COVERING SPACES and FUNDAMENTAL GROUPS

Vikram T. Aithal
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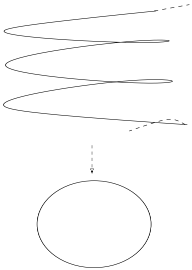
The Prototype

- The map $p : \mathbb{R} \longrightarrow S^1$, which sends $t \mapsto e^{2\pi i t}$



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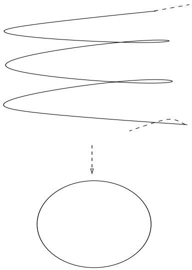
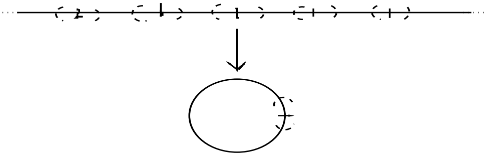
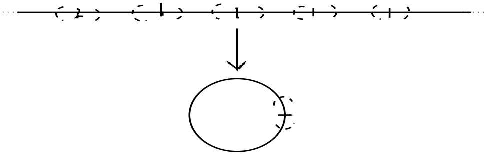


- This map has some interesting properties.



For every point on the circle, there is an open arc U such that :

- $p^{-1}(U) = \bigsqcup_{i \in I} U_i$, where for every $i \in I$, $U_i \subseteq \mathbb{R}$ is an open interval



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- $p|_{U_i} : U_i \longrightarrow U$ is a homeomorphism.

Let $X, \widetilde{X}$ be topological spaces. A map $p : \widetilde{X} \longrightarrow X$ is said to be a *covering map* if for every $x \in X$, there is an open neighbourhood $U \ni x$ such that,

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- $\widetilde{X}$ is said to cover X.
- The open set $U \subseteq X$ defined above is called an *evenly covered* neighbourhood of x.
- The map $p : \mathbb{R} \longrightarrow S^1$, $t \longrightarrow e^{2\pi it}$ is a covering map.

- Let $p_n : S^1 \longrightarrow S^1$ be such that $p_n : z \mapsto z^n$.

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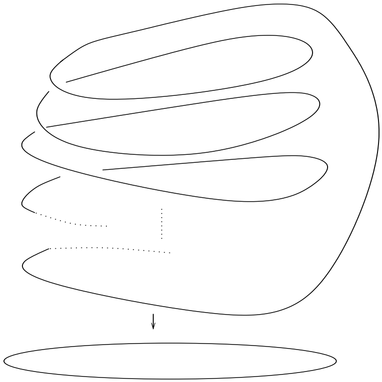

- Let $\widetilde{X} := \{\,(x, y) \mid 0 < x < 1 \,\text{and}\, y = 1, 2, 3, 4\,\}$ and $X := \{\,(x, 0) \mid 0 < x < 1\,\}$

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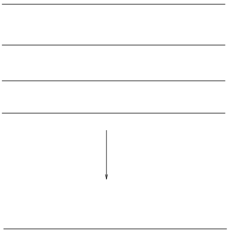

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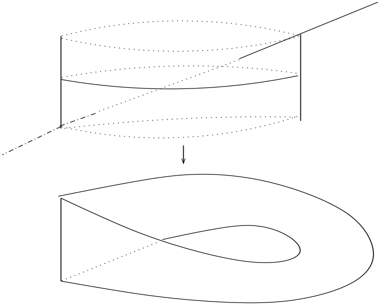

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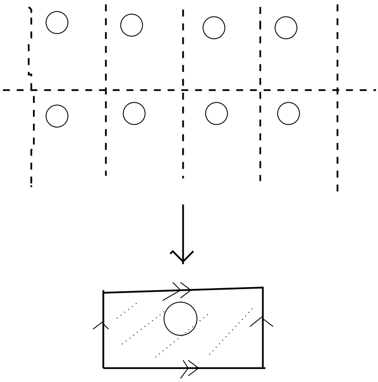
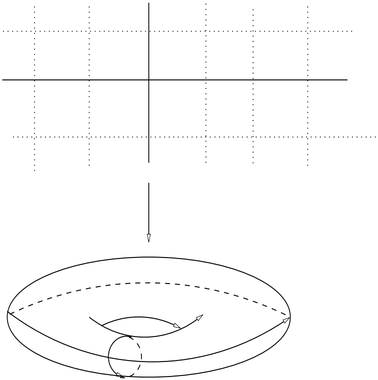


- In fact, after making the appropriate identifications, we get a covering map $p : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$.



