LORENTZIAN GEOMETRY

VIRGINIE CHARETTE AND TODD A. DRUMM

Abstract. These notes are an introduction to Lorentzian geometry, based on lectures given in Almora, India in December 2012. Included, is a discussion of group actions on Lorentzian manifolds and notes on the conformal compactification of Lorentzian space, the Einstein Universe. Most of the discussion centers on the case of three dimensions, two spatial dimensions and one time dimension, but some of the discussion can be easily extended to higher dimensions.

Contents

1. The basics 2
1.1. Affine space and its tangent space 2
1.2. The inner product and $(2+1)$-dimensional Minkowski space 3
1.3. Light, space and time: the causal structure of Minkowski space 3
1.4. Relationship to the hyperbolic plane 5
1.5. Isometries and similarities of Minkowski space 6
2. Proper actions and locally homogeneous Lorentzian 3-manifolds 8
2.1. Groups of isometries 9
2.2. Examples of spacetimes 10
2.3. Examples of Margulis spacetimes; crooked planes 10
2.4. Crooked halfspaces and disjointness 12
3. Deformations 15
3.1. Lorentzian transformations and affine deformations 15
3.2. The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as V 15
3.3. The Margulis invariant 16
4. Length changes in deformations 17
4.1. Deformed hyperbolic transformations 19
4.2. Deformed parabolic transformations 20
5. Einstein Universe 21

The first author gratefully acknowledge partial support from the Natural Sciences and Engineering Research Council of Canada.

5.1. Conformally flat Lorentzian structure on Ein_3 23
5.2. Photons and lightcones 23
5.3. Einstein torus 24
6. Crooked surfaces 24
6.1. Crooked surfaces as conformal compactifications of crooked planes 25
6.2. A basic example 25
6.3. Synthetic description of a crooked surface 26
References 27

1. The basics

In this first section, we introduce elementary notions on 3-dimensional Minkowski space, its relationship to the hyperbolic plane, and its isometries.

1.1. **Affine space and its tangent space.**

We define *n-dimensional affine space* $\mathbb{A}^n$ to be the set of all n-tuples of real numbers $(p_1, \dots, p_n)$. An affine space could be defined over any field, but we will restrict to the field of real numbers. Elements of affine space will be called *points*.

We will use plain font to denote points in affine space: p, q, r, etc. and bold font to denote vectors in a vector space: $\mathbf{t}, \mathbf{u}, \mathbf{v}$, etc.

The point $(0, \dots, 0)$ appears to be rather special, but as far as affine space is concerned, it is no different from any other point. Affine space should not be confused with a vector space!

However, affine spaces do transact with vector spaces. For $p = (p_1, \dots, p_n) \in \mathbb{A}^n$ and $\mathbf{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$, we define:

$$p + \mathbf{t} = (p_1 + t_1, \dots, p_n + t_n) \in \mathbb{A}^n$$

Thus the vector space $\mathbb{R}^n$, considered as a Lie group, acts transitively on $\mathbb{A}^n$ by *translations*; the translation by $\mathbf{t} \in \mathbb{R}^n$, denoted $\tau_{\mathbf{t}}$, is defined as follows:

$$\begin{aligned} \tau_{\mathbf{t}} : \mathbb{R}^n \times \mathbb{A}^n &\longrightarrow \mathbb{A}^n \\ (\mathbf{t}, p) &\longmapsto p + \mathbf{t} \end{aligned}$$

The stabilizer of a point is the trivial subgroup $\{\mathbf{0}\}$; as a homogeneous space, $\mathbb{A}^n$ identifies with $\mathbb{R}^n$. But the homogeneity of $\mathbb{A}^n$ means that every point is the same, including $(0, \dots, 0)$.

Affine space, as opposed to a vector space, lacks a notion of sum, but the action of $\mathbb{R}^n$ by translations yields a notion of *difference*:

$$p - q = \mathbf{t} \quad \text{if and only if } p = q + \mathbf{t}$$

where, of course, $p, q \in \mathbb{A}^n$ and $\mathbf{t} \in \mathbb{R}^n$.

Affine space is an n-dimensional manifold with trivial tangent bundle. The action by translation allows a precise description of the tangent space to a point in $\mathbb{A}^n$, which is evidently $\mathbb{R}^n$. Adding additional structure to $\mathbb{R}^n$, such as an inner product, allows us to endow the tangent bundle with a like structure – as we will do next. We then say that the affine space is *modeled* on the inner product space.

1.2. **The inner product and** (2+1)**-dimensional Minkowski space.** A *Lorentzian vector space of dimension 3* is a real 3-dimensional vector space V endowed with an inner product of signature $(2, 1)$. The Lorentzian inner product will be denoted:

$$\mathsf{V} \times \mathsf{V} \longrightarrow \mathbb{R}$$
$$(\mathbf{v}, \mathbf{u}) \longmapsto \mathbf{v} \cdot \mathbf{u}$$

We also fix an orientation on V. The orientation determines a nondegenerate alternating trilinear form

$$\mathsf{V} \times \mathsf{V} \times \mathsf{V} \xrightarrow{\mathsf{Det}} \mathbb{R}$$

which takes a positively oriented orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with inner products

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \; \mathbf{e}_3 \cdot \mathbf{e}_3 = -1$$

to 1. The oriented Lorentzian 3-dimensional vector space determines an alternating bilinear mapping $\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$, called the *Lorentzian cross-product,* defined by

$$\mathsf{Det}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w}. \tag{1}$$

Compare, for example, with [6].

We call the affine space modeled on V *Minkowski space* and denote it E. This is an oriented manifold, since V is oriented. Alternatively, E can be defined as a 3-dimensional, oriented, geodesically complete, 1-connected, flat Lorentzian manifold.

1.3. **Light, space and time: the causal structure of Minkowski space.** The inner product induces a *causal structure* on V: a vector $\mathbf{v} \neq \mathbf{0}$ is called

- *timelike* if $\mathbf{v} \cdot \mathbf{v} < 0$,
- *null* (or *lightlike*) if $\mathbf{v} \cdot \mathbf{v} = 0$, or
- *spacelike* if $\mathbf{v} \cdot \mathbf{v} > 0$.

We will call the corresponding subsets of V respectively V_-,V_0 and V_+. The set V_0 of null vectors is called the *light cone*.

A spacelike vector $\mathbf{v}$ will be called *unit-spacelike* if $\mathbf{v}\cdot\mathbf{v}=1$.

Say that vectors $\mathbf{u},\mathbf{v}\in\mathsf{V}$ are *Lorentzian-perpendicular* if $\mathbf{u}\cdot\mathbf{v}=0$. Denote the linear subspace of vectors Lorentzian-perpendicular to $\mathbf{v}$ by $\mathbf{v}^\perp$.

The set of timelike vectors admits two connected components. Each component defines a *time-orientation* on V, and the time-orientation structure on V is carried on to E. We select one of the components and call it Future. Call a non-spacelike vector $\mathbf{v}\neq\mathbf{0}$ and its corresponding ray *future-pointing* if $\mathbf{v}$ lies in the closure of Future.

The time-orientation can be defined by a choice of a timelike vector $\mathbf{t}$ as follows. Consider the linear functional $\mathsf{V}\longrightarrow\mathbb{R}$ defined by:

$$\mathbf{v}\longmapsto\mathbf{v}\cdot\mathbf{t}.$$

Then the future and past components can be distinguished by the sign of this functional on the set of timelike vectors.

1.3.1. *Null frames.* The restriction of the inner product to the orthogonal complement $\mathbf{s}^\perp$ of a spacelike vector $\mathbf{s}$ is indefinite, having signature $(1,1)$. The intersection of the light cone with $\mathbf{s}^\perp$ consists of two null lines intersecting transversely at the origin. Choose a linearly independent pair of future-pointing null vectors $\mathbf{s}^\pm\in\mathbf{v}^\perp$ such that: $\{\mathbf{s},\mathbf{s}^-,\mathbf{s}^+\}$ is a positively oriented basis for V (with respect to a fixed orientation on V). The null vectors $\mathbf{s}^-$ and $\mathbf{s}^+$ are defined only up to positive scaling. Here is a useful identity (compare [6] , for example), for a unit spacelike vector $\mathbf{s}$:

$$\begin{aligned}\mathbf{s}\times\mathbf{s}^-&=\mathbf{s}^-\\ \mathbf{s}\times\mathbf{s}^+&=-\mathbf{s}^+,\end{aligned}\tag{2}$$

We call the positively oriented basis $\{\mathbf{s},\mathbf{s}^-,\mathbf{s}^+\}$ a *null frame* associated to $\mathbf{s}$. Margulis [13, 14] takes the null vectors $\mathbf{s}^-,\mathbf{s}^+$ to have unit Euclidean length; in this paper we will not specify the vectors, except that they are future-pointing. In this basis the corresponding Gram matrix (the symmetric matrix of inner products) has the form:

$$\begin{bmatrix}a^2 & 0 & 0\\ 0 & 0 & -k^2\\ 0 & -k^2 & 0\end{bmatrix}$$

The off-diagonal entry, $\frac{1}{2}\mathbf{s}^-\cdot\mathbf{s}^+$, is negative since both $\mathbf{s}^-$ and $\mathbf{s}^+$ are future-pointing. Often we normalize $\mathbf{s}$ to be unit-spacelike, so that $a=1$, and choose $\mathbf{s}^-$ and $\mathbf{s}^+$ so that $k=1$.

