Manifolds and Geometry: A quick warm-up

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Outline of the talk

- Quick review of the definition of a manifold

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- Smooth functions, tangent vectors and tensor fields

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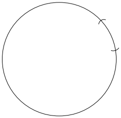

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Real line $\mathbb{R}$

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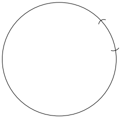

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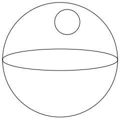
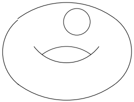


Sphere S^2



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And, in the 2-dimensional examples, each point $p \in M$ has a neighbourhood in M that is homeomorphic to an open disc.

More precisely, following is a formal definition of an n-dimensional manifold.

Definition: An n-dimensional *manifold*, $n \in \mathbb{N}$, is a set M together with a family $\{(\varphi_\alpha, U_\alpha)\}$ of 1-1 maps $\varphi_\alpha : U_\alpha \to M$ of open balls U_α in $\mathbb{R}^n$ into M, called *coordinate charts*, such that

1. $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$
2. For each pair of indices α, β with $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = W (\neq \emptyset)$, the sets $\varphi_\alpha^{-1}(W)$ and $\varphi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and the maps $\varphi_\beta^{-1} \circ \varphi_\alpha$ and $\varphi_\alpha^{-1} \circ \varphi_\beta$, called *transition maps*, are continuous.
3. The family $\{(\varphi_\alpha, U_\alpha)\}$ is maximal with respect to conditions (1) and (2).

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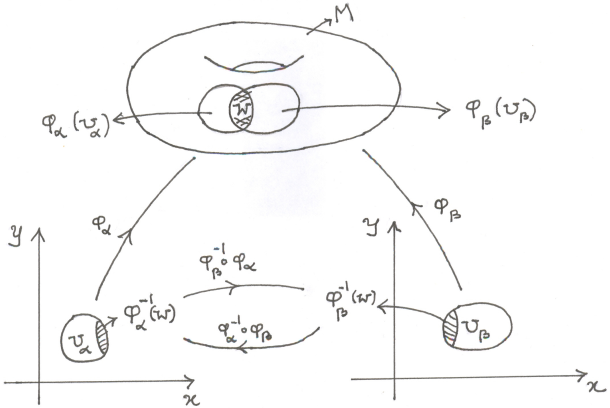

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If the transition maps are differentiable of any order then M is called a *smooth manifold*.

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(4) If $p, q \in M$, with $p \neq q$, then either p and q are in a single $\varphi_\alpha(U_\alpha)$ or there are indices α, β such that $p \in \varphi_\alpha(U_\alpha)$, $q \in \varphi_\beta(U_\beta)$, with $\varphi_\alpha(U_\alpha)$ and $\varphi_\beta(U_\beta)$ disjoint.

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(5) There are countably many $\varphi_\alpha(U_\alpha)$ that cover M.

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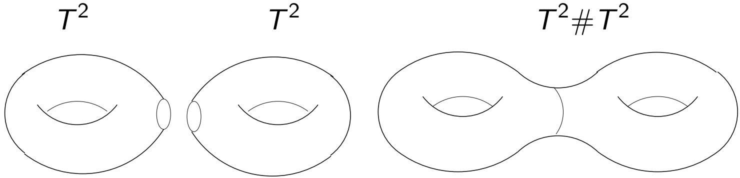
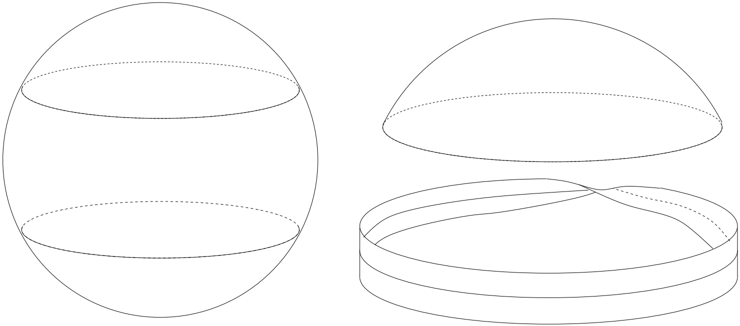


