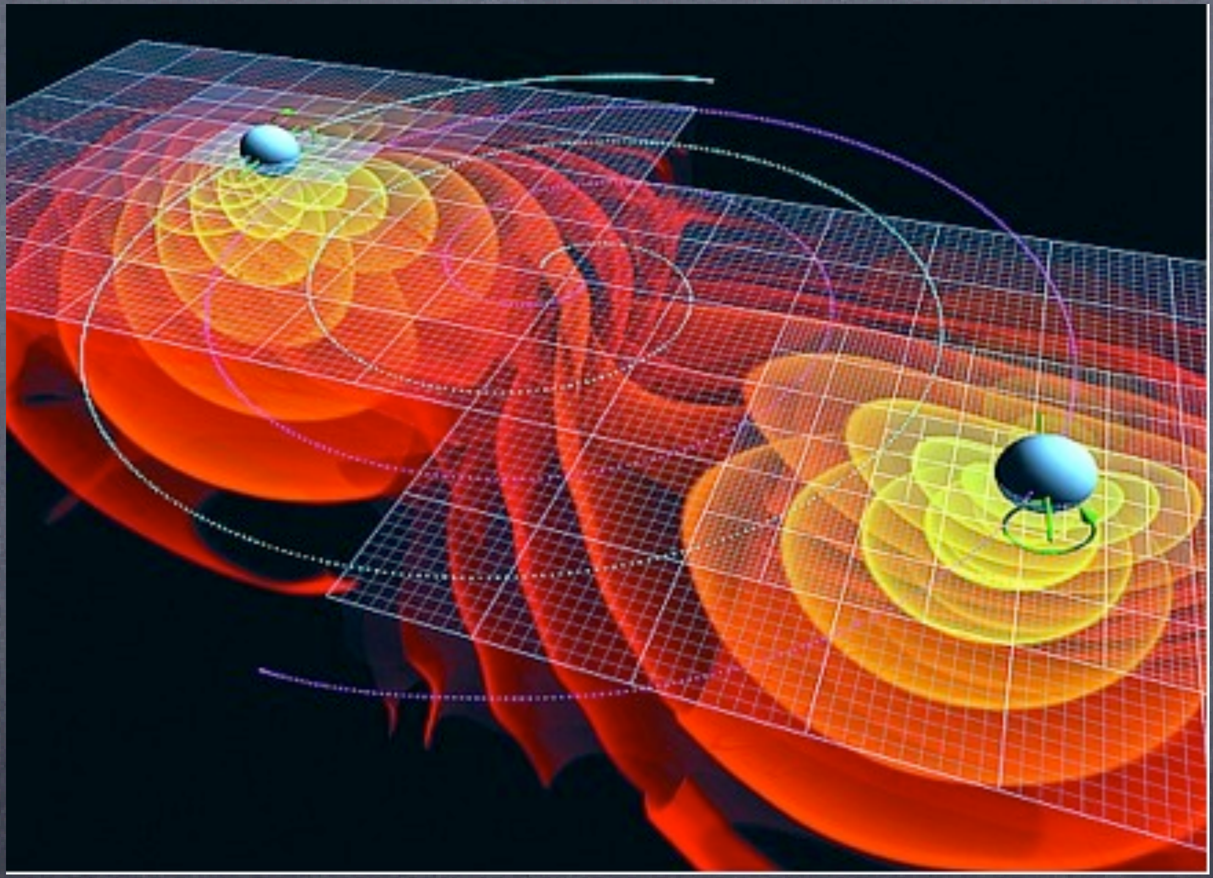
Introduction to Theory and Numerics of Partial Differential Equations III: Properties and Stability of Finite Difference Schemes



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Lab goals for today

- Code up the wave equation with periodic boundary conditions.
- Derive the second order accurate centered finite difference formula in one space dimension using Taylor expansions.