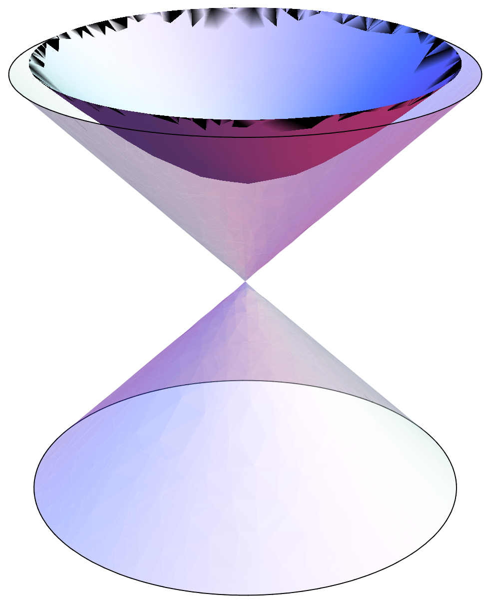

FIGURE 1. The hyperboloid model of the hyperbolic plane.

The basis defines linear coordinates (a, b, c) on V:

$$\mathbf{v} := a\mathbf{s} + b\mathbf{s}^- + c\mathbf{s}^+$$

so the corresponding Lorentz metric on E is:

$$da^2 - db\ dc. \tag{3}$$

1.4. **Relationship to the hyperbolic plane.** Let $\mathsf{H}^2 \subset \mathsf{V}_-$ denote the set of unit future-pointing timelike vectors, that is

$$\mathsf{H}^2 = \{\mathbf{v} \in \mathsf{V}_- \mid \mathbf{v} \cdot \mathbf{v} = -1\}$$

The restriction of the Lorentzian metric to H^2, denoted d_{H^2}, is positive definite. For $\mathbf{u}, \mathbf{v} \in \mathsf{H}^2$:

$$\cosh\left(d_{\mathsf{H}^2}(\mathbf{u}, \mathbf{v})\right) = \mathbf{u} \cdot \mathbf{v}$$

The resulting metric is a Riemannian metric with constant curvature -1, and we identify H^2 with the hyperbolic plane.

Geodesics in the hyperbolic plane correspond to indefinite planes in V, which are precisely the planes that intersect H^2. Equivalently, these

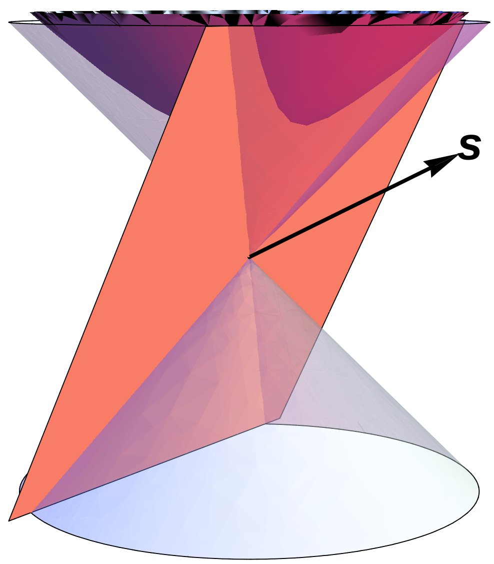

FIGURE 2. The identification between a spacelike vector $\mathbf{s}$ and a line in H^2.

are Lorentzian-perpendicular planes to spacelike vectors. Thus each spacelike vector $\mathbf{s}$ is identified with a geodesic in H^2:

$$\ell_{\mathbf{s}} = \mathbf{s}^{\perp} \cap \mathsf{H}^2$$

The spacelike vector $\mathbf{s}$ also identifies with one of the open halfplanes bounded by the $\ell_{\mathbf{s}}$. Namely, the halfplane $\mathfrak{h}_{\mathbf{s}}$ is the set of vectors $\mathbf{v} \in \mathsf{H}^2$ such that $\mathbf{v} \cdot \mathbf{s} > 0$.

A more detailed and complete correspondence between Lorentzian geometry and hyperbolic spaces can be found in [16].

1.5. **Isometries and similarities of Minkowski space.** Identify E with V by choosing a distinguished point $o \in \mathsf{V}$, which we call an *origin.* For any point $p \in \mathsf{E}$ there is a unique vector $\mathbf{v} \in \mathsf{V}$ such that $p = o + \mathbf{v}$. Thus the choice of origin defines a bijection

$$\begin{aligned} \mathsf{V} &\xrightarrow{A_o} \mathsf{E} \\ \mathbf{v} &\longmapsto o + \mathbf{v}. \end{aligned}$$

For any $o_1, o_2 \in \mathsf{E}$,

$$A_{o_1}(\mathbf{v}) = A_{o_2}\big(\mathbf{v} + (o_1 - o_2)\big)$$

where $o_1 - o_2 \in \mathsf{V}$ is the unique vector translating o_2 to o_1. A transformation $\mathsf{E} \xrightarrow{\gamma} \mathsf{E}$ is called *affine* if and only if it normalizes the group V of translations. Equivalently, γ is an affine transformation if there is a linear transformation $\mathsf{L}(\gamma)$, called its *linear part*, and $\mathbf{u} \in \mathsf{V}$, called its *translational part*, such that:

$$\gamma(p) = o + \mathsf{L}(\gamma)(p - o) + \mathbf{u}$$

(The linear part does not depend on the choice of origin but the translational part does.)

Denote the group of orientation-preserving linear isometries of V by $\mathsf{SO}(2,1)$. The group of orientation-preserving linear *conformal automorphisms* of V is the product $\mathsf{SO}(2,1) \times \mathbb{R}^+$, where $\mathbb{R}^+$ is the group of *positive homotheties* $\mathbf{v} \mapsto \lambda\mathbf{v}$, where $\lambda > 0$. Denote the group of orientation-preserving linear automorphisms of V by $\mathsf{GL}^+(3,\mathbb{R})$.

A linear automorphism of V preserves orientation if and only if it has positive determinant. An affine automorphism of E preserves orientation if and only if its linear part lies in the subgroup $\mathsf{GL}^+(3,\mathbb{R})$ of $\mathsf{GL}(3,\mathbb{R})$ consisting of matrices of positive determinant. The group of orientation-preserving affine automorphisms of E thus decomposes as a semidirect product:

$$\mathsf{Aff}^+(\mathsf{E}) \;=\; \mathsf{V} \rtimes \mathsf{GL}^+(3,\mathbb{R}).$$

Denote the group of orthogonal automorphisms (linear isometries) of V by $\mathsf{O}(2,1)$. Let

$$\mathsf{SO}(2,1) := \mathsf{O}(2,1) \cap \mathsf{GL}^+(3,\mathbb{R})$$

denote, as usual, the subgroup of orientation-preserving linear isometries. Orientation-preserving isometries of E constitute the subgroup:

$$\mathsf{Isom}^+(\mathsf{E}) \;:=\; \mathsf{V} \rtimes \mathsf{SO}(2,1)$$

and the subgroup of orientation-preserving conformal automorphisms is:

$$\mathsf{Conf}^+(\mathsf{E}) \;=\; \mathsf{V} \rtimes \big(\mathsf{SO}(2,1) \times \mathbb{R}^+\big)$$

1.5.1. *Components of the isometry group.* The group $\mathsf{O}(2,1)$ has four connected components. The identity component $\mathsf{SO}^0(2,1)$ consists of orientation-preserving linear isometries preserving time-orientation. It is isomorphic to the group $\mathsf{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of the hyperbolic plane. (Recall §1.4.) The group $\mathsf{O}(2,1)$ is a semidirect product

$$\mathsf{O}(2,1) \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathsf{SO}^0(2,1)$$

where $\pi_0\big(\mathsf{O}(2,1)\big) \cong \mathbb{Z}/2\times\mathbb{Z}/2$ is generated by reflection in a point (the antipodal map $\mathbb{A}$, which reverses orientation) and reflection in a spacelike line (which preserves orientation, but reverses time-orientation).

1.5.2. *Transvections, boosts, homotheties and reflections.* In the null frame coordinates of §1.3.1 , the one-parameter group of linear isometries

$$\xi_t \;:=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \tag{4}$$

(for $t \in \mathbb{R}$) fixes $\mathbf{s}$ and acts on the (indefinite) plane $\mathbf{s}^\perp$. These transformations, called *boosts,* constitute the identity component $\mathsf{SO}^0(1,1)$ of the isometry group of $\mathbf{v}^\perp$.

The one-parameter group $\mathbb{R}^+$ of *positive homotheties*

$$\eta_s \;:=\; \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^s \end{bmatrix}$$

(where $s \in \mathbb{R}$) acts conformally on Minkowski space, preserving orientation. The involution

$$\rho \;:=\; \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{5}$$

preserves orientation, reverses time-orientation, reverses $\mathbf{s}$, and interchanges the two null lines $\mathbb{R}\mathbf{s}^-$ and $\mathbb{R}\mathbf{s}^+$.

