GAP DISTRIBUTIONS AND HOMOGENEOUS DYNAMICS

or, How Random are the Rationals?

Motivating Question:

How random are rational numbers?

Hall's Distribution and Horocycle Flow

Notation:

$$F(Q) = \left\{\gamma_0 = \frac{0}{1} < \gamma_1 = \frac{1}{Q} < \dots \gamma_i = \frac{p_i}{q_i} < \dots \gamma_N = \frac{1}{1}\right\}$$

Normalized Farey Gaps:

$$G(Q) = \{Q^2(\gamma_{i+1} - \gamma_i) : 0 < i < N(Q)\}$$

Exotic Gap Distributions

- Elkies-Mcullen used homogeneous dynamics to study the unusual behavior of fraction parts of n^(1/2), writing: "...the uniform distribution of lattices explains the exotic distribution of gaps."
- This behavior was originally noted by Boshernitzan.
- Marklof-Strombergsson studied affine lattices, and found that for a lattice translated by an irrational vector, the gap distribution for visible directions is the same as the Elkies-McMullen distribution.
- Motivated by results of Veech and Vorobets, A-Chaika and A-Chaika-Lelievre studied gap distribution for saddle connection directions on translation surfaces

Translation Surfaces

Saddle Connections

Further Reading:

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- Marklof-Strombergsson. . The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems. http://www.maths.bris.ac.uk/~majm/bib/lorentz-ONE.pdf
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Groups, Geometry, and Dynamics

Almora, Uttarakhand

15th December 2012

Jayadev S. Athreya

Includes joint work with
Y. Cheung;
J. Chaika;
J. Chaika and S. Lelievre

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Farey Sequences

Q a positive integer,

F(Q) = { all fractions in [0, 1], with denominator at most Q when written in lowest terms}

{F(Q): Q a positive integer} gives an **exhaustion** of the rationals between [0, 1] by finite lists

$$|F(Q)| = \sum_{q=1}^{Q} \varphi(q)$$

Here, $\varphi(q)$ is the Euler totient function

TRUE RANDOMNESS

Let X_1, X_2, ... X_n, ..., be independent, identically distributed (i.i.d.) uniform [0, 1] random variables.

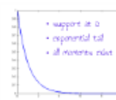
The spacings, or **gaps** for this sequence also contain interesting information, when appropriately normalized.

Namely, let X(1) < X(2) < ... < X(n) be the X_1, ..., X_n written in increasing order. Then if we consider the normalized gap set

G(n) = {n (X(i+1) -X(i)): 0< i < n}

we have that, for any 0 < t < oo

$$\frac{1}{n}|G(n) \cap (t, \infty)| \to e^{-t}$$



That is, a truly random sequence has **exponential** gap distribution.

RANDOMNESS REFINED

DO Farey Sequences have **Exponential Gap Distribution?**

...

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Here, $\varphi(q)$ is the Euler totient function

Exercise: $|F(Q)| \approx \frac{3}{\pi^2}Q^2$

EQUIDISTRIBUTION

Farey Sequences spread out **evenly**:

For any 0 < a < b < 1 we have:

$$\frac{1}{|F(Q)|}|F(Q) \cap (a,b)| \to b - a$$

as Q → ∞

So in some first-order sense, they look random.

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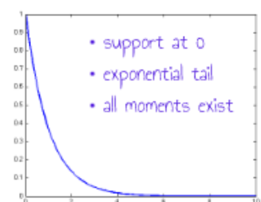

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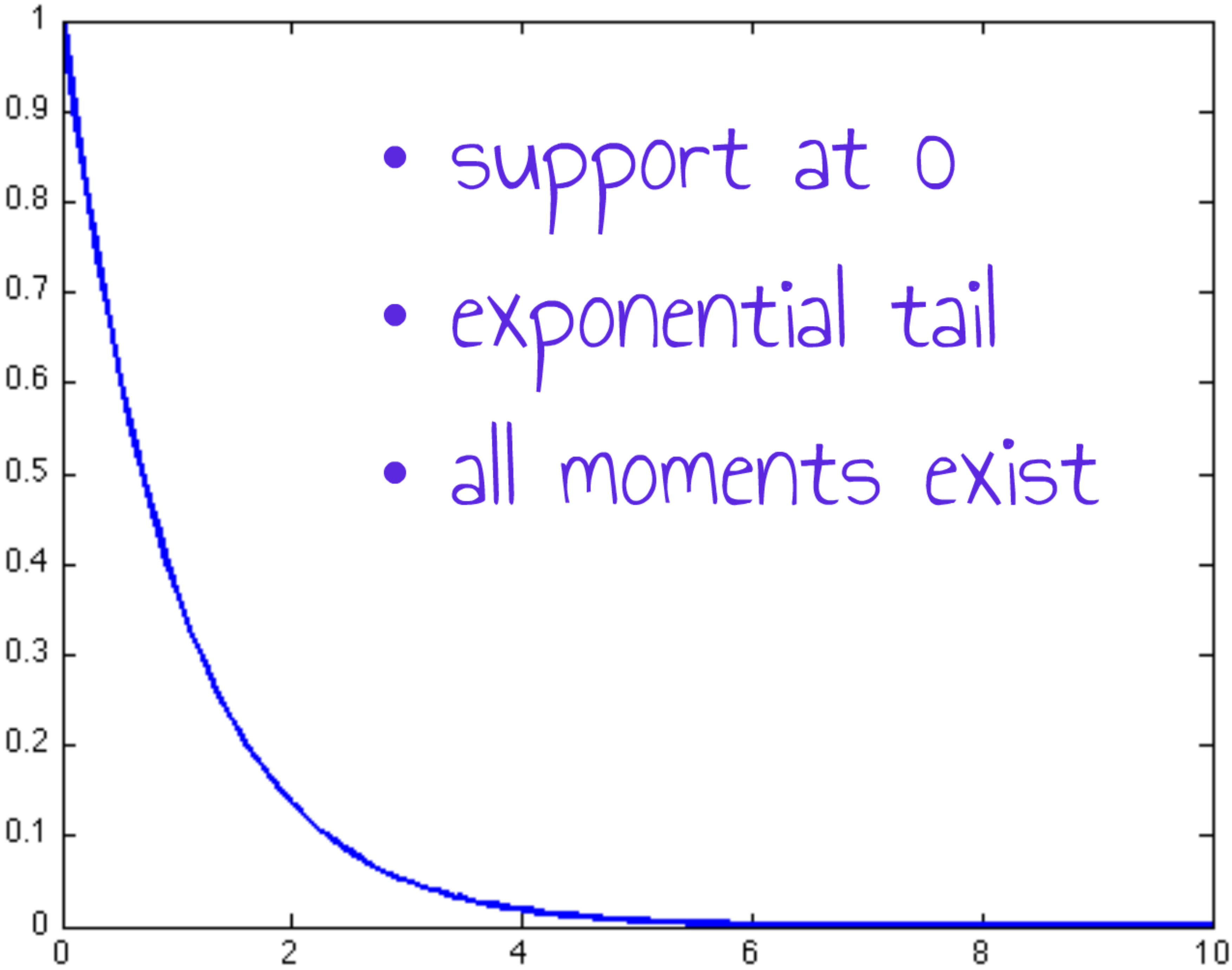

1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
• support at 0
• exponential tail
• all moments exist

RANDOMNESS REFINED

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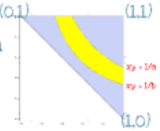
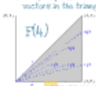
Theorem. (R. R. Hall)

$$\lim_{Q \to \infty} \frac{|G(Q) \cap (a,b)|}{N(Q)} = 2 \text{ area}$$



Geometry of Farey Fractions



Differences in Farey Fractions correspond to differences in slopes.

