# Optionality and Volatility

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# Problem Specification

- Suppose we are given a continuum of option prices of different maturities at the same strike.
- Assuming zero cost of carry for the underlying and for the options, no arbitrage implies that the option prices must be (weakly) above intrinsic value and (weakly) increasing in time to maturity.
- For the rest of this talk, we will favor puts over calls and we will consider an underlying whose price is real-valued rather than non-negative (eg temperatures, spreads).
- So suppose we are given a continuum of put prices on a real-valued underlying. Can we convert this information into a curve that is unconstrained eg allowed to be constant?

# Forward Implied Bachelier Volatilities

- The time-dependent Bachelier model assumes that the risk-neutral process for the underlying forward price $F$ is Gaussian with time-dependent volatility viz:

$$dF_t = a(t)dW_t, \qquad t \geq 0.$$

- For each $T > 0$, there is a closed form formula relating the put price $P(T)$ to the (deterministic) quadratic variation $\langle F \rangle_t \equiv \int_0^T a^2(t)dt$.
- As a result, one can (numerically) convert the given curve of arbitrage-free put prices $\{P(T), T > 0\}$ into a term structure $\{a(t), t \geq 0\}$ of forward implied Bachelier volatilities.
- Conversely, if one starts with *any* term structure of forward implied Bachelier volatilities $\{a(t), t \geq 0\}$, one can convert into an arbitrage-free curve of put prices $\{P(T), T > 0\}$.

# Problem Specification

- Now suppose instead that we are given a continuum of put prices of different strikes at one maturity.
- No arbitrage implies that the put prices must be above intrinsic value, increasing in strike price, and convex in strike price.
- One can convert the given curve of arbitrage-free put prices into a smile of implied Bachelier volatilities.
- The converse does not hold. If one starts with *any* smile, and uses the Bachelier valuation formula to convert to put prices, the resulting curve is not necessarily arbitrage-free.
- If the underlying has non-negative price, the same issues arise if we use Black in place of Bachelier.

# Single Maturity Setting

- To summarize thus far, a term structure of arbitrage-free put prices is one-to-one with a term structure of forward implied Bachelier volatilities. A strike structure of arbitrage-free put prices can be converted into a strike structure of implied Bachelier volatilities, *but not conversely*.
- For some underlyings eg. commodity futures, the market provides simultaneous option quotes at several strikes, but only one maturity date.
- If we pretend that we are given a continuum of co-terminal option prices that are arbitrage-free, we may ask if there exists a one-to-one transformation which need only be real-valued.
- Schweitzer and Wissel (F&S 2008) answer in the affirmative with a concept they call strike vol.
- This talk proposes a simpler alternative which we call money vol.

# Definition of Optionality

- Before we can define money vol, we first need to define a concept called Optionality.
- Assume no arbitrage and hence the existence of a forward probability measure $\mathbb{Q}_{\mathbb{T}}$.
- Let $F_t$ be the forward price at time $t \in [0, T]$ with maturity date $T$.
- Let $P(K)$ and $C(K)$ respectively denote the initial forward prices of European options maturing at $T$ and paying off $(K - F_T)^+$ and $(F_T - K)^+$ respectively.
- Optionality at strike $K \in \mathbb{R}$ for fixed maturity date $T$ is defined as: $\mathcal{O}(K) \equiv \mathbb{Q}_{\mathbb{T}}\{F_T \leq K\}C(K) + \mathbb{Q}_{\mathbb{T}}\{F_T > K\}P(K)$.
- In words, Optionality at strike $K \in \mathbb{R}$ is a convex combination of the forward prices of the put and call at that strike. Each option's forward price is weighted by its (risk-neutral) probability of finishing *out* of the money (OTM). For $K \neq F_0$, the higher priced option gets the lower weight and vice versa.