2. Proper actions and locally homogeneous Lorentzian 3-manifolds

In this section, we use what we know about Lorentzian isometries to construct manifolds which are modeled on E. More specifically, we will consider manifolds of the form E/G, where $G < \mathsf{Isom}^+(\mathsf{E})$ acts "nicely". The fact that G consists of isometries means that the quotient space inherits a causal structure from E. Some of the features of homogeneity survive as well: this is what we mean by "locally homogeneous".

We will pay particular attention to the case where the linear part $\mathsf{L}(G)$ is a free group: these manifolds will be called *Margulis spacetimes.*

2.1. **Groups of isometries.** We start by making precise what we mean by a "nice action".

Definition 2.1. Let X be a locally compact space and G a group acting on X. We say that G acts *properly discontinuously* on X if for every compact $K \subset X$, the set:

$$\{\gamma \in G \mid \gamma K \cap K \neq \emptyset\}$$

is finite.

The following important result is a good exercise. (Recall that a group acts *freely* if it fixes no points.)

Theorem 2.2. *Let X be a Hausdorff manifold and let G be a group that acts freely and properly discontinuously on X. Then X/G is a Hausdorff manifold.*

Kulkarni [12] studied proper actions in the more general context of *pseudo-Riemannian manifolds.*

Remark 2.3. A group that acts properly discontinuously on E is discrete. But the converse, which holds for Riemannian manifolds, is false for group actions on E.

Example 2.4. *Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_2 \in \mathsf{V}$ be three linearly independent vectors. Then $G = \langle \tau_{\mathbf{t}_1}, \tau_{\mathbf{t}_2}, \tau_{\mathbf{t}_3} \rangle$ acts properly discontinuously on E. In fact, one easily sees that E/G is obtained by taking a parallelipiped generated by the three translations and then gluing opposite sides.*

Example 2.4 illustrates the next criterion for a proper action. Denote the interior of a set A by $int\,(A)$.

Definition 2.5. Let X be a topological space and G a group acting on X. Let $F \subset X$ be a closed subset with non-empty interior. We say that F is a *fundamental domain* for the G-action on X if:

- $X = \bigcup_{\gamma \in G} \gamma F$;
- for all $\gamma \neq \eta \in G$, $int\,(\gamma F) \cap int\,(\eta F) = \emptyset$.

Theorem 2.6. *Let X be a topological space and G a group acting on X. Suppose there exists a fundamental domain F for the G-action on X. Then G acts properly discontinuously on X and:*

$$X/G = F/G$$

2.2. **Examples of spacetimes.** Here is a fancier version of Example 2.4. Let $\mathbf{t} \in \mathsf{V}$ be a timelike vector, and let σ be a *screw motion* of order 4 about $\mathbb{R}\mathbf{t}$. Specifically, the linear part of σ is elliptic and its translational part is $\mathbf{t}$. Let $\mathbf{v} \neq \mathbf{0}$ be a vector in $\mathbf{t}^{\perp}$ (it is necessarily spacelike). Consider the following group:

$$G = \langle \tau_{\mathbf{v}}, \tau_{\sigma\mathbf{v}}, \sigma \rangle$$

This group admits a fundamental domain as in Example 2.4: a parallelipiped generated by $\mathbf{v}, \sigma\mathbf{v}, \mathbf{t}$. The group thus acts properly discontinuously on E and furthermore acts freely. Thus E/G is a Lorentzian manifold.

The group G is a very basic example of a *virtually solvable group*, since it contains a finite-index subgroup of translations. Fried and Goldman [9] proved the following important classification result for groups of affine transformations acting properly discontinuously on $\mathbb{R}^3$. If a group of affine transformations G acts properly discontinuously on $\mathbb{R}^3$, either is is virtually solvable or its linear part is conjugate to a subgroup of $\mathsf{O}(2,1)$.

Our example has linear part in $\mathsf{O}(2,1)$ *and* is virtually solvable. Are there examples that are not virtually solvable? This was a question posed by Milnor in the 1970s [15]. Margulis discovered such examples [13, 14], where the linear part was a *Schottky group*, or a free, non-abelian discrete subgroup of $\mathsf{SO}^0(2,1)$.

Definition 2.7. A *Margulis spacetime* is a Hausdorff manifold E/G where G is free and non-abelian.

Given a group G with Schottky linear part, it is difficult to determine whether it acts properly discontinuously on $\mathbb{R}^3$. We would like a "ping-pong" lemma as for Schottky groups acting on the hyperbolic plane. However, the absence of a Riemannian metric makes this challenging. The remedy was the introduction of fundamental domains for these actions [5], bounded by piecewise linear surfaces called *crooked planes.*

2.3. **Examples of Margulis spacetimes; crooked planes.** Now we introduce crooked planes and crooked fundamental domains. The reader interested in more details should consult [2].

Definition 2.8. Let $\mathbf{x} \in \mathbb{R}^{2,1}$ be a future-pointing null vector. Then the closure of the following halfplane:

$$\mathrm{Wing}(\mathbf{x}) = \{\mathbf{u} \in \mathbf{x}^{\perp} \ \mid \ \mathbf{x} = \mathbf{u}^{+}\}$$

is called a *positive linear wing.*

In the affine setting, given $p \in \mathsf{E}$, $p + \mathrm{Wing}(\mathbf{x})$ is called a *positive wing.*

Observe that if $\mathbf{u} \in \mathbb{R}^{2,1}$ is spacelike:

$$\mathbf{u} \in \mathrm{Wing}(\mathbf{u}^+)$$
$$-\mathbf{u} \in \mathrm{Wing}(\mathbf{u}^-)$$
$$\mathrm{Wing}(\mathbf{u}^+) \cap \mathrm{Wing}(\mathbf{u}^-) = \mathbf{0}$$

The set of positive linear wings is $\mathsf{SO}(2,1)$-invariant.

Definition 2.9. Let $\mathbf{u} \in \mathbb{R}^{2,1}$ be spacelike. Then the following set:

$$\mathrm{Stem}(\mathbf{u}) = \{\mathbf{x} \in \mathbf{u}^\perp \ \mid \ \mathbf{x} \cdot \mathbf{x} \leq 0\}$$

is called a *linear stem*. For $p \in \mathsf{E}$, $p + \mathrm{Stem}(\mathbf{u})$ is called a *stem*.

Observe that $\mathrm{Stem}(\mathbf{u})$ is bounded by the lines $\mathbb{R}\mathbf{u}^+$ and $\mathbb{R}\mathbf{u}^-$ and thus respectively intersects the closures of $\mathrm{Wing}(\mathbf{u}^+)$ and $\mathrm{Wing}(\mathbf{u}^-)$ in these lines.

Definition 2.10. Let $p \in \mathsf{E}$ and $\mathbf{u} \in \mathbb{R}^{2,1}$ be spacelike. The *positively extended crooked plane* with *vertex* p and *director* $\mathbf{u}$ is the union of:

- the stem $p + \mathrm{Stem}(\mathbf{u})$;
- the positive wing $p + \mathrm{Wing}(\mathbf{u}^+)$;
- the positive wing $p + \mathrm{Wing}(\mathbf{u}^-)$.

It is denoted $\mathcal{C}(p, \mathbf{u})$.

A crooked plane is depicted in Figure 3.

Remark 2.11. A negatively extended crooked plane is obtained by replacing positive wings with negative wings. One obtains a *negative wing* by choosing the other connected component of $\mathbf{x}^\perp \setminus \mathbb{R}\mathbf{x}$. Without going into details, we will simply state that one can avoid resorting to negatively extended crooked planes by changing the orientation on V and E. Thus we will simply write "wing" to mean positive wing and "crooked plane" to mean positively extended crooked plane. The same convention is used in [2].

Theorem 2.12 (Drumm [5]). *Let $G = \langle \gamma_1, \ldots, \gamma_n \rangle < \mathsf{Isom}^+(\mathsf{E})$ with linear part in $\mathsf{SO}^0(2,1)$. Suppose there exists a simply connected region Δ bounded by $2n$ pairwise disjoint crooked planes $\mathcal{C}_1^-, \mathcal{C}_i^+, \ldots, \mathcal{C}_n^-, \mathcal{C}_n^+$ such that:*

$$\gamma_i \mathcal{C}_i^- = \mathcal{C}_i^+, \ i = 1, \ldots, n$$

Then Δ is a fundamental domain for G, which acts freely and properly discontinuously on E.