Horocycle flow on the Space of Lattices

Horocycle flow: left multiplication by the matrix

$$h_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$$

Why this matrix ?

It's a vertical shear, which subtracts s from slopes, and thus, preserves differences in slopes!

GENERIC VERSUS PERIODIC

- The equidistribution of non-periodic orbits (Furstenberg) gives us our generic lattice theorem
- The equidistribution of long periodic orbits (Sarnak) gives us Hall's theorem (and many, many other results).

Principle: For L(x,y), a < S_n+1 - S_n < b iff a < R(T^n (x,y)) < b

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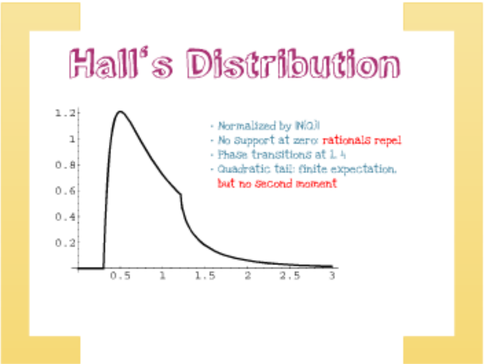

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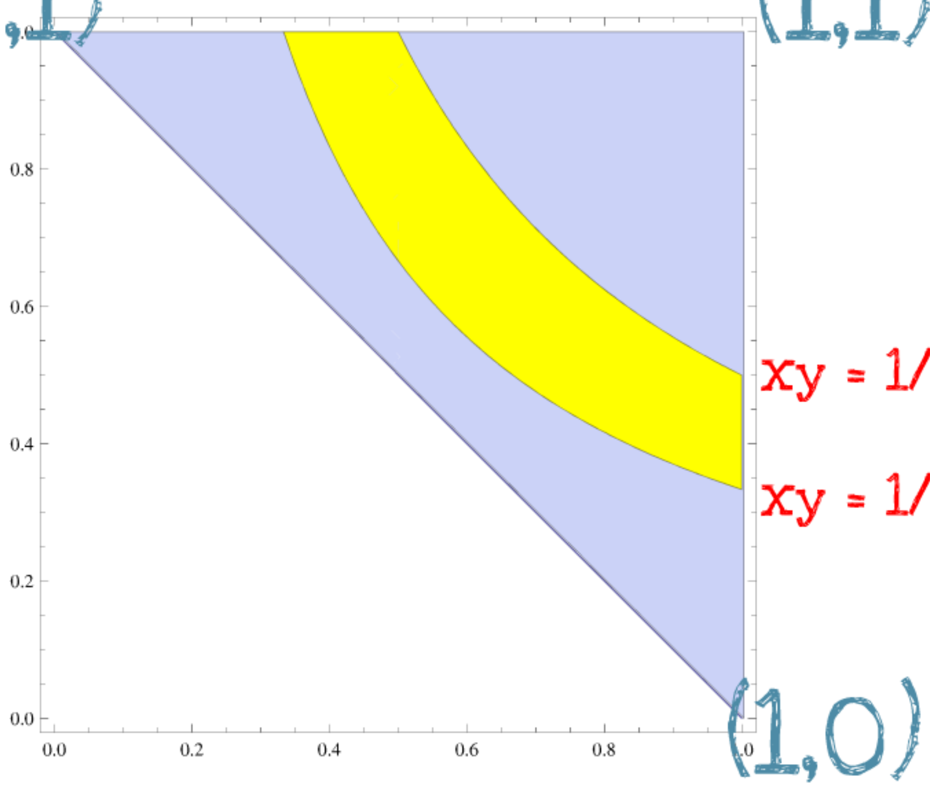
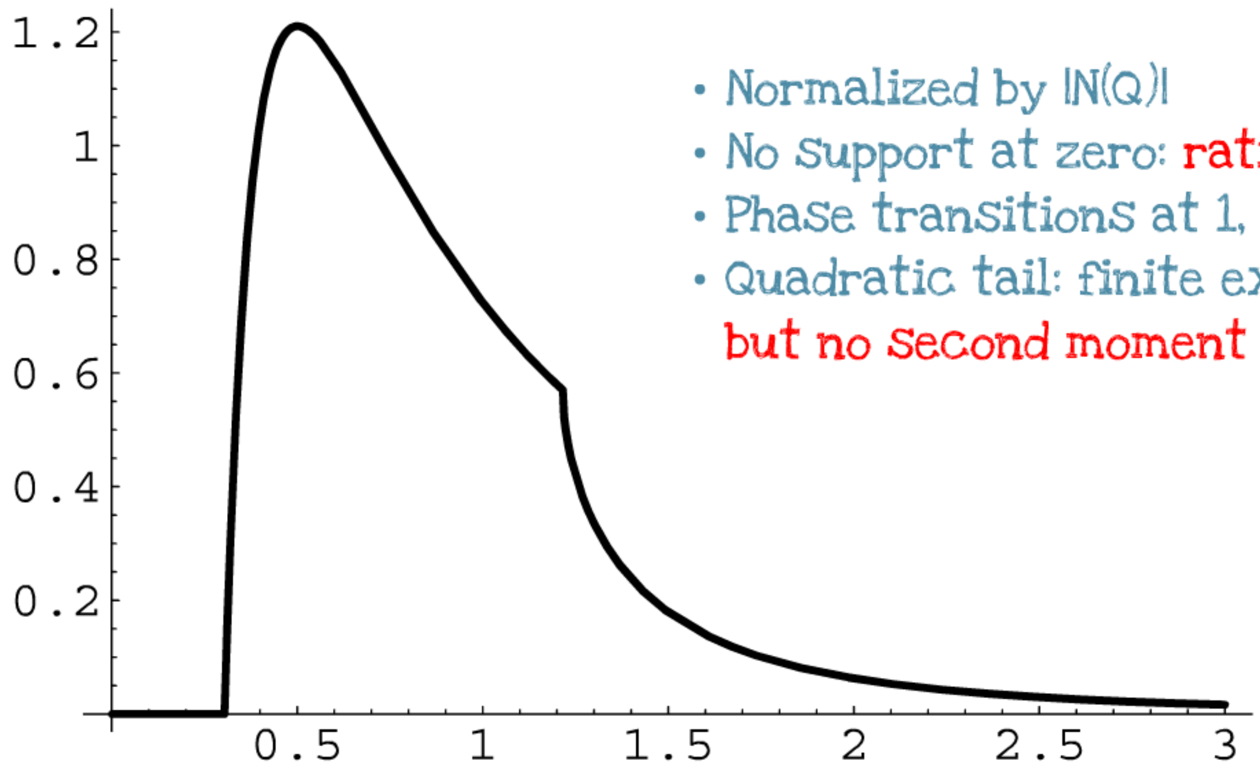