example: wave equation in 1D

- Start with 2nd order form: $\phi_{,tt} = c^2 \phi_{,xx}$
- Can obtain a mixed first/second order form: $\phi_{,t} = c\pi, \quad \pi_{,t} = c\phi_{,xx}$
- or complete first order reduction:

$$\phi_{,t} = c\pi, \quad \phi_{,x} = \psi, \quad \pi_{,t} = c\psi_{,x}, \quad \psi_{,t} = c\pi_{,x}$$

  - where $\phi_{,x} = \psi$ now plays the role of a constraint which is preserved by the evolution equations:

$$\partial_t(\phi_{,x} - \psi) = \partial_x \partial_t \phi - \partial_t \psi = \partial_x \pi - \partial_x \pi = 0.$$

  - The evolution equation for ϕ decouples, and we may focus on the system of equations for ψ and π, which has the form

$$\partial_t u = A\, \partial_x u, \quad u = \{\pi, \psi\} \qquad A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

  - A has eigenvalues $\pm c$, and eigenvectors $(-1,1)$ & $(1,1)$, correspondingly the characteristic variables are $u_\pm = \psi \pm \pi$ and satisfy advection equations $\partial_t u_\pm = \pm c\, \partial_x u_\pm$
  - Solution preserves norm -> WP, as seen before.

example: weakly hyperbolic system

- Consider the following simple system in 1D:

$$\partial_t u = \partial_x (u + v), \quad \partial_t v = \partial_x v$$

- In our matrix notation this becomes:

$$\partial_t u = A\,\partial_x u, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- A has only 1 proper eigenvector (1,0), with eigenvalue 1 (and "trivial" eigenvector 0), thus no complete set of eigenvectors and the system is only **weakly hyperbolic**.

- Explicit solution with frequency ω, U = (u,v):

$$u = \omega t \sin \omega(t + x)\,, \quad v = \sin \omega(t + x)$$

- Compute L^2 norm for data with u(0) = 0 and frequency ω

$$\frac{||U(t)||}{||U(0)||} = \sqrt{1 + t^2 \omega^2}$$

Linear, frequency dependent growth -> ~~W~~P

example: York-ADM in 1D

- York-ADM, $g_{ij} = g_{ij}(x,t)$ - plane wave traveling in x-direction
- gauge condition: densitized lapse: $\alpha = \sqrt{\det {}^3 g}$
- linearized around flat space:

$$\begin{aligned}
&\dot{h}_{ii} = 2K_{ii} \\
&\dot{K}_{xx} = \tfrac{1}{2}\partial_{xx} h_{xx} + \partial_{xx}\left(h_{yy} + h_{zz}\right) \\
&\dot{K}_{jj} = \tfrac{1}{2}\partial_{xx} h_{jj} \qquad (j = y, z)
\end{aligned}$$

- Jordan normal form of first order reduction: all characteristic speeds real, but 2 Jordan blocks