Moreover, the quotient can be seen to be a solid handlebody. Figure 4 shows four pairwise disjoint crooked planes; these bound a fundamental domain for a group whose linear part is the holonomy of a one-holed

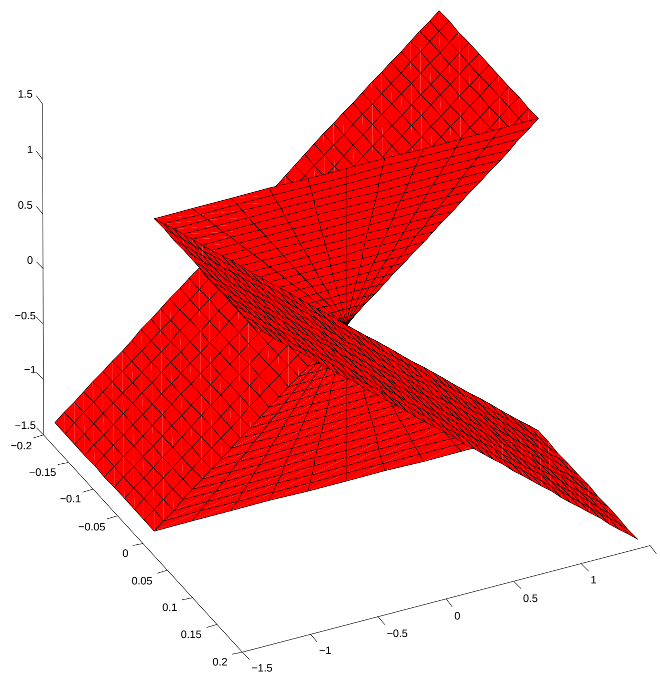

FIGURE 3. A crooked plane.

torus or a three-holed sphere. A fundamental domain bounded by (disjoint) crooked planes is called a *crooked fundamental domain*.

2.4. **Crooked halfspaces and disjointness.** We will discuss here criteria for disjointness of crooked planes, as it plays a vital role in constructing crooked fundamental domains.

The complement of a crooked plane in $\mathcal{C}(p, \mathbf{u}) \in \mathsf{E}$ consists of two crooked halfspaces, respectively corresponding to $\mathbf{u}$ and $-\mathbf{u}$. A crooked halfspace will be determined by the appropriate *stem quadrant*, which we introduce next. Our notation for the stem quadrant is slightly different from that adopted in [2], where it is defined in terms of the crooked halfspace.

Definition 2.13. Let $\mathbf{u} \in \mathsf{V}$ be spacelike and $p \in \mathsf{E}$. The associated *stem quadrant* is:

$$\mathsf{Quad}(p, \mathbf{u}) = p + \{a\mathbf{u}^- - b\mathbf{u}^+ \ \mid \ a, b \geq 0\}$$

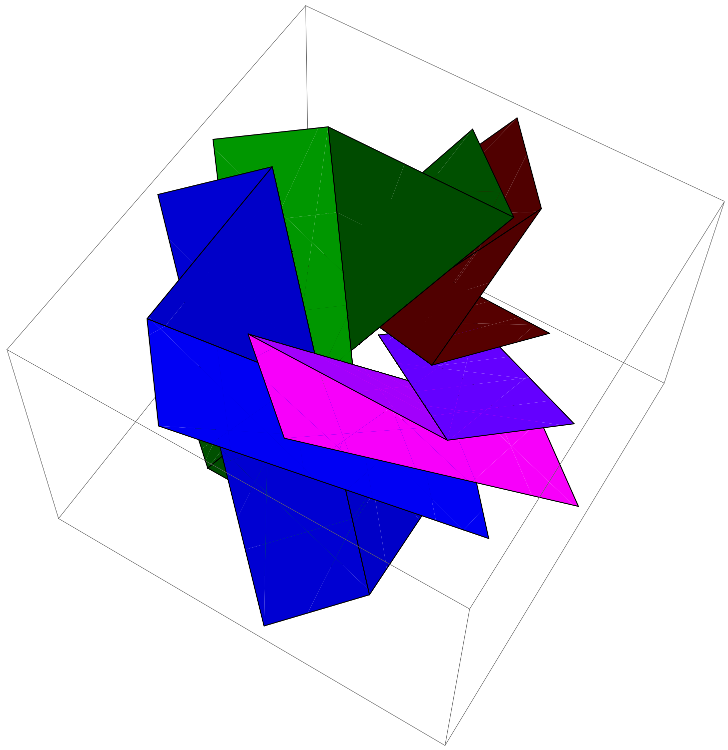

FIGURE 4. Four pairwise disjoint crooked planes. Identifying pairs of "adjacent" crooked planes yields a fundamental domain in the case where the linear part corresponds to a three-holed sphere. Identifying pairs of "opposite" crooked planes yields one in the case of a one-holed torus.

The stem quadrant $\mathsf{Quad}(p, \mathbf{u})$ is bounded by light rays parallel to $\mathbf{u}^-$ and $-\mathbf{u}^+$.

Definition 2.14. Let $p \in \mathsf{E}$ and $\mathbf{u} \in \mathsf{V}$ be spacelike. The *crooked halfspace* $\mathcal{H}(p, \mathbf{u})$ is the component of the complement of $\mathcal{C}(p, \mathbf{u})$ containing *int* $(\mathsf{Quad}(p, \mathbf{u}))$.

By definition, crooked halfspaces are open. While the crooked planes $\mathcal{C}(p, \mathbf{u})$, $\mathcal{C}(p, -\mathbf{u})$ are equal, the crooked halfspaces $\mathcal{H}(p, \mathbf{u})$, $\mathcal{H}(p, -\mathbf{u})$ are disjoint, sharing $\mathcal{C}(p, \mathbf{u})$ as a common boundary.

Definition 2.15. Let $o \in \mathsf{E}$ and $\mathbf{u}_1, \mathbf{u}_2 \in \mathsf{V}$ be spacelike. The vectors are said to be *consistently oriented* if the closures of the crooked halfspaces $\mathcal{H}(o, \mathbf{u}_1)$ and $\mathcal{H}(o, \mathbf{u}_2)$ intersect only in o.

Remark 2.16. This is independent of the choice of o. Also, this definition is equivalent to that of Drumm-Goldman [6]. Let us point out

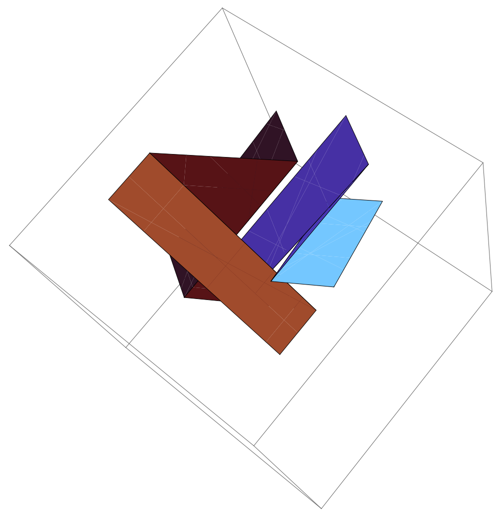

FIGURE 5. A slab bounded by a pair of disjoint crooked planes.

that the definition, as originally stated, requires among other things that $\mathbf{u}_1^\perp \cap \mathbf{u}_2^\perp$ be spacelike.

Definition 2.17. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathsf{V}$ be a pair of consistently oriented ultraparallel spacelike vectors. The set of *allowable translations* for $\mathbf{u}_1, \mathbf{u}_2$ is:

$$\mathsf{A}(\mathbf{u}_1, \mathbf{u}_2) = int\left(\mathsf{Quad}(p, \mathbf{u}_1) - \mathsf{Quad}(p, \mathbf{u}_2)\right) \subset \mathsf{V}$$

where $p \in \mathsf{E}$ can be arbitrarily chosen.

Theorem 2.18 (Drumm-Goldman [6])**.** *Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathsf{V}$ be a pair of consistently oriented ultraparallel spacelike vectors. Then the closures of the crooked halfspaces $\mathcal{H}(p_1, \mathbf{u}_1)$ and $\mathcal{H}(p_2, \mathbf{u}_2)$ are disjoint if and only if $p_1 - p_2 \in \mathsf{A}(\mathbf{u}_1, \mathbf{u}_2)$.*

Choosing $p_1 - p_2 \in \mathsf{A}(\mathbf{u}_1, \mathbf{u}_2)$ means that the complements of the crooked halfspaces $\mathcal{H}(p_1, \mathbf{u}_1)$ and $\mathcal{H}(p_2, \mathbf{u}_2)$ intersect nicely, in a set which we call a *slab*. A slab is a crooked fundamental domain for a cyclic group. Indeed, let $\gamma \in \mathsf{Isom}^+(\mathsf{E})$ be any isometry whose linear part maps $\mathbf{u}_1$ to $-\mathbf{u}_2$ and such that $\gamma(p_1) = p_2$. There is one degree of freedom in choosing the linear part, but the translation part is entirely determined by the condition on p_1 and p_2. Then the slab:

$$\mathsf{E} \setminus \left(\mathcal{H}(p_1, \mathbf{u}_1) \cup \mathcal{H}(p_2, \mathbf{u}_2)\right)$$

is a crooked fundamental domain for $\langle\gamma\rangle$. See Figure 5.

Crooked fundamental domains for higher rank groups are then obtained by intersecting slabs whose boundary components are pairwise disjoint.

3. Deformations

3.1. **Lorentzian transformations and affine deformations.** Let $\mathsf{Isom}^+(\mathsf{E})$ denote the group of all orientation-preserving affine transformations that preserve the Lorentzian inner product on the space of directions; $\mathsf{Isom}^+(\mathsf{E})$ is isomorphic to $\mathsf{SO}(2,1) \ltimes \mathbb{R}^{2,1}$. We shall restrict our attention to those transformations whose linear parts are in $\mathsf{SO}^0(2,1)$, thus preserving orientation and time-orientation.