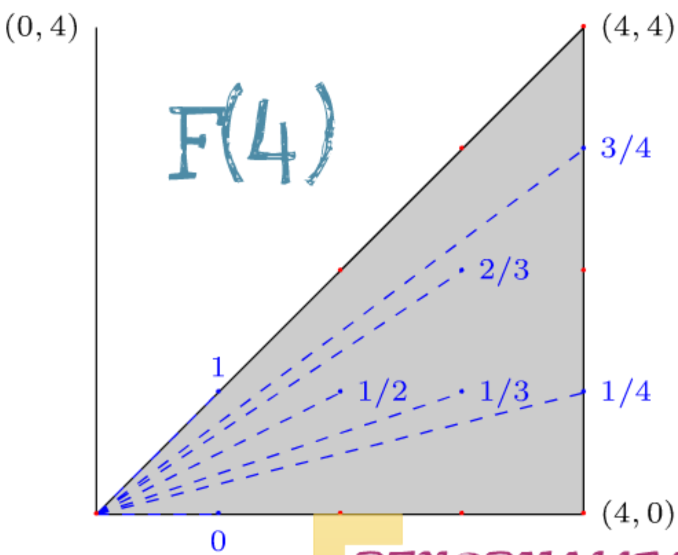
Hall's Distribution



- Normalized by |N(Q)|
- No support at zero: rationals repel
- Phase transitions at 1, 4
- Quadratic tail: finite expectation, but no second moment

Geometry of Farey Fractions

The Farey Sequence F(Q) consists of slopes of (primitive) integer vectors in the triangle with vertices at (0,0), (Q, 0), (Q,Q).



Differences in Farey Fractions correspond to differences in slopes.

RENORMALIZATION

To multiply our slopes by Q^2, we renormalize by the matrix

$$\begin{pmatrix} 1/Q & 0 \\ 0 & Q \end{pmatrix}$$ to get

a triangle with vertices at

(0,0), (1, 0), (1, Q^2)

We've traded slopes smaller than 1 for horizontal components less than 1

Lattice Slope Gaps

Be wise, Generalize! - anon.

Let L be a unimodular lattice in the plane, so

L = gZ^2, g in SL(2, R).

Let s_1, s_2, s_3 .... denote the sequence of slopes of vectors from L in the vertical strip

V = { (x, y): y>0, 0<x<1}

Consider the gap set

G(L, N) = {s_{n+1} - s_n: 0 < n < N}

Aperiodic Lattices

What is the distribution of G(L, N) as N -> 00?

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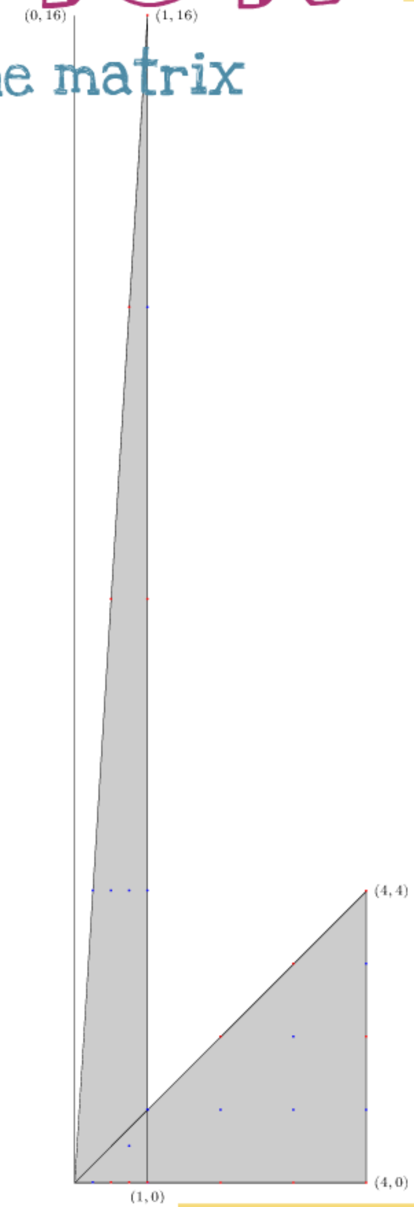

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Theorem (A-Cheung). **If L does not have vertical vectors, then the probability measure supported on G(N, L) converges to Hall's distribution.**

We will prove this theorem and Hall's theorem using the ergodic theory of horocycle flow on the space of lattices.

Aperiodic Lattices

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We will prove this theorem and Hall's theorem using the ergodic theory of horocycle flow on the space of lattices.

Horocycle flow on the Space of Lattices

The space of (unimodular) lattices: X_2 = SL(2, R)/SL(2, Z),

via gSL(2, Z) <--> g Z^2

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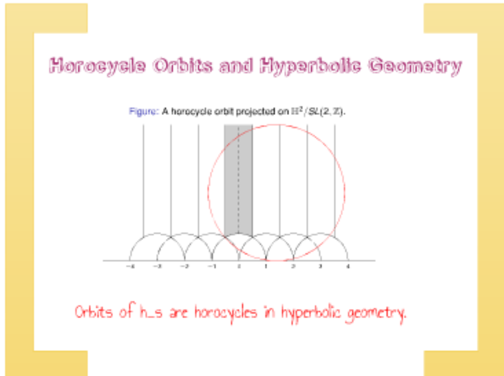

