Maximum Drawdown Protection

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Introduction to Semi-static Hedging

- The recent turmoil has revived interest in hedging, especially for the few path-dependent claims that remain on the market.

- In the classic Black Scholes model, vanillas and barrier options can each be hedged by dynamic trading in the two currencies. This model also allows barrier options to be semi-statically hedged with vanillas.

- We will introduce options on (maximum) drawdown and drawup that can be hedged with barrier options (and hence vanillas or the underlying currencies).

- All of the hedges are either purely static and hence completely model-free, or else semi-static and hence semi-robust.

- In the semi-robust case, the hedges survive an unknown independent time change. Under zero carrying costs for the underlying, the instantaneous volatility is an unknown stochastic process evolving independently of the shocks to an underlying FX rate.

Related Literature

- Working in the Black Model, Sbuelz (1998) constructs a semi-static hedge of a double barrier option using single barrier options
- Working in the Black Scholes model, Vecer, (2006/7) and Pospisil and Vecer(2008/9) value claims paying maximum drawdown using risk-neutral valuation.
- Douady, Shiryaev, and Yor (2000) give formulas for the expected value of the range, the maximum drawdown, and other path statistics of a standard Brownian motion.
- Zhang and Hadjiliadis (2009) extend many of these results to geometric Brownian motion.

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Limitations of Semi-static Hedging

- The semi-static hedges that we propose only work perfectly if a symmetry is present somewhere.
- Although this symmetry is usually lacking in real world markets, simulations on real world data have universally shown that the hedges nonetheless provide better tracking.
- Instead of relying on symmetry, one can put on the proposed hedge and fix it up with a classical dynamic hedge of the residual.
- For this reason, we will make restrictive assumptions that lead to simple hedges.

Standing Assumptions

- Our robust hedges place no dynamical restrictions on the underlying, but assume that barrier options trade.
- Our semi-robust hedges assume that the underlying is a forward FX rate F, so that it has zero risk-neutral drift under the forward measure.
- The hedges further suppose that the process is skip-free, i.e. no jumps over the barrier(s).
- Finally, we propose a notion of symmetry that need only hold when the underlying first visits the barrier(s) of a barrier option.

Arithmetic Symmetry

- For simplicity, we first consider a single barrier H, which WLOG we take to be higher than $F_0 = 100$.
- Let τ_H denote the first passage time to H ($\tau_H = \infty$ if never hit).
- Assume that if $\tau_H < T$, then at time τ, the conditional risk-neutral density governing the terminal price F_T is symmetric about $F_{\tau_H} = H$. Hence, a Digital Put (DP) has the same price at hit as a co-terminal Digital Call (DC) of the same moneyness:

$$DP_\tau(K, T) = DC_\tau(2H - K, T) \text{ for all } K \in \mathbb{R}.$$

- For example, when a barrier of 110 is first hit, a DP struck at 90 has the same price as a co-terminal DC struck at 130.
- As the 90 vanilla put also has the same price as the 130 vanilla call, we refer to this result as Arithmetic Put Call Symmetry (APCS).

Implication of APCS for Up-and-In Digital Put

- If Arithmetic Put Call Symmetry holds at the first passage time τ_H to a barrier H, then a Digital Put has the same price at τ_H as the co-terminal Digital Call of the same moneyness:

$$DP_{\tau_H}(K, T) = DC_{\tau_H}(2H - K, T) \text{ for all } K \in \mathbb{R}.$$

- Suppose we sell an Up-and-In Digital Put (UIDP) with barrier $H = 110$ and strike $K = 90$. To hedge, we can buy a co-terminal Digital Call with strike $2H - K = 130$.

- If the underlying forward rate never hits 110, the DC expires worthless and if F does hit 110, the DC can be costlessly swapped for the DP.

- As a result, for $t \in [0, T \wedge \tau_H]$, no arbitrage implies:

$$UIDP_t(K, T; H) = DC_t(2H - K, T).$$

Implications of APCS for a Double One Touch

- With the underlying forward FX rate starting at $F_0 = 100$, consider a Double One Touch with barriers at $L = 90$ and $H = 110$ and payment at expiry. Define $d = 20$ as the distance between barriers.

- To hedge the sale of a DOT, buy a rectangular wave (RW) whose heights alternate between 0 and 2. This RW pays 0 if $F_T \in (L, H)$, \$2 if $F_T \in (L - d, L)$ or $F_T \in (H, H + d)$, 0 if $F_T \in (L - 2d, L - d)$ or $F_T \in (H + d, H + 2d)$, etc.

- If the underlying avoids both barriers, then the RW pays 0. If either barrier is touched, then the symmetry of the F_T distribution about the hit barrier implies that the RW is worth the PV of \$1.