Denote projection onto the *linear part* of an affine transformation by:

$$\mathsf{Isom}^+(\mathsf{E}) \xrightarrow{\mathsf{L}} \mathsf{SO}(2,1).$$

Recall that the upper nappe sheet of the hyperboloid of unit-timelike vectors in V is a model for the hyperbolic plane H^2. The resulting isomorphism between $\mathsf{SO}^0(2,1)$ and $\mathsf{Isom}^0(\mathcal{H})$ gives rise to the following terminology. (Consult [11], for example, for an explicit isomorphism.)

Definition 3.1. Let $g \in \mathsf{SO}^0(2,1)$ be a nonidentity element;

- g is *hyperbolic* if it has three, distinct real eigenvalues;
- g is *parabolic* if its only eigenvalue is 1;
- g is *elliptic* otherwise.

We also call $\gamma \in \mathsf{Isom}^+(\mathsf{E})$ *hyperbolic* (respectively *parabolic*, *elliptic*) if its linear part $\mathsf{L}(\gamma)$ is hyperbolic (respectively parabolic, elliptic).

Let $\Gamma_0 \subset \mathsf{SO}(2,1)$ be a subgroup. An *affine deformation* of Γ_0 is a representation

$$\rho : \Gamma_0 \longrightarrow \mathsf{Isom}^+(\mathsf{E}).$$

For $g \in \Gamma_0$, write

$$\rho(g)(x) = g(x) + u(g)$$

where $u(g) \in \mathsf{V}$. Then u is a cocycle of Γ_0 with coefficients in the Γ_0-module V corresponding to the linear action of Γ_0. In this way affine deformations of Γ_0 correspond to cocycles in $\mathrm{Z}^1(\Gamma_0, \mathsf{V})$ and translational conjugacy classes of affine deformations correspond to cohomology classes in $\mathrm{H}^1(\Gamma_0, \mathsf{V})$.

By extension, if $\Gamma_0 = \rho_0(\pi_1(\Sigma))$, we will call ρ an affine deformation of the holonomy representation ρ_0.

3.2. **The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as V.** The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is the tangent space to $\mathsf{PSL}(2,\mathbb{R})$ at the identity and consists of the set of traceless 2×2 matrices. The three-dimensional vector space has a natural inner product, the Killing form, defined to be

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{2} \mathrm{Tr}(\mathbf{v} \cdot \mathbf{w}). \tag{6}$$

A basis for $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; E_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{7}$$

Evidently, $\langle E_1, E_1\rangle = \langle E_2, E_2\rangle = 1$, $\langle E_3, E_3\rangle = -1$ and $\langle E_i, E_j\rangle = 0$ for $i \neq j$. That is, $\mathfrak{sl}(2,\mathbb{R})$ is isomorphic to V as a vector space

$$\left\{ \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\} \leftrightarrow \{ xE_1 + yE_2 + zE_3 = \mathbf{v} \}.$$

The adjoint action of $\mathsf{PSL}(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R})$:

$$g(\mathbf{v}) = g\mathbf{v}g^{-1}$$

corresponds to the linear action of $\mathsf{SO}^0(2,1)$ on V.

Using these identifications, set:

$$G \cong \mathsf{PSL}(2,\mathbb{R}) \cong \mathsf{SO}^0(2,1)$$
$$\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathsf{V}.$$

3.3. **The Margulis invariant.** The Margulis invariant is a measure of an affine transformation's signed Lorentzian displacement in E. Originally defined by Margulis for hyperbolic transformations [13, 14], it admits an extension to parabolic transformations [3].

Let $g \in G$ be a non-elliptic element. Lift g to a representative in $\mathsf{SL}(2,\mathbb{R})$; then the following element of $\mathfrak{g}$ is a g-invariant vector which is independent of choice of lift:

$$F_g = \sigma(g)\left(g - \frac{\mathrm{Tr}(g)}{2} I \right)$$

where $\sigma(g)$ is the sign of the trace of the lift.

Now let $\Gamma_0 \subset G$ such that every element other than the identity is non-elliptic. Let ρ be an affine deformation of Γ_0, with corresponding $u \in \mathrm{Z}^1(\Gamma_0, \mathsf{V})$. Given the above identification $\mathfrak{g} \cong \mathsf{V}$, we may also write $u \in Z^1(\Gamma_0, \mathfrak{g})$. We define the *non-normalized Margulis invariant* of $\rho(g) \in \rho(\Gamma_0)$ to be:

$$\tilde{\alpha}_\rho(g) = \langle u(g), F_g \rangle. \tag{8}$$

(In [3], the non-normalized invariant is a functional on a fixed line, rather than a value.)

If $\rho(g)$ happens to be hyperbolic, then F_g is spacelike and we may replace it by the unit-spacelike vector:

$$X_g^0 = \frac{2\sigma(g)}{\sqrt{\mathrm{Tr}(g)^2 - 4}} \left(g - \frac{\mathrm{Tr}(g)}{2} I \right)$$

obtaining the *normalized Margulis invariant*:

$$\alpha_\rho(g) = \langle u(g), X_g^0 \rangle. \tag{9}$$

In Minkowski space, $\alpha_\rho(g)$ is the *signed Lorentzian length* of a closed geodesic in $\mathsf{E}/\langle \rho(g) \rangle$ [13, 14].

As a function of word length in the group Γ_0, normalized α_ρ behaves better than non-normalized $\tilde{\alpha}_\rho$. Nonetheless, the sign of $\tilde{\alpha}_\rho(g)$ is well defined and is equal to that of $\alpha_\rho(g)$. So we may extend the definition of α_ρ to parabolic g, for instance by declaring that $F_g = X_g^0$.

Theorem 3.2. [4] *Let Γ_0 be a Fuchsian group whose corresponding hyperbolic surface Σ is homeomorphic to a three-holed sphere. Denote the generators of Γ_0 corresponding to the three components of $\partial\Sigma$ by $\partial_1, \partial_2, \partial_3$. Let ρ be an affine deformation of Γ_0.*

If $\alpha_\rho(\partial_i)$ is positive (respectively, negative, nonnegative, nonpositive) for each i then for all $\gamma \in \Gamma_0 \setminus \{1\}$, $\alpha_\rho(\gamma)$ is positive (respectively, negative, nonnegative, nonpositive).

The proof of Theorem 3.2 relies upon showing that the affine deformation ρ of the Fuchsian group Γ_0 acts properly on E. By a fundamental lemma due to Margulis [13, 14] and extended in [3], if ρ is proper, then α_ρ applied to every element has the same sign. Moreover,

- if $\alpha_\rho(\partial_1) = 0$ and $\alpha_\rho(\partial_2), \alpha_\rho(\partial_3) > 0$ then specifically $\alpha_\rho(\gamma) = 0$ only if $\gamma \in \langle \partial_1 \rangle$, and
- if $\alpha_\rho(\partial_1) = \alpha_\rho(\partial_2) = 0$ and $\alpha_\rho(\partial_3) > 0$ then specifically $\alpha_\rho(\gamma) = 0$ only if $\gamma \in \langle \delta_1 \rangle \cup \langle \delta_2 \rangle$.

4. Length changes in deformations

An *affine* deformation of a holonomy representation corresponds to an *infinitesimal deformation* of the holonomy representation, or a tangent vector to the holonomy representation. In this section, we will further explore this correspondence, relating the affine Margulis invariant to the derivative of length along a path of holonomy representations. We will then prove Theorem 4.1 by applying Theorem 3.2, which characterizes proper deformations in terms of the Margulis invariant, to the study of length changes along a path of holonomy representations.

Let $\rho_0 : \pi_1(\Sigma) \to \Gamma_0 \subset G$ be a holonomy representation and let $\rho : \Gamma_0 \to \mathsf{Isom}^+(\mathsf{E})$ be an affine deformation of ρ_0, with corresponding cocycle $u \in Z^1(\Gamma_0, \mathfrak{g})$.

The affine deformation ρ induces a path of holonomy representations ρ_t as follows:

$$\begin{aligned}\rho_t : \pi_1(\Sigma) &\longrightarrow G\\ \gamma &\longmapsto \exp(tu(g))g,\end{aligned}$$

where $g = \rho_0(\gamma)$, and u is the tangent vector to this path at $t = 0$. Conversely, for any path of representations ρ_t

$$\rho_t(\gamma) = \exp(tu(g) + O(t^2))g,$$

where $u \in Z^1(\Gamma_0, \mathfrak{g})$ and $g = \rho_0(\gamma)$.

Suppose g is hyperbolic. Then the length of the corresponding closed geodesic in Σ is

$$l(g) = 2\cosh^{-1}\left(\frac{|\operatorname{Tr}(\tilde{g})|}{2}\right),$$

where $\tilde{g}$ is a lift of g to $\mathsf{SL}(2,\mathbb{R})$. With ρ, ρ_t as above and $\rho_0(\gamma) = g$, set:

$$l_t(\gamma) = l(\rho_t(\gamma)).$$

Since the Margulis invariant of ρ can also be seen to be a function of its corresponding cocycle u, we may write:

$$\alpha_u(g) := \alpha_\rho(g).$$

Consequently:

$$\left.\frac{d}{dt}\right|_{t=0} l_t(\gamma) = \frac{\alpha_u(g)}{2},$$

so we may interpret α_u as the change in length of an affine deformation, up to first order [11, 10].