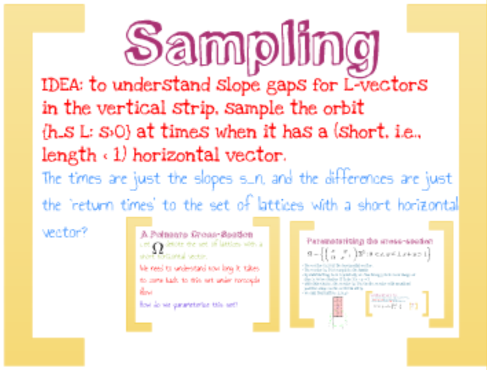

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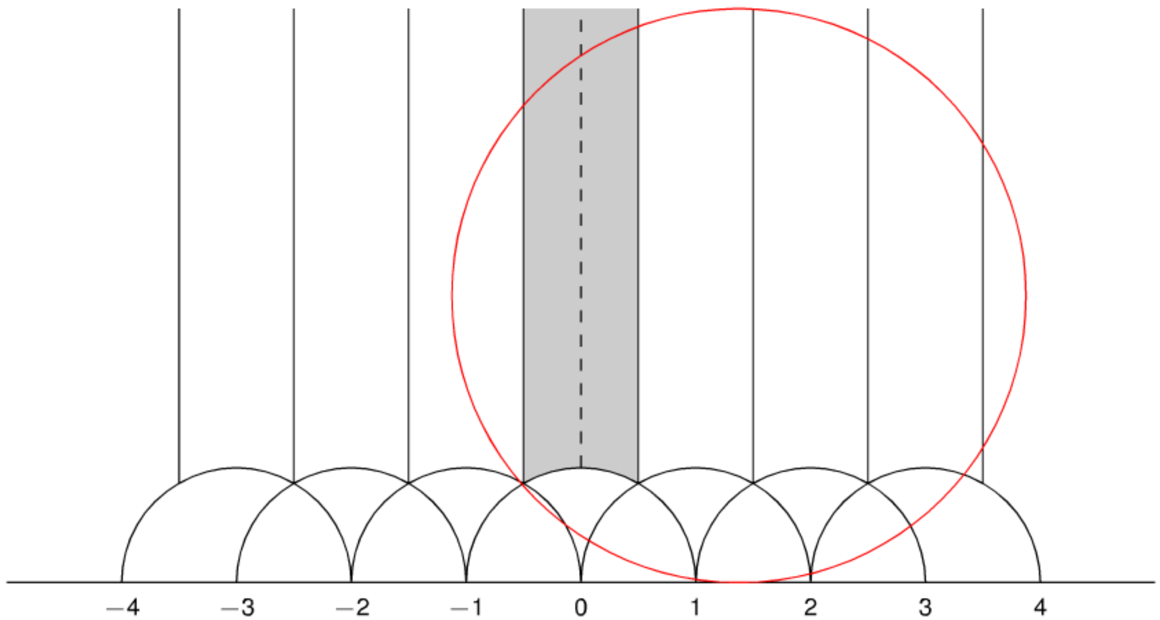
Horocycle Orbits and Hyperbolic Geometry

Figure: A horocycle orbit projected on $\mathbb{H}^2/SL(2,\mathbb{Z})$.



Orbits of h_s are horocycles in hyperbolic geometry.

Sampling

IDEA: to understand slope gaps for L-vectors in the vertical strip, sample the orbit {h_s L: s>0} at times when it has a (short, i.e., length < 1) horizontal vector.

The times are just the slopes s_n, and the differences are just the 'return times' to the set of lattices with a short horizontal vector?

A Poincare Cross-Section

Let Ω denote the set of lattices with a short horizontal vector.

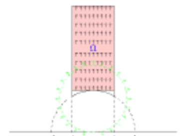
We need to understand how long it takes to come back to this set under horocycle flow

How do we parameterize this set?

Parameterizing the cross-section

$$\Omega = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \mathbb{Z}^2 : 0 < x, y \leq 1, x + y > 1 \right\}$$

- The vector (x,0) is the horizontal vector.
- The vector (y, 1/x) complete the basis
- by subtracting (x,0) repeatedly we can bring y to be in a range of size x. We've chosen it to be 1-x < y <= 1
- With this choice, the vector (y, 1/x) is the vector with smallest positive slope in the vertical strip.
- We call this lattice L_{x,y}



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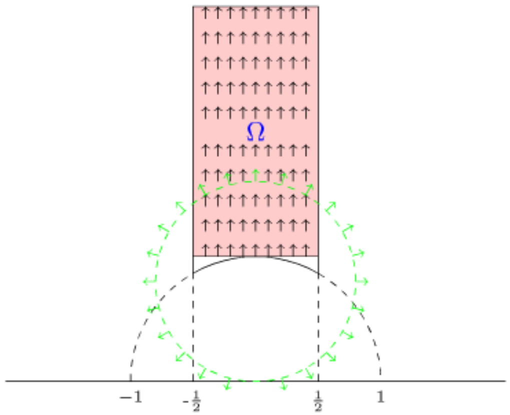
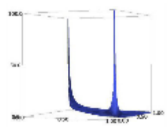


Return Maps and Return Times

- For any (x,y), the orbit {h_s L_{x,y} : s>0} returns to the cross section at time R(x,y) = 1/xy, the slope of the vector (y, 1/x).
- A direct calculation shows that where it returns is given by

$$T(x,y) = \left(y, -x + \left\lfloor \frac{1+x}{y} \right\rfloor y \right)$$



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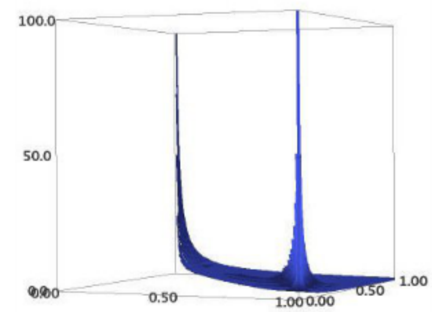

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The BCZ Map

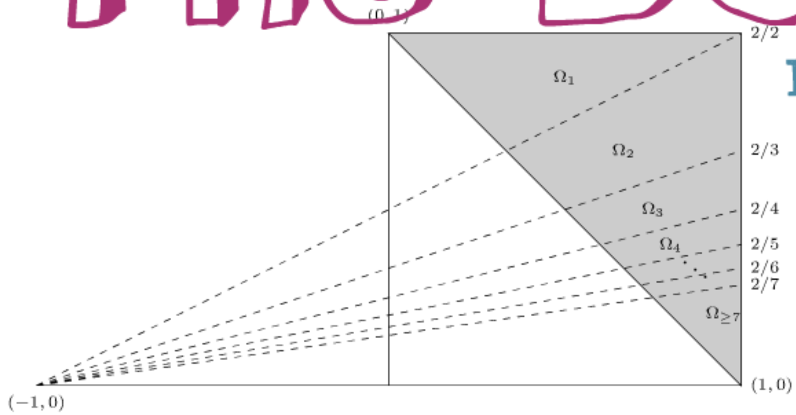
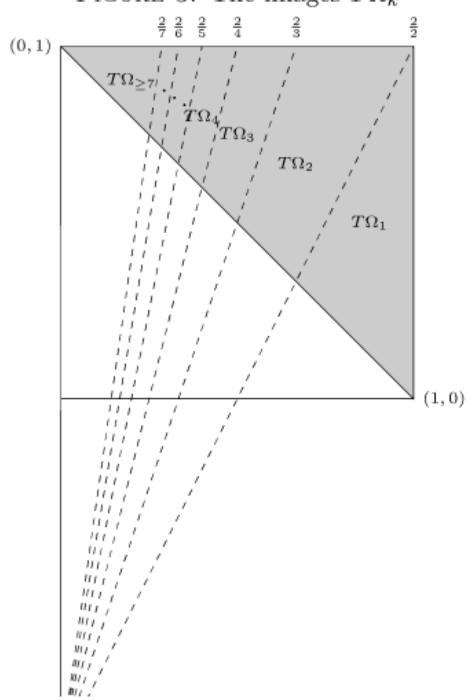


FIGURE 3. The images $T\Omega_k$



Introduced by Boca-Cobeli-Zaharescu in study of Farey sequences.

A.-Cheung expressed this a a return map for h_s, which allows us to use tools from ergodic theory of the horocycle flow, and conclude:

Theorem (A.-Cheung): T is ergodic with respect to Lebesgue measure 2dxdy, and every non-periodic orbit equidistributes. Moreover, long periodic orbits equidistribute.