- From no arb., $DOT_t(T; L, H) = RW_t(T; L, H), t \in [0, T \wedge \tau_L \wedge \tau_H]$.

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\begin{aligned}
RW_t(T; L, H) &= 2\sum_{n=1}^{\infty}[DC_t(T; L + nd) - DC_t(T; L + (n+1)d)] \\
&+ 2\sum^{\infty}[DP_t(T; H - nd) - DP_t(T; H - (n+1)d)].
\end{aligned}
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Drawdown

- Let S_t denote the spot FX rate which we assume can be monitored continuously over a fixed time interval $[0, T]$.

- Let $M_T \equiv \max_{t \in [0,T]} S_t$ be the continuously-monitored maximum of this spot FX rate over $[0, T]$.

- Let $D_T \equiv M_T - S_T$ be the terminal drawdown or just "drawdown" for brevity.

- A call on the drawdown with payoff $(D_T - K_d)^+$ provides insurance for the call buyer against large drawdown realizations, with the maximum loss limited to the initial premium.

- Assuming only frictionless markets and no arbitrage, a new model-free exact hedge for a drawdown call is presented on the next page.

Robust Hedge of Drawdown Call

- Let $B_t(T) > 0$ be the price at $t \in [0, T]$ of a Bond paying \$1 at T. Let $\mathbb{Q}$ be the associated risk-neutral probability measure.
- Let $P_t(K, T) = B_t(T) E_t^{\mathbb{Q}} (K - S_T)^+$ be the vanilla put price at time $t \in [0, T]$.
- Let τ_H be the first passage time of the process S to a barrier $H > S_0$. Let $UIDP_t(K_u, T; H) = B_t(T) E_t^{\mathbb{Q}} 1(M_T > H, S_T < K_u)$ be the value at time $t \in [0, \tau_H]$ of an Up-and-In Digital Put with strike K_u, maturity $T \geq t$, and barrier $H \geq S_0$.
- In frictionless markets, the drawdown call value $C_t^d(K_d, T) \equiv B_t(T) E_t^{\mathbb{Q}} (D_T - K_d)^+ =$

$$P_t(M_t - K_d, T) + \int_{M_t}^{\infty} UIDP_t(H - K_d, T; H) dH, \qquad t \in [0, T], K_d \geq 0.$$

- The next slide explains why.

Interpreting the Drawdown Call Hedge

- Recall that in frictionless markets, the drawdown call value $C_t^d(K_d, T) \equiv B_t(T) E_t^{\mathbb{Q}}(D_T - K_d)^+ =$

$$P_t(M_t - K_d, T) + \int_{M_t}^{\infty} UIDP_t(H - K_d, T; H)dH, \qquad t \in [0, T], K_d \geq 0.$$

- In words, a drawdown call with strike K_d is robustly replicated by keeping a put struck K_d dollars below M_t and also holding dH Up-and-In Digital Puts struck K_d dollars below H for each in-barrier $H > M_t$.

- If $M_T = M_t$, then the put provides the desired payoff, while if $M_T > M_t$, then the Up-and-In Digital Puts which knock in at each rise in M are used to roll up the put strike.

Hedging with Vanillas

- While the proposed hedge is robust, it requires Up-and-In Digital Puts as hedging instruments.
- The needed barrier options may not trade or the bid offer spread may be too wide.
- However under some conditions, principally skip-freedom and symmetry at the barrier, we saw that an Up-and-In Digital Put can be replicated with a Digital Call:

$$UIDP_t(K, T; H) = DC_t(2H - K, T).$$

- The next slide indicates the consequences.

Drawdown Call on Forward Under APCS

- A call on the drawdown of a forward FX rate pays off $(D_T - K_d)^+$ at its expiry T, where $D_t \equiv M_t - F_t$ is the running drawdown and $M_t \equiv \max_{s\in[0,t]} F_s$ is the running maximum over $[0, t]$.

- We will replicate by rolling up strangles of constant width. For $F_t \in (K_p, K_c)$, define the center of the strangle as the average of the two strikes and define the width of the strangle as the difference between K_c and this center.

- We now define: **Assumption set A1:** *The forward FX rate F is a $\mathbb{Q}$ martingale whose running maximum is continuous and for which APCS holds at all times τ when $dM_\tau > 0$.*

- Under frictionless markets &**A1**, no arbitrage implies that:

$$C_t^d(K_d, T) = P_t(M_t - K_d, T) + C_t(M_t + K_d, T), \qquad t \in [0, T], K_d \geq 0.$$

Interpreting the Drawdown Call Hedge

- Recall that if the running maximum of an underlying forward increases only continuously and if APCS holds when it does, then no arbitrage and frictionless markets imply:

$$C_t^d(K_d, T) = P_t(M_t - K_d, T) + C_t(M_t + K_d, T), \qquad t \in [0, T], K_d \geq 0.$$

- In words, a drawdown call is replicated by alway holding a strangle centered at the running maximum M_t, and whose width is the strike K_d of the drawdown call.

