2_{\mathbb{C}}$. We also refer to this as L.
The restriction of the Bergman metric to L is the Poincaré metric on the hyperbolic plane with curvature -1.

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Example. Let $L_0 = \{0\} \times \mathbb{C}$. Then L_0 intersects the unit ball in a disc $\{(0, z_2) \in \mathbb{C}^2 \mid |z_2|^2 < 1\}$.
On L_0 the Hermitian form H_1 has signature $(1, 1)$. Then

$$ds^2 = \frac{4}{(1 - |z_2|^2)^2} dz_2 d\overline{z}_2.$$

L_0 is preserved by block diagonal matrices in $\mathrm{P}\big(\mathrm{U}(1) \times \mathrm{U}(1,1)\big)$.

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Any complex line in $\mathbf{H}^2_{\mathbb{C}}$ is the image of L_0 under some $A \in \mathrm{PU}(2,1)$.
The stabiliser is conjugate via A to $\mathrm{P}\big(\mathrm{U}(1) \times \mathrm{U}(1,1)\big)$.
We sometimes think of the triple cover $\mathrm{S}\big(\mathrm{U}(1) \times \mathrm{U}(1,1)\big)$.

Subspaces of $\mathbf{H}^2_{\mathbb{C}}$ – Lagrangian planes

The ball model and Siegel domain models of $\mathbf{H}^2_{\mathbb{C}}$ are embedded in $\mathbb{C}^2$.
Let R be a totally real Lagrangian plane in $\mathbb{C}^2$.

If $R \cap \mathbf{H}^2_{\mathbb{C}}$ is nonempty, then the intersection is a topological disc. We say this is a Lagrangian plane in $\mathbf{H}^2_{\mathbb{C}}$. We also refer to this as R. The restriction of the Bergman metric to R is the Klein-Beltrami metric on the hyperbolic plane with curvature $-1/4$.

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On R_0 the Hermitian form H_1 is a quadratic form of signature $(2, 1)$.
Then $ds^2 = \dfrac{4}{(1 - x_1^2 - x_2^2)^2}(dx_1^2 + dx_2^2 - (x_1 dx_2 - x_2 dx_1)^2)$.
R_0 is preserved by matrices with real entries in $\mathrm{PO}(2, 1)$.

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On R_0 the Hermitian form H_1 is a quadratic form of signature $(2, 1)$.
Then $ds^2 = \dfrac{4}{(1 - x_1^2 - x_2^2)^2}(dx_1^2 + dx_2^2 - (x_1 dx_2 - x_2 dx_1)^2)$.
R_0 is preserved by matrices with real entries in $\mathrm{PO}(2, 1)$.

Any Lagrangian plane in $\mathbf{H}^2_{\mathbb{C}}$ is the image of R_0 under some $A \in \mathrm{PU}(2, 1)$.
The stabiliser is conjugate via A to $\mathrm{PO}(2, 1)$
We sometimes think of the cover $\mathrm{SO}(2, 1)$.

Cartan's angular invariant

Let z_1, z_2, z_3 be three points in $\partial\mathbf{H}^2_{\mathbb{C}}$.
Let $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ be the corresponding points in $\mathbb{C}^{2,1}$.
Define the Cartan angular invariant of these points to be
$\mathbb{A}(z_1, z_2, z_3) = \arg\big(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle\big) \in [-\pi/2, \pi/2]$.

$\mathbb{A}(w_1, w_2, w_3) = \mathbb{A}(z_1, z_2, z_3) \iff \exists A \in \mathrm{PU}(2,1)$ with $w_j = A(z_j)$.

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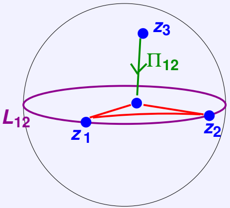
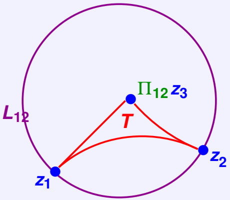
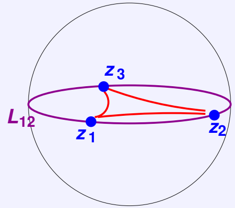


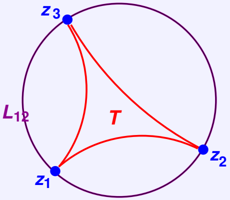


Geometrical interpretation.
z_1 and z_2 lie on a unique complex line L_{12}.
Let Π_{12} be orthogonal projection onto L_{12}.
Let T be the triangle in L_{12} with vertices z_1, z_2, $\Pi_{12}(z_3)$.
Then $\mathbb{A}(z_1, z_2, z_3)$ is half the (signed) Poincaré area of T.

Cartan on complex lines and Lagrangian planes

If z_1, z_2, z_3 all lie on a complex line then $\mathbb{A}(z_1, z_2, z_3) = \pm\pi/2$.
(Sign depends on order of points round the triangle.)
z_3 on L_{12} so $\Pi_{12}(z_3) = z_3$. Then T is an ideal triangle.





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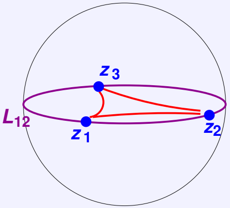

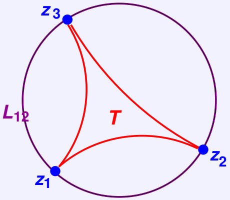

If z_1, z_2, z_3 all lie on a Lagrangian plane then $\mathbb{A}(z_1, z_2, z_3) = 0$.
In this case $\Pi_{12}(z_3)$ lies on geodesic with endpoints z_1 and z_2.
So T has degenerated and has area 0.