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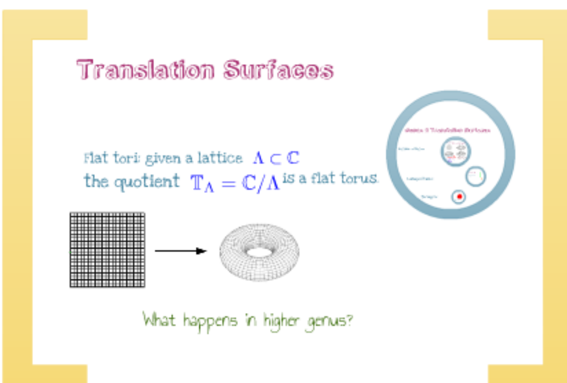
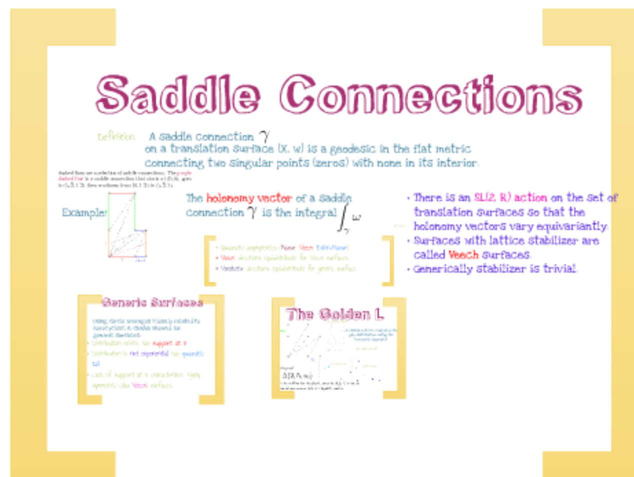
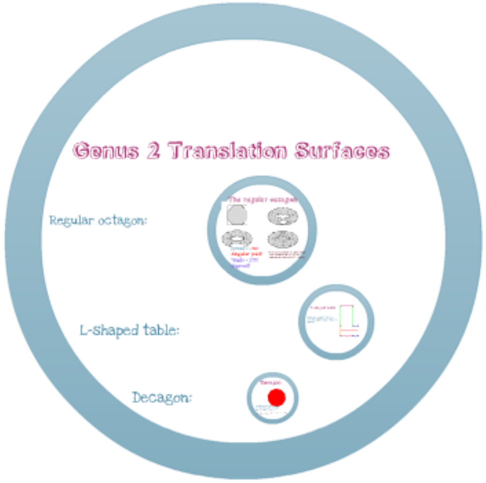
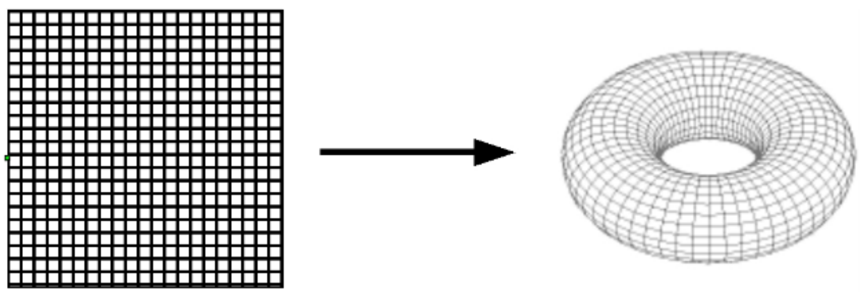




Translation Surfaces



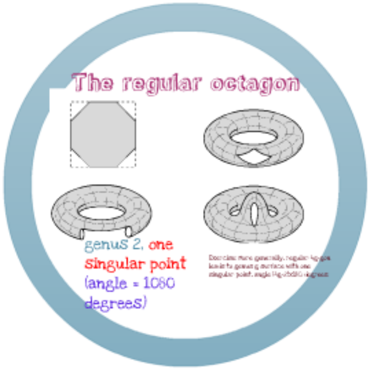
Flat tori: given a lattice $\Lambda \subset \mathbb{C}$
the quotient $\mathbb{T}_\Lambda = \mathbb{C}/\Lambda$ is a flat torus.



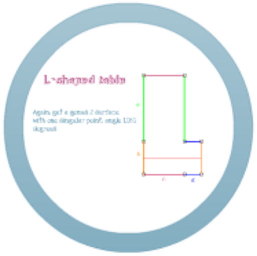
What happens in higher genus?

Genus 2 Translation Surfaces

Regular octagon:



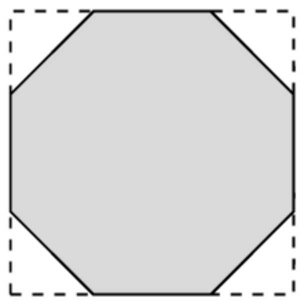
L-shaped table:

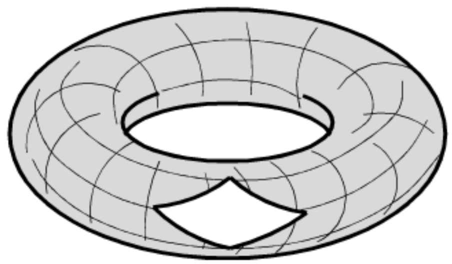


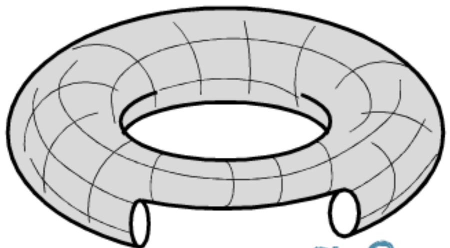
Decagon:

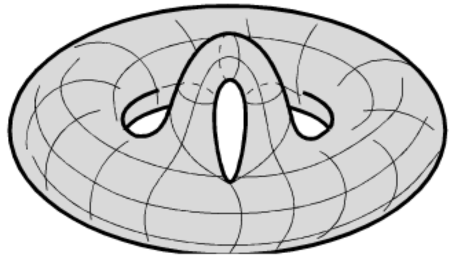
us.

The regular octagon







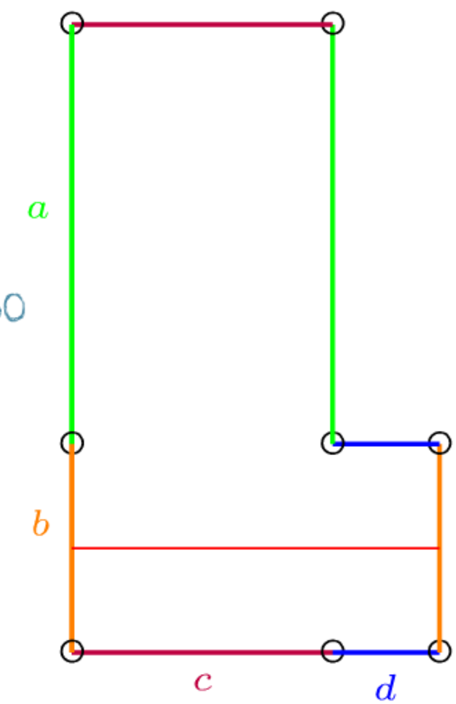


genus 2, one singular point (angle = 1080 degrees)

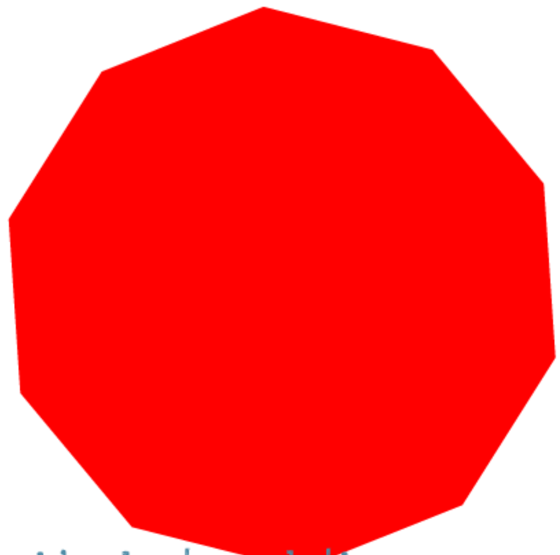
Exercise: More generally, regular 4g-gon leads to genus g surface with one singular point, angle (4g-2)x180 degrees.