Although $l_t(\gamma)$ is not differentiable at 0 for parabolic g,

$$\left.\frac{d}{dt}\right|_{t=0} \frac{\sigma(g)}{2}\operatorname{Tr}(\rho_t(\gamma)) = \tilde{\alpha}_u(g).$$

Thus Theorem 4.1 simply reinterprets Theorem 3.2.

Theorem 4.1. *Let Σ be a three-holed sphere with a hyperbolic structure. Consider any deformation of the hyperbolic structure of Σ where the lengths of the three boundary curves are increasing up to first order, then the lengths of all of the remaining geodesics are also increasing up to first order.*

Proof. Let ρ_t, $-\epsilon \le t \le \epsilon$ be a path of holonomy representations. Since we assume the boundary components to be lengthening, they must have hyperbolic holonomy on $(-\epsilon, \epsilon)$.

Suppose there exists $\gamma \in \pi_1(\Sigma)$ and $T \in (-\epsilon, \epsilon)$ such that the length of $\rho_t(\gamma)$ decreases in a neighborhood of T. Let $u = u_T \in Z^1(\Gamma_0, \mathfrak{g})$ be a cocycle tangent to the path at T; then

$$\alpha_{u_T}(\gamma) < 0.$$

Theorem 3.2 implies that for some $i = 1, 2, 3$:

$$\alpha_{u_T}(\partial_i)) < 0.$$

but then the length of the corresponding end must decrease, contradicting the hypothesis.

□

4.1. **Deformed hyperbolic transformations.** In this and the next paragraph, we explicitly compute the trace of some deformations, to understand first order length changes.

Let $g \in \mathsf{SL}(2, \mathbb{R})$ be a hyperbolic element, thus a lift of a hyperbolic isometry of H^2. Given a tangent vector in $V \in \mathfrak{sl}(2, \mathbb{R})$, consider the following two actions on $\mathsf{SL}(2, \mathbb{R})$:

$$\pi_V : g \to \exp(V) \cdot g, \tag{10}$$

and

$$\pi'_V : g \to g \cdot (\exp(V)^{-1}) = g \cdot \exp(-V). \tag{11}$$

All of our quantities are conjugation-invariant. Therefore, all of our calculations reduce to a single hyperbolic element of $\mathsf{SL}(2, \mathbb{R})$,

$$g = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} = \exp\left(\begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix}\right)$$

whose trace is $\mathrm{Tr}(g) = 2\cosh(s)$. The eigenvalue frame for the action of g on $\mathfrak{sl}(2, \mathbb{R})$ is

$$X_g^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X_g^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X_g^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} gX_g^0 g^{-1} &= X_g^0, \\ gX_g^- g^{-1} &= e^{-2s} X_g^- \\ gX_g^+ g^{-1} &= e^{2s} X_g^+. \end{aligned}$$

Write the vector $V \in \mathfrak{sl}(2, \mathbb{R})$ as

$$V = aX^0(g) + bX^-(g) + cX^+(g) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

By direct computation, the trace of the induced deformation $\pi_V(g)$ is

$$\mathrm{Tr}(\pi_V(g)) = 2\cosh s \cosh\sqrt{a^2+bc} + \frac{2a \sinh s \sinh\sqrt{a^2+bc}}{\sqrt{a^2+bc}}$$

Observe that when $V = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, which is equivalent to $\alpha(\gamma) = 0$:

$$\mathrm{Tr}(\pi_V(g)) = 2\cosh(s)\cosh(\sqrt{bc})$$

Up to first order, $\mathrm{Tr}(\pi_V(g)) = 2\cosh(s)$.

Alternatively, when $b = c = 0$:

$$\mathrm{Tr}(\pi_V(g)) = 2\cosh(s+a)$$

whose Taylor series about $a = 0$ does have a linear term. We assumed that $s > 0$, defining our expanding and contracting eigenvectors. As long as $a > 0$, which corresponds to $\alpha(\gamma) > 0$, the trace of the deformed element $\pi_V(g)$ is greater than the original element g.

Now consider the deformation $\pi'_V(g) = g \cdot (\exp(V))^{-1}$. When $b = c = 0$:

$$\mathrm{Tr}(\pi'_V(g)) = 2\cosh(s-a)$$

whose Taylor series about $a = 0$ has a nonzero linear term. As long as $a > 0$, $\mathrm{Tr}(\pi_V(g))$ is now less than the original element g. So for this deformation, a positive Margulis invariant corresponds to a decrease in trace of the original hyperbolic element.

Lemma 4.2. *Consider a hyperbolic $g \in \mathsf{SL}(2,\mathbb{R})$, with corresponding closed geodesic ∂ and an affine deformation represented by $V \in \mathfrak{sl}(2,\mathbb{R})$. For the actions of V on $\mathsf{SL}(2,\mathbb{R})$ by*

- *$\pi_V(g) = \exp(V) \cdot g$ then a positive value for the Margulis invariant corresponds to first order lengthening of ∂.*
- *$\pi'_V(g) = g \cdot \exp(V)$ then a positive value for the Margulis invariant corresponds to first order shortening of ∂.*

4.2. **Deformed parabolic transformations.** As before, we are interested in quantities invariant under conjugation. Because of this, all of our calculations can be done with the a very special parabolic transformation in $\mathsf{SL}(2,\mathbb{R})$,

$$p = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\right)$$

where $r > 0$ and whose trace is $\mathrm{Tr}(p) = 2$. We choose a convenient frame for the action of p on $\mathfrak{sl}(2,\mathbb{R})$:

$$X^u(g) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X^0(g) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X^c(g) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The trace of the deformation of the element p by the tangent vector V described above is

$$\mathrm{Tr}(\pi_V(p)) = 2\cosh(\sqrt{a^2+bc}) + \frac{cr}{\sqrt{a^2+bc}}\sinh(\sqrt{a^2+bc})$$

When $\alpha(\gamma) = 0$, or equivalently when $V = \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$, is

$$\mathrm{Tr}(\pi_V(p)) = 2\cosh(a)$$

Thus the trace equals 2, in terms of a, to first order.

Alternatively, when $a = b = 0$ in the expression for V,

$$\mathrm{Tr}(\pi_V(p)) = 2 + cr$$

which is linear and increasing in c. As long as $c > 0$, which corresponds to $\alpha(\gamma) > 0$, the trace of the deformed element $\pi_V(p)$ majorizes the original element p.

Lemma 4.3. *Consider a parabolic $g \in \mathsf{SL}(2,\mathbb{R})$, and an affine deformation represented by $V \in \mathfrak{sl}(2,\mathbb{R})$. For the actions of V on $\mathsf{SL}(2,\mathbb{R})$ by*

- $\pi_V(g) = \exp(V)\cdot g$ *then a positive value for the Margulis invariant corresponds to first order increase in the trace of g;*
- $\pi'_V(g) = g\cdot\exp(V)$ *then a positive value for the Margulis invariant corresponds to first order decrease in the trace of g.*

5. Einstein Universe

The Einstein Universe Ein_n can be defined as the projectivisation of the lightcone of $\mathbb{R}^{n,2}$. We will write everything for $n = 3$, as this is the focus of the paper.

Let $\mathbb{R}^{3,2}$ denote the vector space $\mathbb{R}^5$ endowed with a symmetric bilinear form of signature $(3,2)$. Specifically, for $\mathbf{x} = (x_1,\dots,x_5)$ and $\mathbf{y} = (y_1,\dots,y_5) \in \mathbb{R}^5$, set:

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - x_5y_5$$

Let $\mathbf{x}^\perp$ denote the orthogonal hyperplane to $\mathbf{x}$:

$$\mathbf{x}^\perp = \{\mathbf{y} \in \mathbb{R}^{3,2} \mid \mathbf{x}\cdot\mathbf{y} = 0\}$$

Let $\mathcal{N}^{3,2}$ denote the *lightcone* of $\mathbb{R}^{3,2}$:

$$\mathcal{N}^{3,2} = \{\mathbf{x} \in \mathbb{R}^{3,2}\setminus\mathbf{0} \mid \mathbf{x}\cdot\mathbf{x} = 0\}$$

Note that to keep the definitions as simple as possible, we do not consider the zero vector to belong to the lightcone.

The Einstein Universe consists of the projective lightcone, that is, the quotient of $\mathcal{N}^{3,2}$ under the action of $\mathbb{R}^*$ by scaling:

$$\mathrm{Ein}_3 = \mathcal{N}^{3,2}/\mathbb{R}^*$$

Denote by $\pi(\mathbf{v})$ the image of $\mathbf{v} \in \mathcal{N}^{3,2}$ under this projection. Wherever convenient, for $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5)$ we will also write:

$$\pi(\mathbf{v}) = (v_1 : v_2 : v_3 : v_4 : v_5)$$

Denote by $\widehat{\mathrm{Ein}_3}$ the orientable double-cover of Ein_3. Alternatively:

$$\widehat{\mathrm{Ein}_3} = \mathcal{N}^{3,2}/\mathbb{R}^+$$

Any lift of $\widehat{\mathrm{Ein}_3}$ to $\mathcal{N}^{3,2}$ induces a metric on Ein_3 by restricting $,\cdot$ to the image of the lift. For instance, the intersection with $\mathcal{N}^{3,2}$ of the sphere of radius 2, centered at $\mathbf{0}$, consists of vectors $\mathbf{x}$ such that:

$$x_1^2 + x_2^2 + x_3^2 = 1 = x_4^2 + x_5^2$$

It projects bijectively to $\widehat{\mathrm{Ein}_3}$, endowing it with the Lorentzian product metric $dg^2 - dt^2$, where dg^2 is the standard round metric on the 2-sphere S^2, and dt^2 is the standard metric on the circle S^1.