$$\mathrm{JordanForm}(A) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

discretization example: wave equation

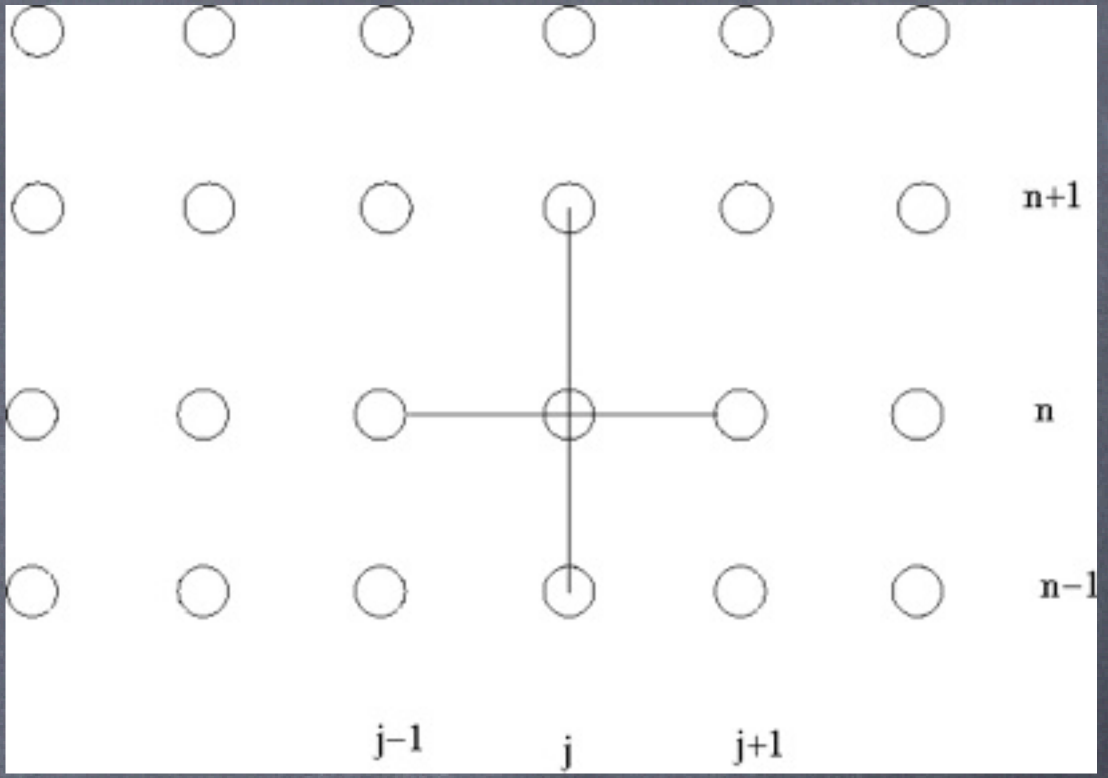
- Discretize wave equation straightforwardly to 2nd order accuracy.

$$u_{tt} = u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0.$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

- Use periodic boundaries for simplicity (all points equal!)

$$u(0,t) = u(1,t)$$



- grid: $\quad t^n = n\Delta t, \quad x_j = (j-1)\,\Delta x, \quad \Delta t = \lambda \Delta x$

- leapfrog algorithm:

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = (u_{tt})_j^n + O(\Delta t^2)$$

$$\lambda = \frac{\Delta t}{\Delta x}$$

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \lambda^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right)$$

discretization example: wave equation II

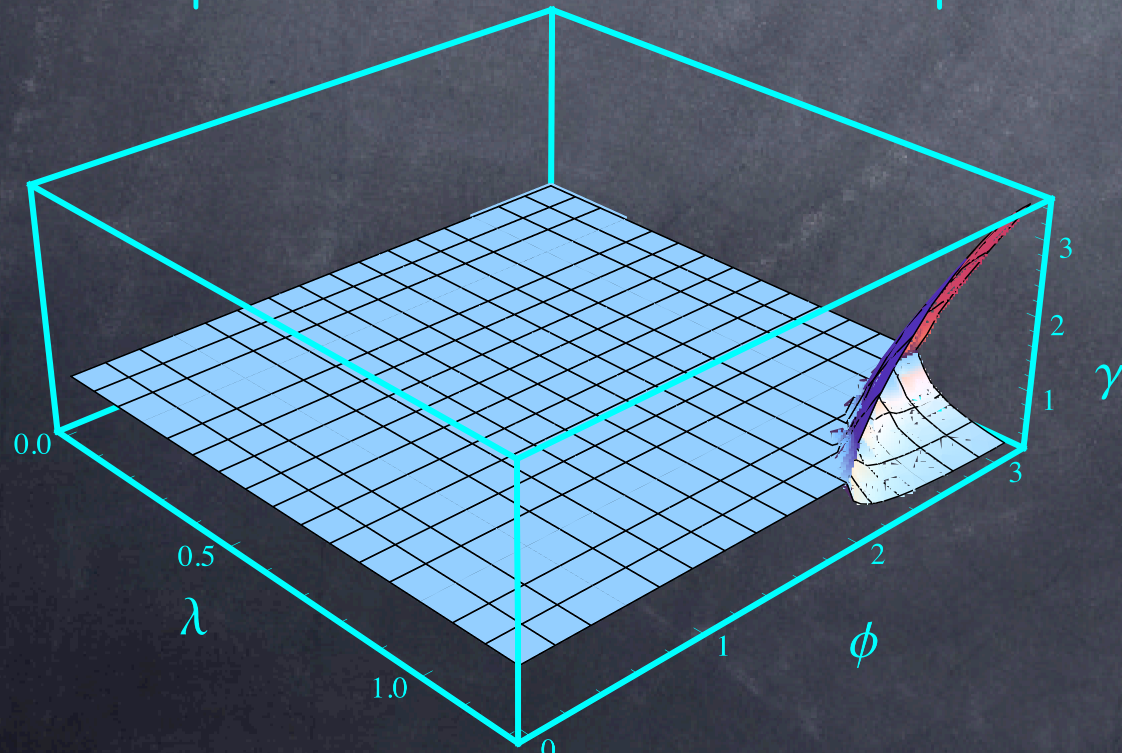
- Superpose solution of Fourier modes $e^{i \omega j \Delta x}$ -> $u = q^n e^{i \omega j \Delta x}$
  - wave number/frequency , $|\omega \Delta x| \leq$ pi
  - call q amplification factor, q > 1 => unstable algorithm
- for a smooth solution, the "signal" is concentrated at small $\xi = \omega \Delta x$
- insert ansatz into discretization