L-shaped table

Again, get a genus 2 surface with one singular point, angle 1080 degrees



Decagon



Identifying parallel sides by translation get genus 2 surface with two singular points, each angle 720 degrees

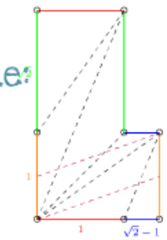
Exercise: More generally, regular (4g+2)-gon leads to genus g surface with two singular points, angle 2g x 180 degrees.

Saddle Connections

Definition: A saddle connection γ on a translation surface (X, w) is a geodesic in the flat metric connecting two singular points (zeros) with none in its interior.

dashed lines are a selection of saddle connections. The purple dashed line is a saddle connection that starts at $(0,0)$, goes to $(\sqrt{2}, 1/2)$ then continues from $(0, 1/2)$ to $(\sqrt{2}, 1)$.

Example:



The holonomy vector of a saddle connection γ is the integral $\int_\gamma \omega$

- Quadratic asymptotics (Masur; Veech; Eskin-Masur)
- Veech: directions equidistribute for Veech surfaces.
- Vorobets: directions equidistribute for generic surface.

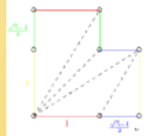
- There is an $SL(2, \mathbb{R})$ action on the set of translation surfaces so that the holonomy vectors vary equivariantly.
- Surfaces with lattice stabilizer are called Veech surfaces.
- Generically stabilizer is trivial.

Generic Surfaces

Using circle averages (closely related to horocycles), A.-Chaika showed, for generic surfaces:

- Distribution exists, has support at 0
- Distribution is not exponential, has quadratic tail.
- Lack of support at 0 characterizes 'highly symmetric' (aka Veech) surfaces.

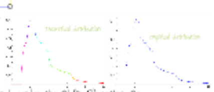
The Golden L



A.-Chaika-Lelievre computed the gap distribution, using the horocycle approach.

A key tool

$\Delta(2, 5, \infty)$



is the stabilizer for the golden L under the $SL(2, \mathbb{R})$ action. So our universe reduces again to a hyperbolic surface.

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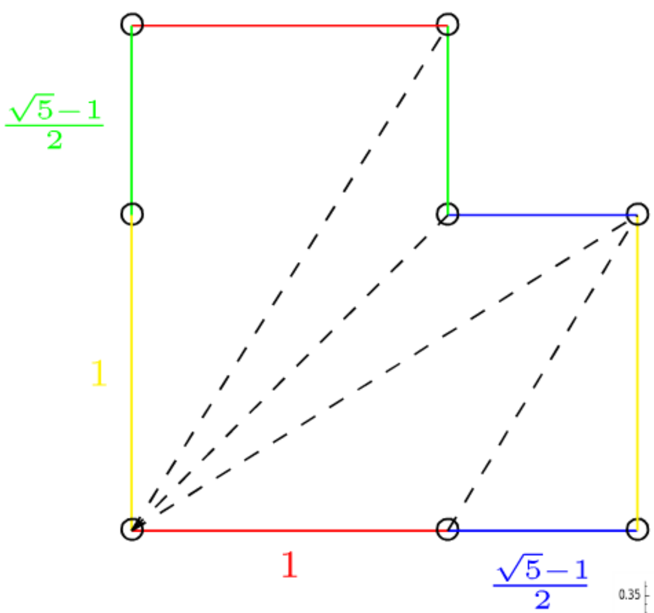
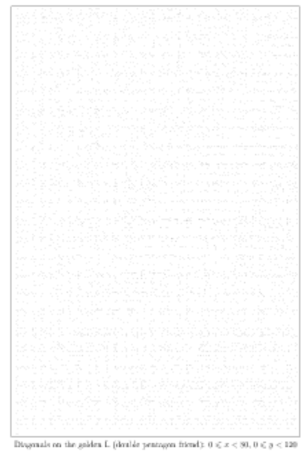


saddle connection holonomies



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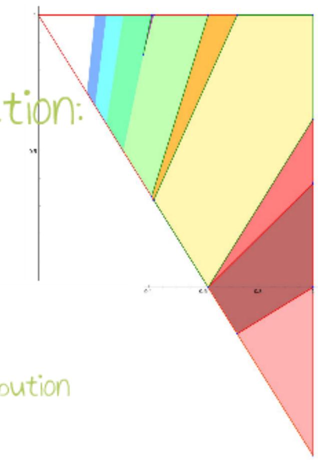
the cross section:



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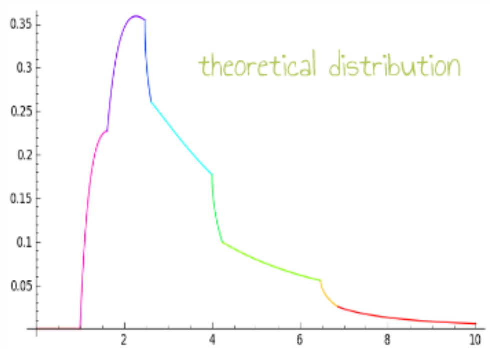
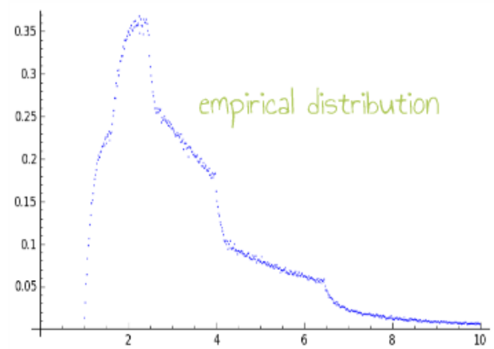
$$\Delta(2, 5, \infty)$$