- The strategy is self-financing because the cash outflow required to move the put strike up when the running maximum increases infinitessimally is financed by the cash inflow received from moving the call strike up (given that APCS is in fact holding at such times).

Maximum Drawdown

- Recall F_t denotes the forward FX rate which we assume can be monitored continuously over a fixed time interval $[0, T]$.
- Now let $M_t \equiv \max_{s \in [0,t]} F_s$ be the continuously-monitored running maximum of this forward FX rate over $[0, t]$.
- Now let $D_t \equiv M_t - F_t$ be the running drawdown for $t \in [0, T]$.
- Let $MD_T \equiv \max_{t \in [0,T]} D_t$ be the Maximum Drawdown experienced over $[0, T]$.
- Maximum Drawdown is a widely used risk measure, as it captures an inability to time, when doing say the carry trade.

Digital Call on Maximum Drawdown

- Recall that $D_t \equiv M_t - F_t$ denotes the running drawdown for $t \in [0, T]$, while $MD_T \equiv \max_{t \in [0,T]} D_t$ denotes the Maximum Drawdown experienced over $[0, T]$.

- A Digital Call on the Maximum Drawdown has terminal payoff $DC_T^{md} = 1(MD_T > K_{md})$. It provides insurance for the buyer against large realizations of Maximum Drawdown, with the maximum loss limited to the initial premium.

- Using the same conditions that lead to a call on drawdown being replicated by rolling up a strangle, a new semi-static exact hedge for a Digital Call on Maximum Drawdown is presented on the next page.

Hedging a DC on MD Under APCS

- Recall that $MD_T \equiv_{t\in[0,T]}^{\max} (M_t - F_t)$ denotes the Maximum Drawdown experienced over $[0, T]$. and that a Digital Call on Maximum Drawdown pays off $DC_T^{md} = 1(MD_T > K_{md})$ at its expiry T.

- We will replicate by rolling up Double One Touches. For $F_t \in (L, H)$, a Double One Touch with price $DOT_t(L, H, T), t \in [0, T]$ pays \$1 at T if either barrier is hit before T and it pays zero otherwise.

- Recall **Assumption set A1:** *The forward FX rate F is a* $\mathbb{Q}$ *martingale whose running maximum is continuous and for which APCS holds at all times* τ *when* $dM_\tau > 0$.

- Under frictionless markets &**A1**, no arbitrage implies that:

$$DC_t^{md}(K_{md}, T) = DOT_t(M_t - K_{md}, M_t + K_{md}, T), \qquad t \in [0, T], K_{md} \geq 0.$$

Interpreting the DC on MD Hedge

- Recall that if the running maximum of an underlying forward increases only continuously and if APCS holds when it does, then no arbitrage and frictionless markets imply:

$$DC_t^{md}(K_{md}, T) = DOT_t(M_t - K_{md}, M_t + K_{md}, T), \qquad t \in [0, T], K_{md} \geq 0.$$

- In words, a Digital Call on the Maximum Drawdown is replicated by alway holding a Double One Touch centered at the running maximum M_t, and whose width is the strike K_{md} of the Digital Call.

- The strategy is self-financing because the cash outflow required to bring the lower barrier nearer by dM when the running maximum increases infinitessimally is financed by the cash inflow received from pushing the upper barrier away by dM (given that APCS is in fact holding at such times).

Using Vanillas Instead of DOT's

- The proposed hedge for the sale of a Digital Call on Maximum Drawdown involves rolling up centered Double One Touches whenever the running maximum creeps up.
- For some underlyings, Double No Touches (DNT's) trade liquidly and since DOT = B - DNT, so do DOT's.
- For other underlyings, neither DNT's nor DOT's trade. Fortunately, we saw that a DOT has a semi-static hedge in terms of a Rectangular Wave.
- As a consequence, the sale of a Digital Call on Maximum Drawdown can also be hedged by rolling up centered Rectangular Waves whenever the running maximum creeps up.
- Hedges are also available using single touches.

Maximum Drawup

- Let S_t denote the spot FX rate which we assume can be monitored continuously over a fixed time interval $[0, T]$.
- Let $m_t \equiv \min_{s \in [0,t]} S_s$ be the continuously-monitored maximum of this spot FX rate over $[0, t]$.
- Let $U_t \equiv S_t - m_t$ be the running drawup or just "drawup" for brevity.
- Let $MU_T \equiv \max_{t \in [0,T]} U_t$ be the maximum drawup over $[0, T]$.
- Just as Maximum Drawdown is a risk measure, Maximum Drawup is a reward measure.