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \lambda^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

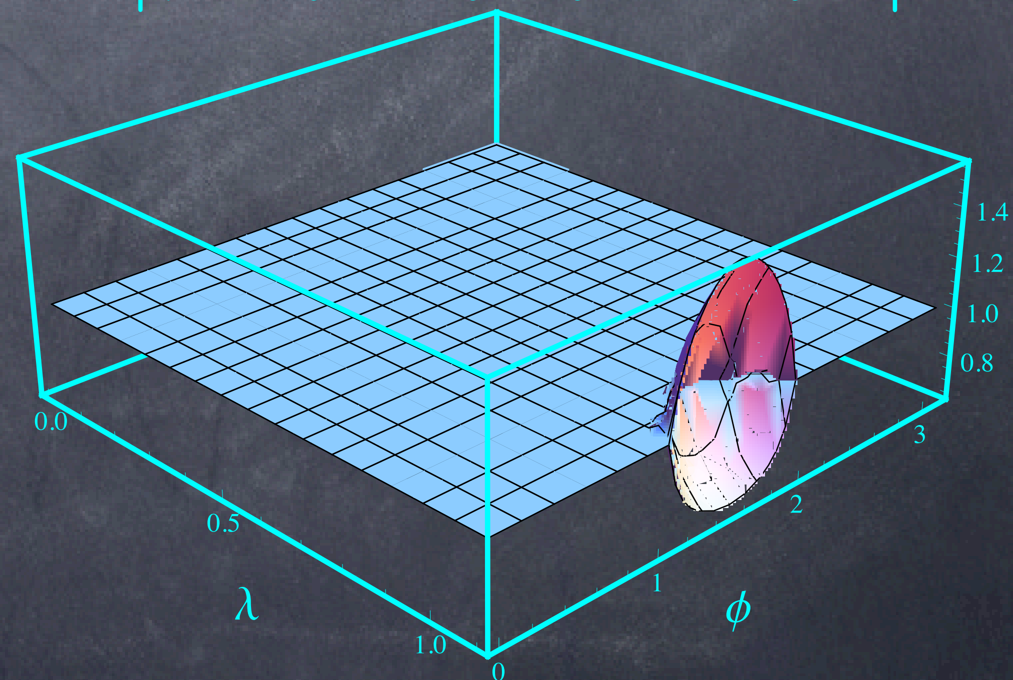
- -> obtain quadratic equation:

$$\xi \left(2\lambda^2 - 2\lambda^2 \cos(k\Delta x) - 2\right) + \xi^2 + 1 = 0$$

amplification factors for wave eq.