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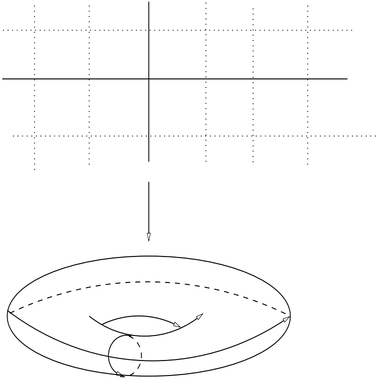

- This can also be realised as $p((s,t)) = (e^{2\pi is}, e^{2\pi it})$

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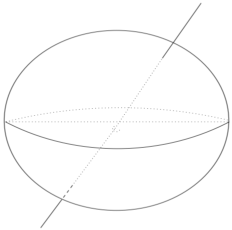
- Note that $S^2/\sim \ \cong \mathbb{RP}^2$



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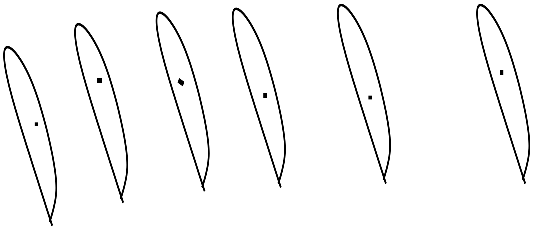
- G is said to act *evenly* on $\widetilde{X}$ if :
- Given any $\widetilde{x} \in \widetilde{X}$, there is an open set $\widetilde{U} \ni \widetilde{x}$ such that $\{ \, g\widetilde{U} \, | \, g \in G \, \}$ is a pairwise disjoint family, i.e. for every $g_1, g_2 \in G$, $g_1\widetilde{U} \cap g_2\widetilde{U} = \phi$.



- Exercise : Let G be a finite group acting on a Hausdorff space $\widetilde{X}$. Assume the action of G on $\widetilde{X}$ is free, i.e. if for any $g \in G$, there exists x such that $g \cdot x = x$, then $g = e$. Then show that G acts evenly on $\widetilde{X}$.

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- Most of the examples we considered above were of this form!

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- Exercise : Assume $\widetilde{X}, X$ to be connected, Hausdorff. Assume $\widetilde{X}$ is compact. Show that any surjective local-homeomorphism $p : \widetilde{X} \longrightarrow X$ is a covering map.

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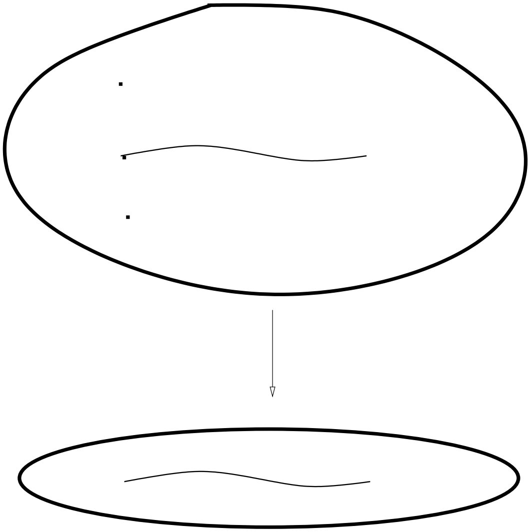
- The following property characterises covering maps :
- **Theorem** Let $p : \widetilde{X} \longrightarrow X$ be a covering map. Let $c : [0,1] \longrightarrow X$ be a curve. Let $\widetilde{x} \in p^{-1}\{c(0)\}$ be given. Then, there exists a unique curve $\widetilde{c} : [0,1] \longrightarrow \widetilde{X}$ such that $\widetilde{c}(0) = \widetilde{x}$ and $p \circ \widetilde{c} \equiv c$.



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- f_0 and f_1 are said to be *homotopic* if there exists a continuous map $H : Y \times [0,1] \longrightarrow X$ such that :
- $H(y,0) = f_0(y)$ and $H(y,1) = f_1(y)$ for every $y \in Y$

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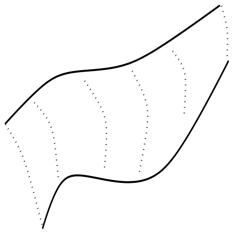
- The map H is called a *homotopy* joining c_0 and c_1



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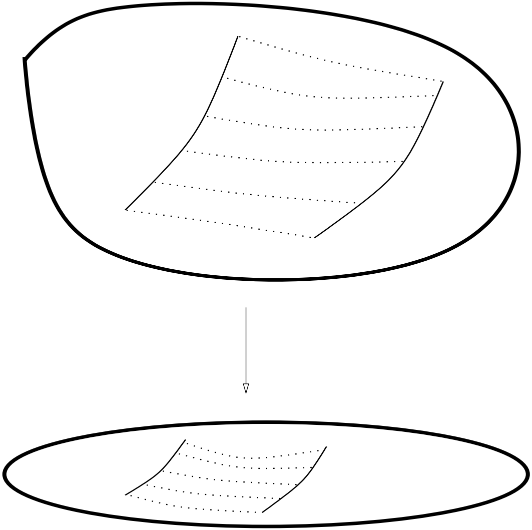
- **Theorem** Let $p : \widetilde{X} \longrightarrow X$ be a covering map. Let $c_0, c_1 : [0,1] \longrightarrow X$ be two homotopic curves, with homotopy $H : [0,1] \times [0,1] \longrightarrow X$. Let $\gamma_0 : [0,1] \longrightarrow \widetilde{X}$ be a lift of the curve c_0. Then there exists a map $G : [0,1] \times [0,1] \longrightarrow \widetilde{X}$, such that $G(t,0) = \gamma_0(t)$ and $p \circ G \equiv H$.



