# Trancendental Dynamics
# Sharing Values and Normality

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- Chen and Hua proved the following

## Theorem

*Let $\mathfrak{F}$ be a family of holomorphic functions in the unit disk $\Delta$.Suppose that there exist a non zero finite value $a$ such that for each function $f \in F$; $f, f'$ and $f''$ share the value $a$ IM in $\Delta$. Then the family $\mathfrak{F}$ is normal in $\Delta$.*

- we improved following result of Fang and Xu

### Theorem

*Let $\mathfrak{F}$ be a family of holomorphic functions on the unit disk $\Delta$ and let a,b be two distinct finite complex numbers such that $b \neq 0$.If for any $f \in \mathfrak{F}$, $f$ and $f'$ share a IM and $f(z){=}b$ whenever $f'(z){=}b$ then $\mathfrak{F}$ is normal in $\Delta$.*

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## Lemma

*Let $\mathfrak{F}$ be a family of holomorphic functions in a domain $\Delta$ with the property that for every function $f \in \mathfrak{F}$ , the zeros of $f$ are of multiplicity at least $k$. If $\mathfrak{F}$ is not normal at $z_0$=0, then for 0$\leq \alpha < k$, there exist*
*(a) a sequennce of complex numbers $z_n \to 0$, $|z_n| < r < 1$*
*(b) a sequence of functions $f_n \in \mathfrak{F}$ and*
*(c) a sequence of positive numbers $\rho_n \to 0$*
*such that $g_n(\zeta) = \rho^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges to a nonconstant entire function $g$ on $\mathbb{C}$. Moreover $g$ is of order at most one . If $\mathfrak{F}$ possesses the additional property that there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ for any $f \in \mathfrak{F}$, then we can take $\alpha = k$.*

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- Now we claim that $g = 0$ *iff* $g^{(k)} = a$ and $g^{(k)} \neq b$
  Suppose $g(\zeta_0) = 0$. then by Hurwitz's theorem there exist $\zeta_n; \zeta_n \to \zeta_0$ such that $g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta_n) - a] = 0$, since $f_n(z_n + \rho_n\zeta_n) = a$ since $f_n$ and $f_n^{(k)}$ share a , we have $g_n^k(\zeta_n) = f_n^{(k)}(z_n\rho_n) = a$ hence

$$g^{(k)}(\zeta_0) = \lim_{n\to\infty} g_n^{(k)}(\zeta_n) = a$$

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- Thus we have proved that $g^{(k)} = a$ *whenever* $g = 0$.

- On the other hand, if $g^{(k)}(\zeta_0) = a$ then there exist $\zeta_n, \zeta_n \to \zeta_0$ such that $g_n^{(k)}(\zeta_n) = f_n^{()k}(z_n + \rho_n\zeta_n) = a,\ \ n = 1, 2, \ldots$ hence $f_n(z_n = \rho_n\zeta_n) = a$ and $g_n(\zeta_n) = 0$ for n=1,2,... thus

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  suppose that there exist $\zeta_0$ satisfying $g^{(k)}(\zeta_0) = b$, by Hurwitz's theorem there exsit a sequence $\zeta_n \to \zeta_0$ and $g_n^{(k)}(\zeta_n) = b; n = 1, 2, \dots$

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- since $f_n(z) = b$ whenever $f_n^{(k)}(z) = b \Rightarrow f_n(z_n + \rho_n\zeta_n) = b$ and $g_n(\zeta_n) = \rho_n^{-k}[f_n(z_n + \rho_n\zeta_n) - a] = \rho_n^{(k)}[b - a] \to \infty$ this contradicts $\lim_{n\to\infty} g_n(\zeta_n) = g(\zeta_0) \neq \infty$
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- Hence we get
  $g^{(k)}(\zeta) = b + e^{A\zeta+B}$

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### Theorem

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**Lemma**

*let f be a nonconstant meromophic function. Then for* $k \geq 1, b \neq 0, \infty,$
$$T(r,f) \leq \overline{N}(r,f) + N(r,\tfrac{1}{f}) + N(r,\tfrac{1}{f^{(k)}-b}) - N(r,\tfrac{1}{f^{(k+1)}}) + S(r,f)$$

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- Now using **lemma and Nevanlinna's first fundamental theorem** , we have