The space obtained by identifying every pair of diametrically opposite points of the sphere S^2 is called a *Real projective plane*; it can be visualised as follows:



Smooth functions, tangent vectors and tensor fields

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A function $f : M \to \mathbb{R}$ is said to be *differentiable* at a point $p \in M$ if for a coordinate neighbourhood $\phi_\alpha(U_\alpha)$ in M that contains the point p, the function $f \circ \phi_\alpha$ from U_α to $\mathbb{R}$ is smooth at the point $\phi_\alpha^{-1}(p) \in U_\alpha \subset \mathbb{R}^n$. A *differentiable map* between differentiable manifolds is defined similarly by invoking local charts.

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The condition 2 in the definition then assures that smoothness of f at p is well-defined.

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In particular, the following lemma holds.

Lemma: *Given any neighbourhood U of a point p in M there is a function $f \in C^\infty(M)$, called a bump function at p, such that*

1. $0 \leq f \leq 1$ on M.
2. $f = 1$ on some neighbourhood of p.
3. $\mathrm{supp}\, f \subset U$.

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It turns out that the set $TM = \cup_{p \in M} T_p M$ of all tangent vectors at all points of M, called the tangent bundle of M, is also a smooth manifold of dimension $2n$ and the projection $\pi : TM \to M$ that sends a tangent vector to its foot-point is a smooth map.

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Let $\mathcal{X}^*(M)$ denote the dual module of $\mathcal{X}(M)$. The elements of $\mathcal{X}^*(M)$ are called *one-forms*.

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On an n-dimensional smooth manifold, any differential k-form, for $k > n$ is zero.

Rule of differentiation or the Connection

We have seen that on a manifold M, in general, apart from the notion of smooth functions, one also has the notion of tensor fields (functions are tensor fields of type $(0, 0)$). Just as one has *vector derivatives* of functions, one can differentiate tensor fields of type $(0, s)$ in the direction of a tangent vector v.

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Let us first see the notion of differentiating a vector field W (which is a tensor field of type $(0,1)$) along a vector field V.

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In particular, it is described as a function $D : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that for any vector fields $X, V, W \in \mathcal{X}(M)$ and any function $f \in C^\infty(M)$, one has

1. $D_V W$ is $C^\infty(M)$-linear in V
2. $D_V W$ is $\mathbb{R}$-linear in W
3. $D_V(fW) = (Vf)W + fD_V W$

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We say that a connection is compatible with the underlying differential structure if

$$D_V W - D_W V = [V, W], \ \textit{where} \ [V, W] = VW - WV$$

Riemannian manifolds and the Levi-Civita Connection

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Further, positive definiteness of g implies that at each point p of U, the matrix $(g_{ij}(p))$ is invertible; its inverse matrix is denoted by $(g^{ij}(p))$.

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Compatibility with the Riemannian metric means that, apart from the conditions that the rule D must satisfy, it should also satisfy

$$X\langle V, W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle$$

It is a fundamental result in Riemannian geometry that there is a unique rule of covariant differentiation D on M, given by the more commonly known *Riemannian connection* or the *Levi-Civita connection*, which is compatible with the Riemannian metric.

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Uniqueness of the Riemannian connection can be seen from the fact that D can be characterised by the formula

$$2\langle D_V W, X\rangle = V\langle W, X\rangle + W\langle X, V\rangle - X\langle V, W\rangle - \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle$$

One of the most important geometric invariants of a Riemannian manifold is the *Riemann curvature tensor* which is a $(1,3)$-tensor field $R : \mathcal{X}(M)^3 \to \mathcal{X}(M)$ defined by

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Given a 2-dimensional subspace P of T_pM and an orthonormal basis $\{v, w\}$ for P, the *sectional curvature* $\kappa(P)$ of the plane P at a point $p \in M$ is given by

$$\kappa(P) = \langle R(v, w)w, v\rangle = \langle R(V, W)W, V\rangle(p)$$

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Let $\widetilde{M}$ be a complete, simply connected Riemannian manifold with constant sectional curvature K. Then $\widetilde{M}$ is isometric to one of the model spaces $\mathbb{R}^n, \mathbb{S}^n$ or $\mathbb{H}^n$ with constant sectional curvature K.