# Optionality from Option Prices

- Recall that Optionality at strike $K \in \mathbb{R}$ is defined as the following convex combination of the forward prices of put and call at that strike:

$$\mathcal{O}(K) \equiv \mathbb{Q}_{\mathbb{T}}\{F_T \leq K\}C(K) + \mathbb{Q}_{\mathbb{T}}\{F_T > K\}P(K).$$

- If (forward) option prices are differentiable in their strike $K$, then the strike derivatives of these prices reveal the risk-neutral probabilities:

$$P'(K) = \mathbb{Q}_{\mathbb{T}}\{F_T \leq K\} \text{ and } -C'(K) = \mathbb{Q}_{\mathbb{T}}\{F_T > K\}.$$

- As a result, Optionality at strike $K \in \mathbb{R}$ is observed from the level and slope information in put and call prices:

$$\mathcal{O}(K) = P'(K)C(K) - C'(K)P(K).$$

- Hence, if option prices are arbitrage-free, then Optionality is non-negative.

# Optionality as a Wronskian

- Recall that Optionality at strike $K \in \mathbb{R}$ is observed from the level and slope information in put and call prices:

$$\mathcal{O}(K) = P'(K)C(K) - C'(K)P(K).$$

- If we form a 2x2 matrix with option prices and their strike derivatives, then Optionality is the determinant of this matrix:

$$\mathcal{O}(K) = \det \begin{pmatrix} C(K) & P(K) \\ C'(K) & P'(K) \end{pmatrix}.$$

- This determinant is called the Wronskian of the put and call prices. We later develop a 2nd order ODE which option prices solve. The positivity of the Wronskian guarantees that the put and call are two linearly independent solutions.

# Single Polarity Representation of Optionality

- Recall that Optionality is a Wronskian of put and call prices:

$$\mathcal{O}(K) = P'(K)C(K) - C'(K)P(K).$$

- Suppose we use Put Call Parity to eliminate the calls:

$$C(K) = F_0 - K + P(K) \text{ so } C'(K) = -1 + P'(K).$$

- Then Optionality is the following functional of put prices:

$$\mathcal{O}(K) = P(K) - (K - F_0)P'(K).$$

- Suppose we had instead used Put Call Parity to eliminate the puts:

$$P(K) = K - F_0 + C(K) \text{ so } P'(K) = 1 + C'(K).$$

- Then optionality is the following functional of call prices:

$$\mathcal{O}(K) = C(K) - (K - F_0)C'(K).$$

- Hence, Optionality is the same price functional for puts and calls.

# Optionality and Time Value

- Recall the single polarity representations of Optionality:

$$\mathcal{O}(K) = C(K) - (K - F_0)C'(K).$$

$$\mathcal{O}(K) = P(K) - (K - F_0)P'(K).$$

- When $K = F_0$, we see that ATM calls and puts have the same value. This common value is both Optionality and time value.
- Differentiating Optionality w.r.t. $K$ implies $\mathcal{O}'(K) = (F_0 - K)q(K)$ where $q(K) \equiv C''(K) = P''(K)$ is the risk-neutral PDF of terminal price. Since $q(K) \geq 0$, Optionality is decreasing for $K > F_0$ and increasing for $K < F_0$.
- Since Optionality is non-negative, the Optionality profile is that of a bell-shaped curve. The area under this curve is positive and finite, but not necessarily one.
- In models where option price is a twice differentiable function of strike price, Optionality is a differentiable function of strike price.

# From Optionality to Option Prices

- Recall again the single polarity representations of Optionality:

$$\mathcal{O}(K) = C(K) - (K - F_0)C'(K).$$

$$\mathcal{O}(K) = P(K) - (K - F_0)P'(K).$$

- Suppose $C(K)$ and $P(K)$ are unknown and that $\mathcal{O}(K) \geq 0$ is given.
- To obtain call prices across strikes, we can solve the first order ODE subject to the condition that $\lim_{K\uparrow\infty} C(K) = 0$. The solution is:

$$C(K) = (K - F_0)\int_K^\infty \frac{\mathcal{O}(J)}{(J - F_0)^2} dJ.$$

- To obtain put prices across strikes, we can solve the first order ODE subject to the condition that $\lim_{K\downarrow-\infty} P(K) = 0$. The solution is:

$$P(K) = (F_0 - K)\int_{-\infty}^K \frac{\mathcal{O}(J)}{(J - F_0)^2} dJ.$$

- We now explore alternative interpretations of Optionality.