Thus Ein_3 is conformally equivalent to:

$$S^2 \times S^1/\sim, \text{ where } \mathbf{x} \sim -\mathbf{x}$$

Here $-I$ factors into the product of two antipodal maps.

Any metric on $\widehat{\mathrm{Ein}_3}$ pushes forward to a metric on Ein_3. Thus Ein_3 inherits a conformal class of Lorentzian metrics from the ambient spacetime $\mathbb{R}^{3,2}$. The group of conformal automorphisms of Ein_3 is:

$$\mathrm{Conf}(\mathrm{Ein}_3) \cong \mathrm{PO}(3,2) \cong \mathrm{SO}(3,2)$$

As $\mathrm{SO}(3,2)$ acts transitively on $\mathcal{N}^{3,2}$, $\mathrm{Conf}(\mathrm{Ein}_3)$ acts transitively on Ein_3.

Slightly abusing notation, we will also denote by $\pi(p)$ the image of $p \in \widehat{\mathrm{Ein}_3}$ under projection onto Ein_3.

The antipodal map being orientation-reversing in the first factor (but orientation preserving in the second), Ein_3 is non-orientable. However, it is *time-orientable*, in the sense that a future-pointing timelike vector field on $\mathbb{R}^{3,2}$ induces one on Ein_3.

5.1. **Conformally flat Lorentzian structure on** Ein_3**.** The Einstein Universe contains a copy of Minkowski spacetime, which we describe here for dimension three. Denote the scalar product on V by $\cdot$. Set:

$$\begin{aligned}\iota : \mathsf{V} &\longrightarrow \mathrm{Ein}_3\\ \mathbf{v} &\longmapsto \pi\left(\frac{1-\mathbf{v}\cdot\mathbf{v}}{2}, \mathbf{v}, \frac{1+\mathbf{v}\cdot\mathbf{v}}{2}\right)\end{aligned}$$

This is a conformal transformation that maps V to a neighborhood of $(1:0:0:0:1)$. In fact, setting:

$$p_\infty = (-1:0:0:0:1)$$

then

$$\iota(\mathsf{V}) = \mathrm{Ein}_3 \setminus \mathsf{L}(p_\infty)$$

Thus Ein_3 is the *conformal compactification of* V.

Since $\mathrm{Conf}(\mathrm{Ein}_3)$ acts transitively on Ein_3, every point of the Einstein Universe admits a neighborhood that is conformally equivalent to V. In other words, Ein_3 is a *conformally flat Lorentzian manifold.*

5.2. **Photons and lightcones.** Let us describe the causal structure of Ein_3, namely photons and lightcones. It is useful to know (see for instance [7]) that conformally equivalent lorentzian metrics give rise to the same causal structure. As a matter of fact, the non-parametrized lightlike geodesics are the same. This will mean that anything defined in terms of the causal structure of a given metric will in fact be well defined in the conformal class of that metric. More details can be found in [1].

Recall that, given a vector space V endowed with a non-degenerate, symmetric bilinear form $\cdot$, a subspace of $W \subset V$ is *totally isotropic* if $\cdot\,|_W$ is identically zero. If $W \subset \mathbb{R}^{3,2}$ is totally isotropic, then $W \setminus \mathbf{0} \subset \mathcal{N}^{3,2}$.

Definition 5.1. Let $W \subset \mathbb{R}^{3,2}$ be a totally isotropic plane. Then $\pi(W \setminus \mathbf{0})$ is called a *photon.*

Alternatively, a photon is an unparametrized lightlike geodesic of Ein_3.

It can easily be shown that no photon is homotopically trivial. The homotopy class of a photon generates the fundamental group of Ein_3.

Definition 5.2. Two points $p, q \in \mathrm{Ein}_3$ are said to be *incident* if they lie on a common photon.

Definition 5.3. Let $p \in \mathrm{Ein}_3$. The *lightcone* at p, denoted $\mathsf{L}(p)$, is the union of all photons containing p.

In other words, $\mathsf{L}(p)$ is the set of all points incident to p. Also:

$$\mathsf{L}(p) = \pi\left(\mathbf{v}^{\perp} \cap \mathcal{N}^{3,2}\right)$$

where $\mathbf{v} \in \mathcal{N}^{3,2}$ is such that $\pi(\mathbf{v}) = p$.

Lemma 5.4. *Suppose $p, q \in \mathrm{Ein}_3$ are non-incident. Then $\mathsf{L}(p) \cap \mathsf{L}(q)$ is a simple closed curve.*

Indeed, the intersection in this case is a *spacelike circle*: it is a smooth curve whose tangent vector at all points is spacelike.

Proof. Suppose without loss of generality that $p = p_\infty$. The intersection of $\mathsf{L}(q)$ with the Minkowski patch $\mathrm{Ein}_3 \setminus \mathsf{L}(p)$ corresponds to a lightcone in V. Applying a translation if necessary, we may suppose that $q = \iota(0,0,0)$. Then $\mathsf{L}(p) \cap \mathsf{L}(q)$ is the so-called circle at infinity:

$$\mathsf{L}(p) \cap \mathsf{L}(q) = \{(0 : \cos t : \sin t : 1 : 0) \mid t \in \mathbb{R}\}$$

$\square$

5.3. **Einstein torus.**

Definition 5.5. An *Einstein torus* is a closed surface in $S \subset \mathrm{Ein}_3$ such that the restriction of the conformal class of metrics to S is of signature $(1,1)$.

Specifically, an Einstein torus is given by a certain configuration of four points as follows. Let:

$$\mathsf{D} = \{p_1, p_2, f_1, f_2\}$$

where:

- p_1, p_2 are non-incident;
- $f_1, f_2 \in \mathsf{L}(p_1) \cap \mathsf{L}(p_2)$.

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}^{3,2}$ such that:

$$\mathbf{v}_i \in \pi^{-1} p_i$$
$$\mathbf{x}_i \in \pi^{-1} f_i$$

The restriction of $, \cdot$ endows the subspace of $\mathbb{R}^{3,2}$ spanned by the four vectors with a non-degenerate scalar product of signature $(2,2)$. Thus its lightcone is a 3-dimensional subset of $\mathcal{N}^{3,2}$. It projects to a torus in Ein_3 that is conformally equivalent to Ein_2.

6. Crooked surfaces

A *crooked surface* is the conformal compactification in Ein_3 of a crooked plane (up to isometry). Crooked planes were generalised by Frances to study the extension of such actions to Ein_3 [8].

6.1. **Crooked surfaces as conformal compactifications of crooked planes.** Let $p_\infty = \pi(-1,0,0,0,1)$; recall that $\iota(\mathsf{V})$ consists of the complement of $\mathsf{L}(p_\infty)$.

Let $\mathbf{u} \in \mathsf{V}$ be spacelike and $p \in \mathsf{E}$. The crooked plane $\mathcal{C}(\mathbf{u}, p)$ admits a conformal compactification, which we denote by $\overline{\mathcal{C}(\mathbf{u},p)}^{\text{conf}}$. Explicitly, setting $o = (0,0,0)$:

$$\overline{\mathcal{C}(\mathbf{u},o)}^{\text{conf}} = \iota(\mathcal{C}(\mathbf{u},o)) \cup \phi \cup \psi$$

where:

- $\phi \subset \mathsf{L}(p_\infty)$ is the photon containing $\pi(0,\mathbf{u}^+,0)$;
- $\psi \subset \mathsf{L}(p_\infty)$ is the photon containing $\pi(0,\mathbf{u}^-,0)$.

The wing $o+\mathrm{Wing}(\mathbf{u}^+)$ is in fact a "half lightcone"; specifically it is one of the two components in $\mathsf{L}(\pi(0,\mathbf{u}^+,0)) \setminus \phi'$, where $\phi' \subset \mathsf{L}(\pi(0,\mathbf{u}^+,0))$ is the photon containing $\iota(0)$. (A similar statement holds for $o + \mathrm{Wing}(\mathbf{u}^-)$.)

Definition 6.1. Let $o \in \mathsf{E}$ and $\mathbf{u} \in \mathsf{V}$ be spacelike. A *crooked surface* is any element in the $\mathrm{SO}(3,2)$-orbit of $\overline{\mathcal{C}(\mathbf{u},o)}^{\text{conf}}$.

Remark 6.2. One could show that the conformal compactifications of negatively extended crooked planes also lie in the $\mathrm{SO}(3,2)$-orbit of $\overline{\mathcal{C}(\mathbf{u},o)}^{\text{conf}}$. If one restricts to the connected component of the identity of $\mathrm{SO}(3,2)$, however, one only reaches what one could call a "positively extended crooked surface".

