Elliptic equations: Exercises

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1. Solve the linear one-dimensional elliptic equation

$$u'' = -e^x(-2 + 2x + 5x^2 + x^3),$$

on the domain $x \in [0, 1]$ with $u(0) = u(1) = 0$. Use a cell-centered grid and the tridiagonal method. Verify second-order convergence.

2. Solve the same problem on a staggered grid, with the Nuemann boundary condition $u'(0) = 1$. (And the same condition at $x = 1$, $u(1) = 0$.) Is there a significant difference in accuracy between the two methods, for the same numbers of grid points?

3. Solve the radial elliptic equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \phi'' + \frac{2}{r}\phi' = -\frac{r^2}{1+r^6},$$

with the boundary conditions $\phi'(0) = 0$ and $\phi(R) = 0$, where R is the outer boundary of the domain. Verify second-order convergence. How does the solution change with the location of R? How much does the solution change with respect to R, compared to changes with respect to spatial resolution?

4. Solve the nonlinear equation

$$\phi'' + \frac{1}{\phi^3} = 0,$$

with $\phi'(0) = 0$ and $\phi(1) = 1$. First linearize the equation, and solve iteratively, with an initial guess of $\bar{\phi} = 1$.

5. Solve the Hamiltonian constraint for a maximal trumpet slice of the Schwarzschild spacetime,

$$\nabla^2\psi + \frac{1}{8}\psi^{-7}\frac{6C^2}{r^6} = 0,$$

where $C^2 = 27M^4/16$. As an ansatz for the conformal factor, use

$$\psi = \psi_0 + u.$$

Near $r = 0$ we want $\psi \to \sqrt{3M/2r}$, and as $r \to \infty$ we want $\psi \to 1 + M/2r$. We can add these two asymptotic contributions in such a way that they do not influence each other's leading-order behaviour. For example,

$$\psi_0 = \frac{1}{1 + r^2/100}\sqrt{\frac{3M}{2r}} + \frac{2r^2}{1 + 2r^2}\left(1 + \frac{M}{2r}\right).$$

Linearize the equation with $u = \bar{u} - \delta u$ and solve for δu. You can use the boundary conditions that $\delta u'(0) = 0$ and either $\delta u(r_{max}) = 0$ or a Robin boundary condition at r_{max}. You can set $M = 1$.

Once you have a solution, locate the horizon at $R = 2M$ in terms of the coordinate r. (You can use the fact that $R = \psi^2 r$!)

More details on this solution are given in gr-qc/0612097 (where the same problem is solved numerically) and arXiv:0804.0628 (for full details on the trumpet solution). An implicit analytic solution is given in gr-qc/0701037.