Cheapening Premium

- Consider a digital call paying \$1 at its maturity date $T < \infty$ if and only if S draws down by at least $K > 0$ before it draws up by K.

- An investor who buys this claim and borrows the cost is clearly betting that S will draw down by at least \$K before T, and before it draws up by \$K.

- The cost of insuring against a Maximum Drawdown of at least K \$ over $[0, T]$ via this digital call is cheaper than just buying Maximum Drawdown protection outright since:

$$1(\tau_K^D < \tau_K^U \wedge T) = 1(MD_T > K) - 1(\tau_K^U < \tau_K^D \wedge T).$$

One Touch Knockouts

- We will show that there exists a robust static hedge of the digital call on the $K-$drawdown preceding a $K-$drawup which uses positions in one-touch knockouts (henceforth OTKO's).
- An OTKO is issued with a lower barrier $L < S_0$, a higher barrier $H \geq S_0$, and a fixed maturity date T.
- Let $\tau_H^S \equiv \{\overset{\max}{t\in[0,T]}\ S_t > H\}$ be the first passage time (FPT) of the spot process S to the higher barrier $H \geq S_0$. Let $\tau_L^S \equiv \{\overset{\max}{t\in[0,T]}\ S_t \leq L\}$ be the FPT of the spot process S to the lower barrier $L < S_0$.
- The terminal payoff of an OTKO is:

$$OTKO_T(L, H, T) = 1(\tau_L^S < \tau_H^S \wedge T).$$

  In words, the OTKO pays one dollar at its expiry date T if and only if the spot price S hits the lower barrier L before hitting the upper barrier H and this first passage time to L occurs before T.

Pricing the Digital Call Using OTKO's

- Let τ_K^D be the first passage time of running drawdown to $K > 0$. Similarly, let τ_K^U be the first passage time of running drawup to K.

- We seek to replicate a claim paying \$1 if $\tau_K^D \leq \tau_K^U \wedge T$ & 0 o.w.

- The theorem below is a consequence of a model-free static hedge of this digital call in terms of one-touch knockouts.

  Theorem: Robust Pricing of Digital Call *Under frictionless markets, no arbitrage implies:*

$$\begin{aligned} E_t^{\mathbb{Q}} 1(\tau_K^D \leq \tau_K^U \wedge T) &= 1(\tau_K^D \leq t \wedge \tau_K^U \wedge T) B_t(T) \\ &+ 1(t < \tau_K^D \wedge \tau_K^U \wedge T) \Big[OTKO_t(M_t - K, M_t, T) \\ &+ \int_{(M_t - K)+}^{m_t -} \frac{\partial}{\partial K} OTKO_t(L, L + K, T) dL \Big]. \end{aligned}$$

Does APCS Hold?

- The conditions that generate semi-static hedges are sufficient but not necessary.
- An example of a model that meets all of the sufficient conditions is the Bachelier model, i.e. constant normal volatility.
- As is well known, this model produces a negative skew when the latter is expressed in terms of the usual Black implied vol.
- The semi-static hedges all succeed if the normal instantaneous volatility follows an unspecified stochastic process that evolves independently of the noise driving FX rates.
- As is well known, this stochastic volatility effect produces curvature in the usual Black implied vol.

Geometric Symmetry and Beyond

- A drawback of Arithmetic Put Call Symmetry is that positive probability of arbitrarily high FX rates implies positive probability of arbitrarily low FX rates, including negative FX rates.
- Fortunately, all of the results have their geometric counterpart.
- In fact, all of the claims have semi-static hedges for a mixed SV/LV framework, $dF_t = \sqrt{V_t} a(F_t) dW_t$, whenever a is affine in F and when V evolves independently of F and W.
- If these sufficient conditions are still considered too restrictive, then as mentioned in the introduction, nothing prevents putting on the semi-static hedge and then delta and vega hedging the residual.

Summary

- We showed that single and double barrier options could be used as robust hedges for new exotics whose payoff depends on drawdown, maximum drawdown, and/or maximum drawup.
- We gave sufficient conditions under which the payoff from single and double barrier options could be replicated by semi-static positions in vanilla options.
- It follows that by restricting the dynamics of the underlying, the new exotics can be semi-statically hedged with vanillas.
- Besides improved hedging, these results also have implications for software architecture and for symbolic solvers, in particular understanding *why* some exotics can be priced in closed form and others can't.