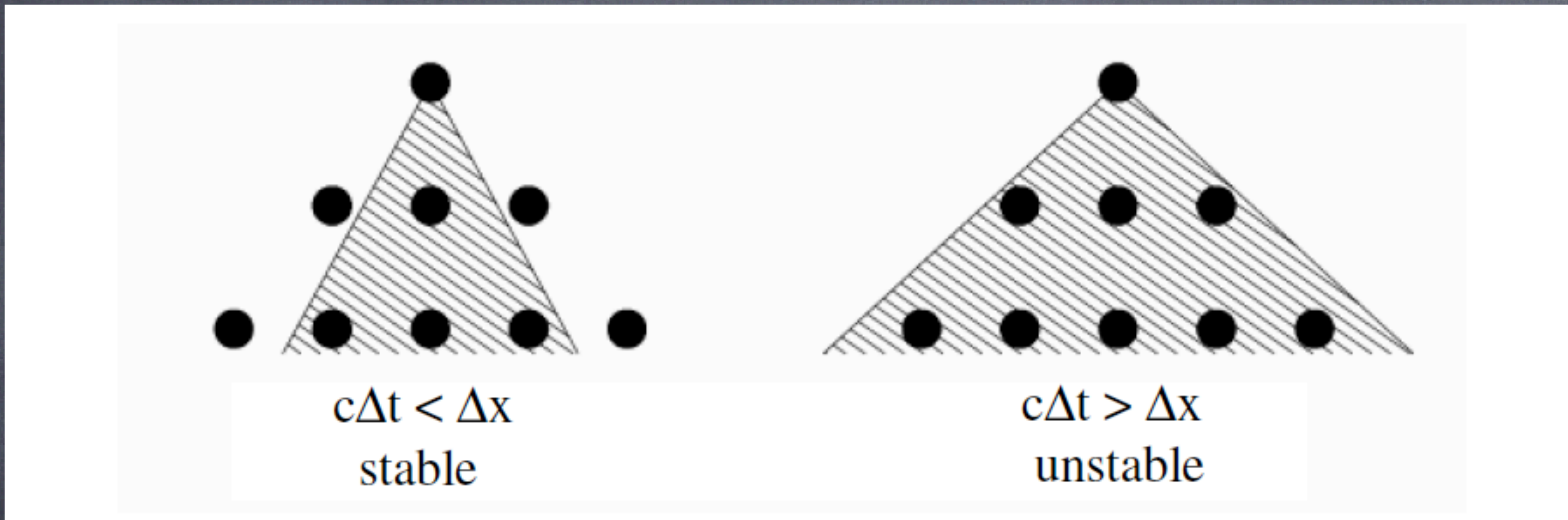


amplification factors for advection eq.



Courant-Friedrichs-Lewy condition

- Explicit time stepping schemes impose limits on Δt.