6.2. **A basic example.** We will describe $\mathcal{S} = \overline{\mathcal{C}(o,\mathbf{u})}^{\text{conf}}$, where $o = (0,0,0)$ and $\mathbf{u} = (1,0,0)$. We identify Ein_3 with a quotient of $S^2 \times S^1$, which admits the following parametrization (which can be recognized as a permuted version of the usual parametrizations):

$$(\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta, \sin t, \cos t)\,,\ 0 \le \phi \le \pi,\ 0 \le \theta, t \le 2\pi \tag{12}$$

Since $\mathbf{u}^\pm = (0, \mp 1, 1)$, the compactification of $\iota\left(\mathbf{u}^\perp\right)$ is the Einstein torus determined by:

$$\begin{aligned}
\mathbf{v}_1 &= \iota(o) = (1:0:0:0:1)\\
\mathbf{v}_2 &= (-1:0:0:0:1)\\
\mathbf{x}_1 &= (0:0:1:1:0)\\
\mathbf{x}_2 &= (0:0:-1:1:0)
\end{aligned}$$

Thus it is $\pi\left((0,1,0,0,0)^\perp \cap \mathcal{N}^{3,2}\right)$, which can be parametrized as:

$$(\sin t : 0 : \cos t : \sin s : \cos s)\,,\ -\pi/2 \le s \le \pi/2,\ 0 \le t \le 2\pi$$

And the stem corresponds to restricting t to lie between $\pi/2 - s$ and $\pi/2 + s$.

The wing $o + \mathrm{Wing}(\mathbf{u}^-)$ is a subset of the lightcone $\mathsf{L}((0:0:1:1:0))$, which can be parametrized as follows:

$$(\sin s\cos t : \sin s\sin t : \cos s : \cos s : -\sin s)\,, -\frac{\pi}{2} \leq s \leq \frac{\pi}{2}, 0 \leq t \leq 2\pi$$

Photons are parametrized as $t =$ constant. The photon incident to $(-1:0:0:0:1)$ corresponds to $t = 0$ and the photon incident to $(1:0:0:0:1)$, to $t = \pi$. This wing contains:

$$\iota(-\mathbf{u}) = (0:-1:0:0:1)$$

which lies on the photon $t = \pi/2$. Therefore, the wing is the half lightcone $0 \leq t \leq \pi$.

In a similar way, we find that the wing $o + \mathrm{Wing}(\mathbf{u}^+)$ is the half lightcone parametrized as follows:

$$(\sin s\cos t : -\sin s\sin t : -\cos s : \cos s : -\sin s)\,, -\frac{\pi}{2} \leq s \leq \frac{\pi}{2}, 0 \leq t \leq 2\pi$$

Given that all crooked surfaces are conformally equivalent to the basic example, the following two theorems are easily verified.

Theorem 6.3. *A crooked surface is a surface homeomorphic to a Klein bottle.*

Proof. This is a simple cut-and-paste argument, as can be found in [1]. □

Theorem 6.4. *A crooked surface separates* Ein_3.

6.3. **Synthetic description of a crooked surface.** We will now show how a crooked surface in Ein_3 is determined by a set like the following:

$$\mathsf{D} = \{p_1, p_2, f_1, f_2\}$$

where:

- $p_1, p_2 \in \mathrm{Ein}_3$ are a pair of non-incident points ;
- f_1, f_2 are two points on the spacelike circle which is the intersection of $\mathsf{L}(p_1)$ and $\mathsf{L}(p_2)$.

The data D determines two pairs of photons:

- one pair of photons ϕ_1, ϕ_2 belonging to $\mathsf{L}(p_1)$;
- one pair of photons ψ_1, ψ_2 belonging to $\mathsf{L}(p_2)$;
- each pair of photons ϕ_i, ψ_i intersects in f_i.

Alternatively, ϕ_i and ψ_i belong to the lightcone of f_i.

Here is how we may define the stem of the crooked surface. Choose a conformal map of $\mathrm{Ein}_3 \setminus \mathsf{L}(p_2)$ to V, and set $\mathcal{NF}(p_1)$ be the set of points mapping to points related to the image of p_1 by a timelike vector. (Alternatively, $\mathcal{NF}(p_1)$ is the set of endpoints of timelike geodesic paths starting from p_1, given an arbitrary choice of Lorentz metric in the conformal class.) Observe that this is symmetric in the pair $\{p_1, p_2\}$, so that we can write unambiguously:

$$\mathcal{NF}(\mathsf{D}) := \mathcal{NF}(p_1) = \mathcal{NF}(p_2)$$

On the other hand, as seen in §5.3, D determines an Einstein torus, $\mathrm{Ein}_2(\mathsf{D})$. The *stem*, denoted $\mathrm{Stem}(\mathsf{D})$, consists of the intersection:

$$\mathrm{Stem}(\mathsf{D}) = \mathrm{Ein}_2(\mathsf{D}) \cap \mathcal{NF}(\mathsf{D})$$

As for the wings, consider the following. The pair of photons ϕ_1 and ϕ_2 bounds a pair half lightcones in $\mathsf{L}(f_1)$, as does the pair ψ_1 and ψ_2 in $\mathsf{L}(f_2)$. Each half lightcone intersects the common spacelike circle in a... half circle, bounded by p_1 and p_2. We want to choose one half lightcone from each pair, in such a way that they intersect only in p_1 and p_2. In fact, there are two ways to do this; conformally identifying $\mathrm{Ein}_3 \setminus \mathsf{L}(p_2)$ with V via an element of $\mathrm{SO}^0(3,2)$:

- one way yields a pair of positive wings, $\mathrm{Wing}(\mathsf{D}, f_1)$ and $\mathrm{Wing}(\mathsf{D}, f_2)$;
- the other way yields a pair of negative wings.

Thus the crooked surface determined by D is:

$$\mathcal{S} = \mathcal{S}(\mathsf{D}) = \mathrm{Stem}(\mathsf{D}) \cup \mathrm{Wing}(\mathsf{D}, f_1) \cup \mathrm{Wing}(\mathsf{D}, f_2)$$

References

1. Thierry Barbot, Virginie Charette, Todd A. Drumm, William M. Goldman, and Karin Melnick, *A primer on the (2+1) Einstein universe*, Recent Developments in Pseudo-Riemannian Geometry (Dmitri V. Alekseevsky and Helga Baum, eds.), European Mathematical Society, 2008, pp. 179–225.
2. Jean-Philippe Burelle, Virginie Charette, Todd A. Drumm, and William M. Goldman, *Crooked halfspaces*, in preparation.
3. Virginie Charette and Todd A. Drumm, *The Margulis invariant for parabolic transformations*, Proc. Amer. Math. Soc. **133** (2005), no. 8, 2439–2447.
4. Virginie Charette, Todd A. Drumm, and William M. Goldman, *Affine deformations of a three-holed sphere*, Geom. Topol. **14** (2010), 1355–1382.
5. Todd A. Drumm, *Fundamental polyhedra for Margulis space-times*, Topology **31** (1992), no. 4, 677–683.
6. Todd A. Drumm and William M. Goldman, *The geometry of crooked planes*, Topology **38** (1999), no. 2, 323–351.
7. Charles Frances, *Géometrie et dynamique lorentziennes conformes*, Ph.D. thesis, E.N.S. Lyon, 2002.

8. ______, *The conformal boundary of Margulis space-times*, C. R. Acad. Sci. Paris Sér. I **332** (2003), 751–756.
9. David Fried and William M. Goldman, *Three-dimensional affine crystallographic groups*, Adv. in Math. **47** (1983), no. 1, 1–49. MR 689763 (84d:20047)
10. William M. Goldman, *The Margulis invariant of isometric actions on Minkowski* (2+1)*-space*, Rigidity in dynamics and geometry (Cambridge, 2000), Springer, Berlin, 2002, pp. 187–201. MR 1919401 (2003i:53107)
11. William M. Goldman and Gregory A. Margulis, *Flat Lorentz 3-manifolds and cocompact Fuchsian groups*, Crystallographic groups and their generalizations (Kortrijk, 1999), Contemp. Math., vol. 262, Amer. Math. Soc., Providence, RI, 2000, pp. 135–145. MR 1796129 (2001m:53124)
12. Ravi S. Kulkarni, *Proper actions and pseudo-Riemannian space forms*, Adv. in Math. **40** (1981), no. 1, 10–51. MR 616159 (84b:53047)
13. Gregory Margulis, *Free properly discontinuous groups of affine transformations*, Dokl. Akad. Nauk SSSR **272** (1983), 937–940.
14. ______, *Complete affine locally flat manifolds with a free fundamental group*, J. Soviet Math. **134** (1987), 129–134.
15. John Milnor, *On fundamental groups of complete affinely flat manifolds*, Advances in Math. **25** (1977), no. 2, 178–187.
16. John G. Ratcliffe, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006. MR 2249478 (2007d:57029)

Département de mathématiques, Université de Sherbrooke, Sherbrooke, Quebec, Canada
E-mail address: v.charette@usherbrooke.ca

Department of Mathematics, Howard University, Washington, DC, USA
E-mail address: tdrumm@howard.edu