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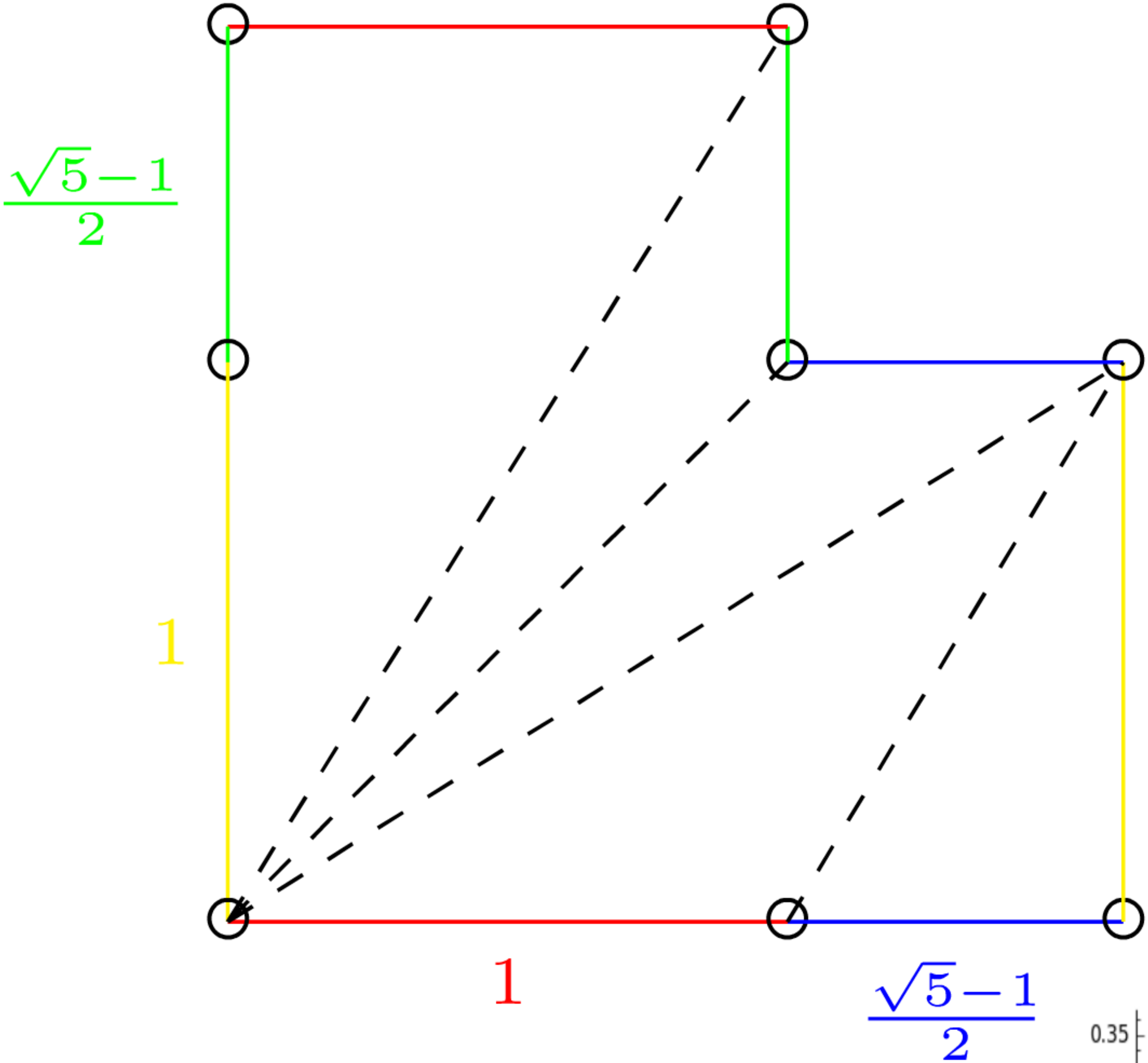
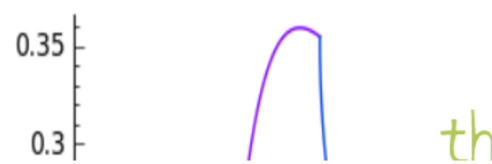
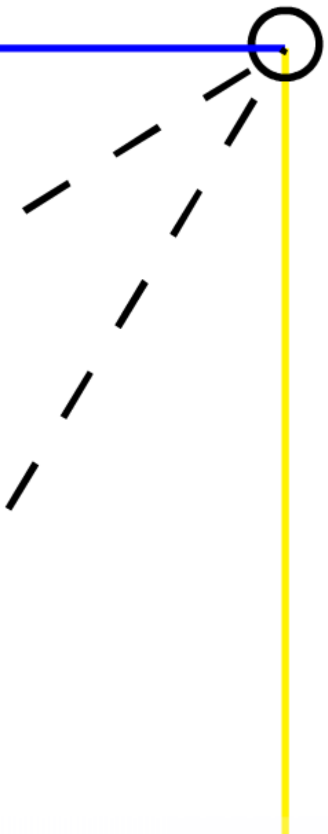