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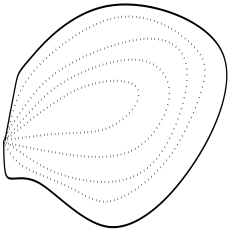
- A connected space $\widetilde{X}$ is said to be *simply connected* if for any $\widetilde{x} \in \widetilde{X}$ we have : every loop $\gamma : [0,1] \longrightarrow \widetilde{X}$ based at $\widetilde{x}$ is homotopic to the constant loop $\alpha \equiv \widetilde{x}$



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- $\widetilde{X}$ is called the universal cover of X.
- Let $f : Y \longrightarrow X$ be a continuous map. Let $p : \widetilde{X} \longrightarrow X$ be any covering map. Assume Y is simply connected. Then there exists a map $\widetilde{f} : Y \longrightarrow \widetilde{X}$ such that $p \circ \widetilde{f} \equiv f$.

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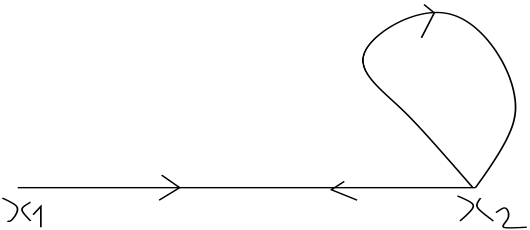
- $\pi_1(X, x_0) := \{ \ [\alpha] \mid \alpha \text{ is a loop based at } x_0 \ \}$
- $\pi_1(X, x_0)$ is called the fundamental group of X with basepoint x_0

- If X is path connected, then for any $x_1, x_2 \in X$, $\pi_1(X, x_1) \cong \pi_1(X, x_2)$.



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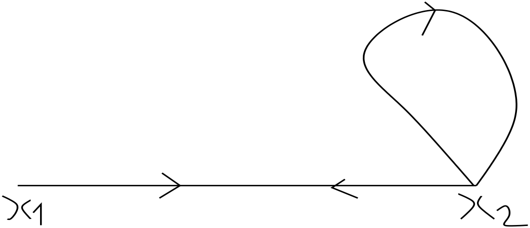

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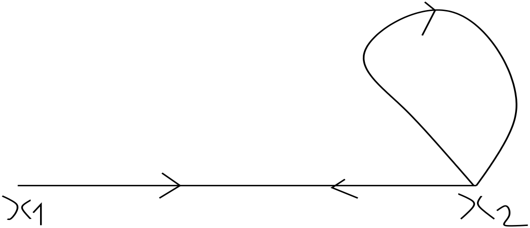

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- If $\widetilde{X}$ is simply connected, then $\pi_1(\widetilde{X}, *)$ is trivial.

-

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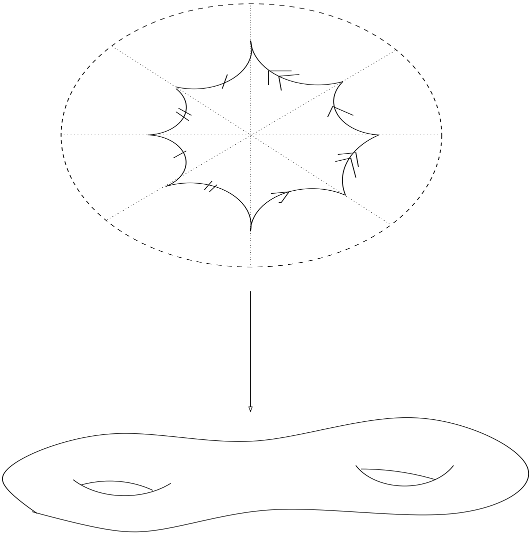
- $$\pi_1(X, *) \cong \left\langle a_1, b_1, \cdots a_g, b_g \mid \prod_{1=1}^{g} [a_i, b_i] = 1 \right\rangle$$



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- Let $p : \widetilde{X} \longrightarrow X$ be a covering such that $\widetilde{X}$ is simply connected. (X connected). For every subgroup $H \leq \pi_1(X,*)$ there is a connected covering $q : Y \longrightarrow X$ such that $\pi_1(Y,*) \cong H$.

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- $$\left\{ \begin{array}{ccc} \{1\} & \leftrightarrow & \widetilde{X} \\ \downarrow & \vdots & \downarrow \\ H & \leftrightarrow & Y \\ \downarrow & \vdots & \downarrow \\ \pi_1(X,*) & \leftrightarrow & X \end{array} \right\}$$

THANK YOU