- $T(r,g) \leq$
$\overline{N}(r,g) + N(r,\frac{1}{g}) + N(r,\frac{1}{g^{(k)}-a}) - N(r,\frac{1}{g^{(k+1)}}) + S(r,g)$
$= N(r,\frac{1}{g}) + N(r,\frac{1}{g^{(k)}-a} - N(r,\frac{1}{g^{(k+1)}}) + S(r,g)$
$\leq N(r,\frac{1}{g^{k}-a}) - \overline{N}(r,\frac{1}{g^{(k+1)}}) + S(r,g)$
$\leq T(r,\frac{1}{g^{k}-a}) - \overline{N}(r,\frac{1}{g^{(k+1)}}) + S(r,g) \leq$
$T(r,g^{(k)}-a) - \overline{N}(r,\frac{1}{g^{(k+1)}}) + S(r,g)$
$\leq T(r,g) - \overline{N}(r,\frac{1}{g^{(k+1)}}) + S(r,g)$
Thus we get
$\overline{N}(r,\frac{1}{g^{(k+1)}}) = S(r,g)$
by this and the
claim$(g = 0 \Leftrightarrow g' = a, g^{(k)} = g^{(k+1)} = 0 \ whenever \ g = 0)$
we get a contradiction: $T(r,g) = S(r,g)$.
Hence the theorem.

## Definition

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$F(f) = \{z \in \mathbb{C} : \{f^n\}_{n\in\mathbb{N}}$ is normal in some neighborhood of $z\}$ is called the Fatou set of $f$ or the set of normality of $f$.

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3. The Fatou set is open and completely invariant: $z \in F(f)$ if and only if $f(z) \in F(f)$ and consequently $J(f)$ is completely invariant.

4. A component $U$ of the Fatou set is called a **wandering domain** if $U_k \cap U_l = \emptyset$ for $k \neq l$, where $U_k$ denotes the component of $F(f)$ containing $f^k(U)$, otherwise
$U$ is called a **preperiodic component** of $F(f)$ $f^k(U_l) = U_l$, for some $k, l \geq 0$. If $f^k(U) = U$, for some $k \in \mathbb{N}$, then $U$ is called a **periodic component** of $F(f)$.

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Singh A.P. constructed several examples where the dynamics of $f$ and $g$ vary largely from the dynamics of the composite entire functions. In fact he proved:

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- In the construction of the proof SINGH has exhibited entire functions $f$ and $g$ with one domain $G_1$ satisfying the conditions of Theorem . In this connection, one would also be interested in knowing whether it is possible to have entire functions $f$ and $g$ having more than one domain satisfying the conditions of Theorem . This is possible. We have shown the existence of entire functions having infinitely many domains satisfying the conditions of Theorem .

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**Theorem**

*There exists infinite number of domains which lies in the wandering component of $f$ and wandering component of $g$ and lies in the periodic component of $g \circ f$.*

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  Then $S$ is called a Carleman set (for $\mathbb{C}$) if for any $f \in C(S)$

  and any positive continuous function $\epsilon$ on $S$, there exists an entire function $g$ such that $|f(z) - g(z)| < \epsilon(z)$ for all $z \in S$.

- **Lemma**[*Gair*] . Let $S$ be a closed proper subset of $\mathbb{C}$. Then $S$ is a Carleman set in $\mathbb{C}$ if and only if $S$ satisfies:
  (i) $\widetilde{\mathbb{C}} \setminus S$ is connected;
  (ii) $\widetilde{\mathbb{C}} \setminus S$ is locally connected at $\infty$;
  (iii) for every compact subset $K$ of $\mathbb{C}$ there exists a neighborhood $V$ of $\infty$ in $\widetilde{\mathbb{C}}$ such that no component of $S^{\circ}$ intersects both $K$ and $V$.

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Then using Lemma, we get $S$ is a Carleman set.

□

- It is known that the set of all natural numbers $\mathbb{N}$ can be expressed in an infinite array of numbers as

$$\{\frac{q(q-1)}{2} + 1 + pq + \frac{p(p+1)}{2} : p = 0, 1, \ldots, \ q = 1, 2, \ldots\}$$

Infact a natural number lying in row $p$ and column $q$ $(p = 0, 1, \ldots, \ q = 1, 2, \ldots)$ would be $\frac{q(q-1)}{2} + 1 + pq + \frac{p(p+1)}{2}$. Next if $n \in \mathbb{N}$, let $r$ be the least positive integer such that $\frac{r(r+1)}{2} \geq n$ and $s = \frac{r(r+1)}{2} - n$. Then $n$ lies in row $n_r = r - s - 1$ and column $n_c = s + 1$. Thus without any loss of generality we may denote the set $G_n$ by its place position $G_{n_r, n_c}$ say, or more simply by $G_{i,j}$ for suitable $i, j$ and $G_{i,j}$ may be denoted by $G_n$ for suitable $n$, and similarly for other terms.