# Optionality as Truncated Forward Price

- Recall the representation of Optionality in terms of call price:

$$\mathcal{O}(K) = C(K) - (K - F_0)C'(K).$$

- The RHS is the risk-neutral exp'd value of a truncated forward payoff:

$$E^{\mathbb{Q}_T}[(F_T - K)^+ + (K - F_0)1(F_T > K)] = E^{\mathbb{Q}_T}[(F_T - F_0)1(F_T > K)].$$

- Since: $\lim_{K \downarrow -\infty} E^{\mathbb{Q}_T}(F_T - F_0)1(F_T > K) = 0$, it is no surprise that Optionality is non-negative.
- As $K$ increases from negative infinity towards $F_0$, negative outcomes are taken away from the calculation and so Optionality increases.
- When $K$ passes $F_0$ positive outcomes are taken away from the calculation and so Optionality decreases. As $K \uparrow \infty$, all outcomes are taken away from the calculation and so Optionality approaches zero.
- As a result, Optionality is non-negative and the optionality profile is that of a bell shaped curve.

# Optionality as Profit from a Partial Buy-Write

- A buy-write is defined as the sale of a call combined with a purchase of one unit of the underlying asset.
- If an investor sells a call and buys $-C'(K) \in (0,1)$ shares instead, we term it a "partial buy-write".
- The terminal $P\&L$ if the position is held static to maturity is:

$$P\&L_T(F_T) = C(K) - (F_T - K)^+ + C_0'(K)(F_T - F_0).$$

- If $F_T = K$, then the terminal $P\&L$ simplifies to:

$$P\&L_T(K) = C(K) - C_0'(K)(K - F_0) = \mathcal{O}(K).$$

- As a result, Optionality is just the profit from a partial buy write when the underling finishes at-the-money. This profit is non-negative for every strike price.

# Optionality as the Far Convexity Part of ATM Value

- The Optionality at $K \neq F_0$ can be interpreted as the part of the ATM Value due to the convexity of option prices in $K$ for all strikes more distant from $F_0$ than $K$.
- For example, for $K > F_0$, Taylor expanding $C(K)$ about $K > F_0$ expresses the ATM value $C(F_0)$ as:

$$C(F_0) = C(K) + C'(K)(F_0 - K) + \int_{-\infty}^{K} C''(L)(L - F_0)^+ dL.$$

- The first two terms on the RHS are just the Optionality at $K$, so:

$$\mathcal{O}(K) = C(F_0) - \int_{F_0}^{K} C''(L)(L - F_0)dL.$$

- Since $\lim_{K\uparrow\infty} C(K) = \lim_{K\uparrow\infty} C'(K) = 0$, $C(F_0) = \int_{F_0}^{\infty} C''(L)(L - F_0)dL$.
- Thus, Optionality is just the far convexity contribution to ATM Value:

$$\mathcal{O}(K) = \int_{K}^{\infty} C''(L)(L - F_0)dL, \qquad K > F_0.$$

# Optionality as Covariance of $F$ with Call's Exercise Prob.

- Let $BC(K) \equiv -C'(K)$ be the call's risk-neutral exercise probability.
- The value of the forward contract $F - K$ and the binary call's price $BC(K)$ are both martingales under forward measure $\mathbb{Q}_T$. Their bracket process is the stochastic process that must be subtracted from their product to produce a martingale. The martingale so produced has terminal payoff $(F_T - K)^+ - [F - K, BC(K)]_T$ and initial value $(F_0 - K)BC_0(K)$.
- Using the martingale property implies that Optionality is just the risk-neutral covariance of $F$ with the call's exercise probability:

$$\mathcal{O}(K) = C_0(K) - (F_0 - K)BC_0(K) = E_0^{\mathbb{Q}_T}[F, BC(K)]_T.$$

- While it is possible for the call's exercise probability to occasionally fall as the forward price rises, no arbitrage implies that the mean covariation has to be positive.

