Lie Groups: Quick overview with examples

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Outline of the talk

- Some standard examples

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- Groups of isometries and Symmetric spaces

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- Generalities on Lie groups and Lie Algebras

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The next natural example then would be the space $GL(n, \mathbb{R})$ of all non-singular matrices. Once we note that taking determinants is a smooth real-valued function, it easily follows that $GL(n, \mathbb{R})$ is an open subspace of $M(n, \mathbb{R})$, and hence carries an (induced) smooth structure; with respect to matrix multiplication which is again a smooth operation, it follows that $GL(n, \mathbb{R})$ is another example.

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Since the groups $O(n, \mathbb{R}), SO(n, \mathbb{R})$ act on $\mathbb{R}^n$ and preserve the euclidean norm, they also act particularly on the unit sphere S^{n-1} in $\mathbb{R}^n$. In addition, they act as a group of isometries of S^{n-1}. And, S^{n-1} can also be as the quotient space $SO(n)/SO(n-1)$.

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$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$$

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It turns out that the group $SO(n,1)$ acts as group of isometries of the hyperbolic space H^n. And, $H^n = SO(n,1)/SO(n)$.

Groups of isometries and Symmetric spaces

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In case $I(M)$ acts transitively on M, that is, for every pair of distinct point $x, y \in M$ there is an isometry that takes x to y, then the manifold M is called a **Homogeneous space** and can be expressed as the quotient space $M = I(M)/Iso_p$ where Iso_p is the subgroup that fix a point $p \in M$.

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Generalities on Lie groups and Lie Algebras

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1. $[a_1 v_1 + a_2 v_2, W] = a_1[v_1, w] + a_2[v_2, w]$
2. $[v, w] = -[w, v]$
3. $[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$

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Example: If M is a smooth manifold, the space $\mathcal{X}(M)$ of smooth vector fields on M is a Lie algebra with respect to the bracket operation $[V, W] = VW - WV$ defined in the previous lecture.

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Thus, the left invariant vector fields form an n-dimensional subspace of $\mathcal{X}(G)$.

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Similarly, the right invariant vector fields also form a Lie algebra isomorphic to the left invariant vector fields.

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Let $ad = D(Ad)$ be the differential of Ad; hence, $ad : \mathfrak{g} \to gl(\mathfrak{g})$.

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One has the so called Killing form on $\mathfrak{g}$ defined by $\langle g_1, g_2 \rangle = -\text{trace}\,(ad(g_1)ad(g_2))$, which gives rise the standard Riemannian metric on G if and only if G is compact and semisimple.

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The case of $G = SO(n+1, \mathbb{R}), SO(n, 1)$ and $H = SO(n)$ giving rise to the homogeneous spaces S^n and H^n were mentioned earlier.

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Further, the case of $H^n = SO(n, 1)/SO(n)$ admits interesting discrete subgroups Γ whose quotient spaces carry finite Riemannian volume.

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