# continued...

- We can write $G_k = G_{p,q}$ for suitable $p, q$. Using the continuity of $e^z$, for each $k = 1, 2, \ldots$ choose $\eta_{p,q}$ and $\xi_{p,q}$ so that

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- $|e^w + (4(\frac{q(q+1)}{2} + 1 + p(q+1) + \frac{p(p+1)}{2}) + 2)| < \frac{1}{2}$, whenever $|w - (\pi i + \log(4(\frac{q(q+1)}{2} + 1 + p(q+1) + \frac{p(p+1)}{2}) + 2))| < \eta_{p,q}$, and

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- $|e^w - (4(\frac{q(q+1)}{2} + 1 + p(q+1) + \frac{p(p+1)}{2}) + 2)| < \frac{1}{2}$, whenever $|w - \log(4(\frac{q(q+1)}{2} + 1 + p(q+1) + \frac{p(p+1)}{2}) + 2)| < \xi_{p,q}$
  Also choose $\delta_0, \delta_q, \delta'_q$ so that
  $|e^w - 2| < \frac{1}{2}$, whenever $|w - \log 2| < \delta_0$
  $|e^w + (4(\frac{q(q-1)}{2} + 1) + 2)| < \frac{1}{2}$, whenever
  $|w - (\pi i + \log(4(\frac{q(q-1)}{2} + 1) + 2))| < \delta_q$, and
  $|e^w - (4(\frac{q(q-1)}{2} + 1) + 2)| < \frac{1}{2}$, whenever
  $|w - \log(4(\frac{q(q-1)}{2} + 1) + 2)| < \delta'_q$.

- Define

$$\begin{aligned}\alpha(z) &= \log 2,\ \text{if } z \in G_0 \cup \{\bigcup_{k=1}^{\infty}(L_k \cup M_k)\} \\ &= \pi i + \log(4(\frac{q(q-1)}{2}+1)+2),\ \text{if } z \in G_{p,q}, \\ &\quad p = 0,1,\ldots, q = 1,2,\ldots \\ &= \pi i + \log(4(\frac{q(q+1)}{2}+1+p(q+1)+\frac{p(p+1)}{2})+2), \\ &\quad \text{if } z \in B_{p,q},\ p = 0,1,\ldots, q = 1,2,\ldots\end{aligned}$$

$$\begin{aligned}\beta(z) &= \log 2,\ \text{if } z \in G_0 \cup \{\bigcup_{k=1}^{\infty}(L_k \cup M_k)\}\\ &= \log(4(\frac{q(q-1)}{2}+1)+2),\ \text{if } z \in B_{p,q},\\ &\quad p = 0, 1, \ldots, q = 1, 2, \ldots\\ &= \log(4(\frac{q(q+1)}{2}+1+p(q+1)+\frac{p(p+1)}{2})+2),\\ &\quad \text{if } z \in G_{p,q},\ p = 0, 1, \ldots, q = 1, 2, \ldots\end{aligned}$$

$$\epsilon(z) = \begin{cases} \delta_0, & z \in G_0 \cup \{\bigcup_{k=1}^{\infty}(L_k \cup M_k)\} \\ \delta_q, & z \in G_{p,q},\ p = 0, 1, \ldots, q = 1, 2, \ldots \\ \eta_{p,q}, & z \in B_{p,q},\ p = 0, 1, \ldots, q = 1, 2, \ldots \end{cases}$$

and

$$\epsilon_1(z) = \begin{cases} \delta_0, & z \in G_0 \cup \{\bigcup_{k=1}^{\infty}(L_k \cup M_k)\} \\ \delta'_q, & z \in B_{p,q},\ p = 0, 1, \ldots, q = 1, 2, \ldots \\ \xi_{p,q}, & z \in G_{p,q},\ p = 0, 1, \ldots, q = 1, 2, \ldots \end{cases}$$

Thank You!