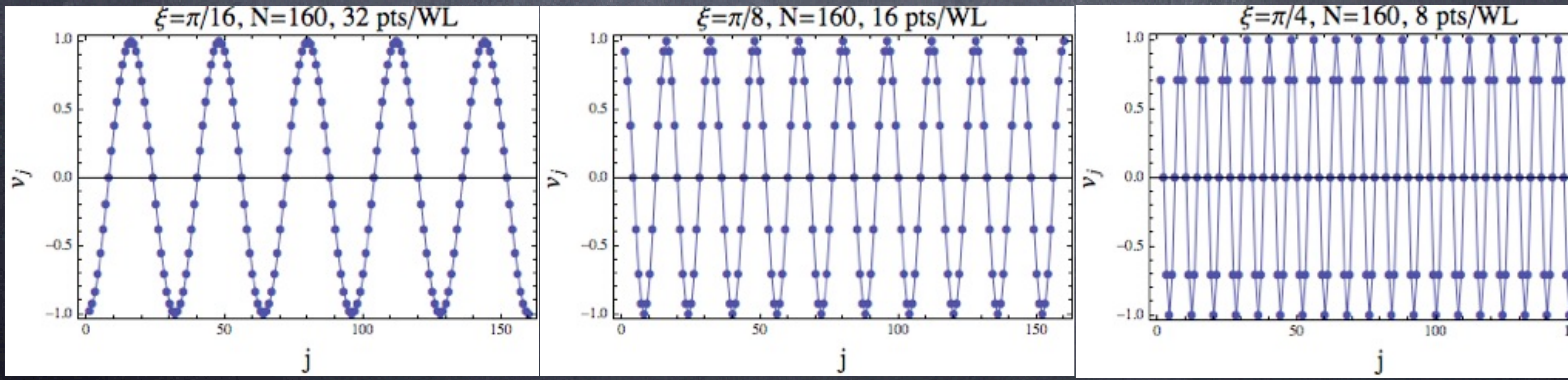
- Geometric interpretation: the numerical domain of dependence should include the physical domain of dependence. If the physical DoD is larger, we can't converge to the correct solution, since relevant physical information is neglected. Lax => unstable

- necessary but not sufficient

- parabolic: const. $\Delta t < \Delta x^2$ -> use implicit methods

Of grids and frequencies

- consider equispaced grid in d dimensions, tensor product of 1-D grids $x_j = j\,h$, $j = 0,1,\ldots,N-1$
    - inner product $(u,v)_h = \Sigma\, u_j\, v_j\, h^d$, $||v||_h = (v,v)^{1/2}$
    - Stability: $\exists$ K, α: $||v^n||_h \leq K\, e^{\alpha\, t_n}\, ||v^0||_h\ \forall\ n$: $t_n = n\, k = n\, \Delta t$, $\forall\ v^0$
- can represent frequencies $\omega_j = -N/2 +1, \ldots, N/2$, $\xi_j = \omega_j\, h = -\pi + 2\pi/N, -\pi + 4\pi/N, \ldots, \pi$ (N even, highest frequency represented)
- grid function v at time step n: $$v_j^n = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_\omega e^{i\omega x_j}\, \hat{v}^n(\omega)$$
- when smooth functions are represented and well resolved (many gridpoints per wavelength) on the grid, "signal" is concentrated at low frequencies.



Deriving finite difference stencils

- Taylor expansions, or approximating polynomials.
- Example: derive second order centered finite difference stencils.
- -> need to approximate solution by second order polynomial.

$$f(x) = a_0 + a_1 x + a_2 x^2$$

- 3 coefficients a_i -> need 3 gridpoints to define their values.
- Consider grid X = {-h, 0, h} -> equations:

$$\{f(-h) = f_1,\, f(0) = f_2,\, f(h) = f_3\}$$

- solution:

$$a_0 = f_2, a_1 = \frac{f_3 - f_1}{2h}, a_2 = -\frac{f_3 + 2f_2 + f_1}{2h^2}$$

- Take derivatives of f to obtain stencil coefficients, and set x=0:

$$\partial_x f \approx \frac{f_3 - f_1}{2h} \qquad \partial_{xx} f \approx \frac{f_3 - 2f_2 + f_1}{h^2}$$

Method of Lines

- Direct space-time discretizations are hard to generalize to higher order, and stability has to be analyzed case by case.
- Convert PDEs to coupled ODEs, discretize space and time separately. Example:

$$\partial_t u(x,t) + \partial_x u(x,t) = 0 \quad \rightarrow \quad \partial_t u(i,t) = -\frac{u(i+1,t) - u(i-1,t)}{2\Delta x}$$

- Integrate ODEs with any stable ODE integrator.
  - explicit: subject to time step conditions, e.g. RK3, RK4, ...
  - implicit: no or negligible time step restriction for stability
- Easy to plug in different time integrators, space discretizations, boundary conditions, ... Flexibility and robustness are key virtues in scientific computing!
- **First order constant coefficient hyperbolic systems are stable with centered finite differencing and simple time step restriction.**