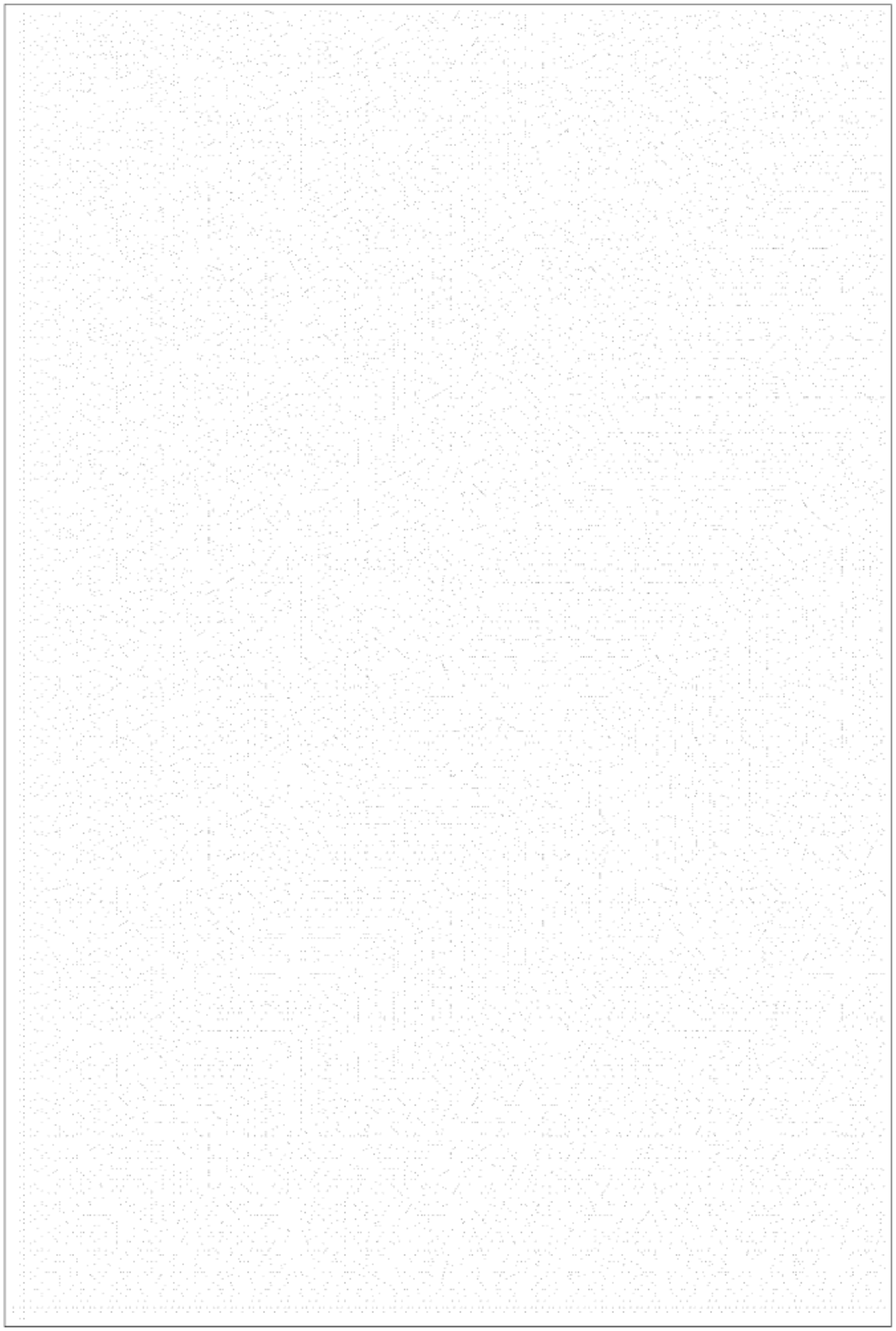
saddle co

Diagonals on the golden



saddle connection holonomies





Diagonals on the golden L (double pentagon friend): $0 \leqslant x < 80$, $0 \leqslant y < 120$

A.-Chai

gap

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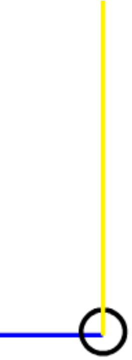

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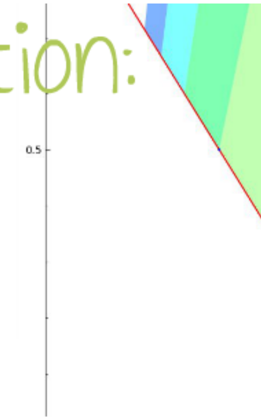

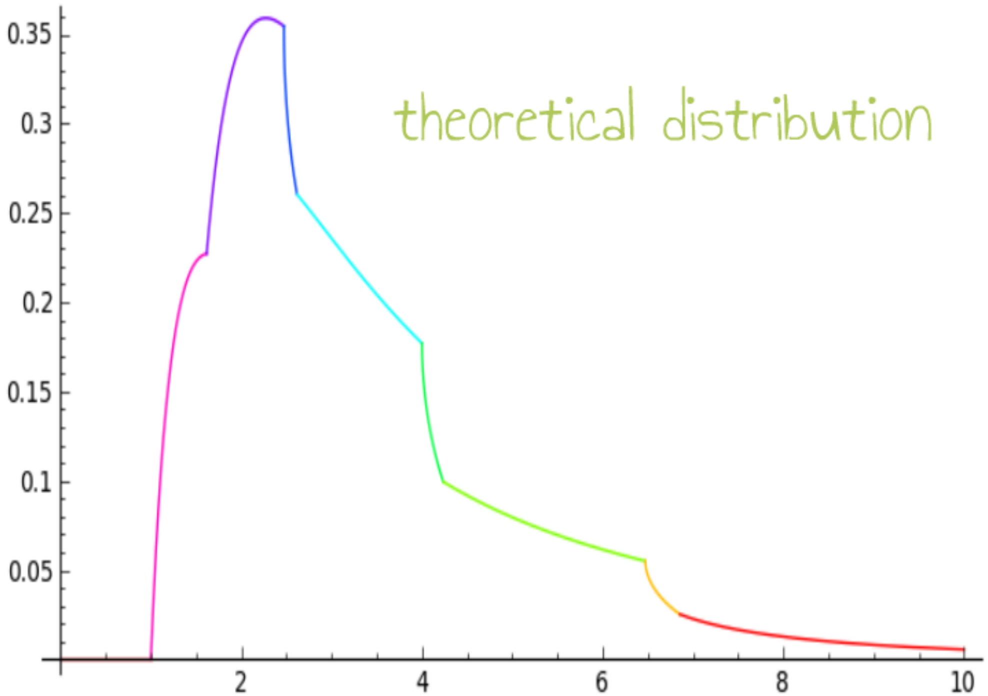

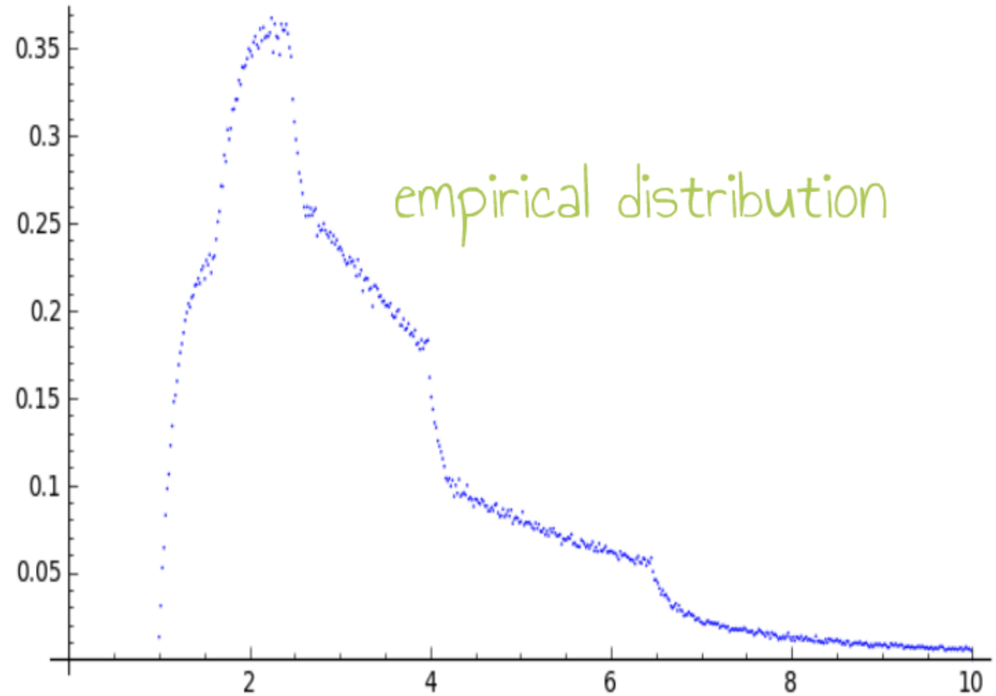

n L under the SL(2, $\mathbb{R}$) action. So

a hyperbolic surface.

Further Reading:

- Elkies-McMullen, Gaps in square root of n mod 1 and ergodic theory, http://goo.gl/bzlls
- Marklof-Strombergsson, , The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems, http://www.maths.bris.ac.uk/~majm/bib/lorentz-ONE.pdf
- Athreya-Cheung, A Poincaré section for horocycle flow on the space of lattices, http://arxiv.org/abs/1206.6597
- Athreya-Chaika, The distribution of gaps for saddle connection directions, http://arxiv.org/abs/1012.4298
- Athreya, Gap distributions and homogeneous dynamics, http://arxiv.org/abs/1210.0816