# Optionality as Expected Rebalancing Cost

- We now suppose that the risk-neutral process for the underlying forward price has translation invariance. In words, option prices depend only on $F$ and $K$ through $F - K$. Hence, the martingale describing the binary call's value also describes the call's delta:

$$BC_t(K) = \partial_F C_t(K), t \in [0, T]$$

- Recall that Optionality is just the risk-neutral covariance of $F$ with the call's exercise probability:

$$\mathcal{O}(K) = C_0(K) - (F_0 - K)BC_0(K) = E_0^{\mathbb{Q}_T}[F, BC(K)]_T.$$

- It follows that Optionality is also the risk-neutral mean of the covariation of $F$ with the call's delta:

$$\mathcal{O}(K) = C_0(K) - (F_0 - K)\partial_F C_0(K) = E_0^{\mathbb{Q}_T}[F, \partial_F C(K)]_T.$$

- This expectation is exactly the expected cost of rebalancing a delta-hedge. It is sometimes mistaken for either option premium $C_0(K)$ or time value $C_0(K) - (F_0 - K)^+$.

# Optionality as the Mean of Gamma Trading Profit

- Still assuming that $F$ has translation invariance, Optionality is still the expected cost of rebalancing a delta-hedge:

$$\mathcal{O}(K) = E_0^{\mathbb{Q}_T}[F, \partial_F C(K)]_T.$$

- We now further assume that $F$ is standard Brownian motion (SBM) running on an independent and continuous stochastic clock:

$$dF_t = a_t dW_t, \qquad da_t = b(a_t, t)dt + \omega(a_t, t)dZ_t,$$

  where $W$ and $Z$ are independent standard Brownian motions.

- Hence the covariation of $F$ with the call's delta $\partial_F C(K)$ simplifies to the mean of the call's gamma trading profit:

$$\mathcal{O}(K) = E_0^{\mathbb{Q}_T} \int_0^T \partial_F^2 C_t(K) d\langle F \rangle_t, \qquad t \in [0, T].$$

# Optionality as the Mean of Sifted Quadratic Variation

- Still assuming that $F$ is SBM running on an independent and continuous stochastic clock, Optionality is still the risk-neutral mean of the call's gamma trading profit:

$$\mathcal{O}(K) = E_0^{\mathbb{Q}_T} \int_0^T \partial_F^2 C_t(K) d\langle F \rangle_t, \qquad t \in [0, T].$$

- Under these assumptions, the call's gamma $\partial_F^2 C(K)$ is a continuous $\mathbb{Q}_T$ martingale, which is just the conditional expected value of its terminal payoff, i.e.:

$$\partial_F^2 C_t(K) = E_t^{\mathbb{Q}_T} \delta(F_T - K), \qquad t \in [0, T].$$

- The law of iterated expectations implies that Optionality is just the risk-neutral mean of sifted Quadratic Variation:

$$\mathcal{O}(K) = E_0^{\mathbb{Q}_T} [\delta(F_T - K) \langle F \rangle_T].$$

# Optionality vs. Time Value for Time Changed SBM

- When $F$ is independently time-changed SBM:

$$dF_t = a_t dW_t, \qquad da_t = b(a_t,t)dt + \omega(a_t,t)dZ_t, \qquad d\langle W,Z\rangle_t = 0,$$

Optionality is just the risk-neutral mean of sifted Quadratic Variation:

$$\mathcal{O}(K) = E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\langle F\rangle_T\right] = E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\int_0^T d\langle F\rangle_T\right].$$

- In contrast, doubling time value permutes sifting and integrating:

$$2\mathcal{TV}(K) \equiv 2[P(K) - (K - F_0)^+] = E_0^{\mathbb{Q}_T}\left[\int_0^T \delta(F_t - K)d\langle F\rangle_t\right].$$

# Local Variance Rate

- The last result holds if $F$ is a continuous martingale:

$$2\mathcal{TV}(K,T) \equiv 2[P(K,T)-(K-F_0)^+] = E_0^{\mathbb{Q}_T}\left[\int_0^T \delta(F_t - K)d\langle F\rangle_t\right].$$

- Differentiating w.r.t. $T$ implies that:

$$2\partial_T \mathcal{TV}(K,T) = 2\partial_T P(K,T) = E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\frac{d\langle F\rangle_T}{dT}\right].$$

- Dupire (94/96) defines the local variance rate $\ell^2(K,T)$ by dividing twice the calendar spread by the transition probability: $\ell^2(K,T) \equiv$

$$2\frac{\partial_T P(K,T)}{\partial_{KK} P(K,T)} = E_0^{\mathbb{Q}_T}\left[\frac{d\langle F\rangle_T}{dT}|F_T = K\right] = \frac{E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\frac{d\langle F\rangle_T}{dT}\right]}{E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\right]}.$$

# Money Variance Rate

- Recall that the local variance rate $\ell^2(K,T) \equiv$

$$2\frac{\partial_T P(K,T)}{\partial_{KK} P(K,T)} = E_0^{\mathbb{Q}_T}\left[\frac{d\langle F\rangle_T}{dT}|F_T = K\right] = \frac{E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\frac{d\langle F\rangle_T}{dT}\right]}{E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\right]}.$$

- Also recall that when $F$ is an independently time-changed SBM:

$$\mathcal{O}(K) = P(K) - (K - F_0)P'(K) = E_0^{\mathbb{Q}_T}[\delta(F_T - K)\langle F\rangle_T].$$

- Suppose Optionality is divided by the transition probability and $T$:

$$\frac{\mathcal{O}(K)}{q(K)T} = E^{\mathbb{Q}_T}\left[\frac{\langle F\rangle_T}{T}\middle| F_T = K\right] = \frac{E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)\langle F\rangle_T\right]}{E_0^{\mathbb{Q}_T}\left[\delta(F_T - K)T\right]}.$$

- Whether $F$ is an independently time changed SBM or not, we define the LHS for each fixed $T > 0$ as the "money variance rate" curve:

$$m^2(K) \equiv \frac{\mathcal{O}(K)}{q(K)T} = \frac{P(K) - (K - F_0)P'(K)}{P''(K)T}, \qquad K \in \mathbb{R}.$$

- If a given $\{P(K), K \in \mathbb{R}\}$ is arb-free, then $\{m^2(K), K \in \mathbb{R}\} \geq 0$.

# Money Vol

- Recall how we defined the "money variance rate" curve:

$$m^2(K) \equiv \frac{P(K) - (K - F_0)P'(K)}{P''(K)T}, \qquad K \in \mathbb{R}.$$

- When $F$ is independently time-changed SBM:

$$dF_t = a_t dW_t, \qquad da_t = b(a_t, t)dt + \omega(a_t, t)dZ_t, \qquad d\langle W, Z\rangle_t = 0,$$

  the money variance rate at strike K is the average of squared price changes over all paths that finish at-the-money.

$$m^2(K) = E_0^{\mathbb{Q}_T}\left[\frac{\langle F\rangle_T}{T}\bigg| F_T = K\right].$$

- The money vol at $K$ is just the square root of the money variance rate at $K$:

$$m(K) = \sqrt{E_0^{\mathbb{Q}_T}\left[\frac{\langle F\rangle_T}{T}\bigg| F_T = K\right]}.$$

# Linear Second Order ODE for Put Prices

- Suppose we are given a money vol curve $\{m(K), K \in \mathbb{R}\}$.
- Recall the relation between the money variance rate and put prices:

$$m^2(K) = \frac{P(K) - (K - F_0)P'(K)}{P''(K)T}, \qquad K \in \mathbb{R}.$$

- It follows that put prices solve a linear second order ODE:

$$m^2(K)TP''(K) + (K - F_0)P'(K) - P(K) = 0, \qquad K \in \mathbb{R},$$

subject to boundary cdns $\lim_{K \downarrow -\infty} P(K) = 0$ and $\lim_{K \uparrow \infty} P(K) \sim K - F_0$.

- We now solve for the put price curve and show that it is arbitrage-free.

# Factorizing the 2nd Order Differential Operator

- Let $\mathcal{G}_K$ be a linear second order differential operator:

$$\mathcal{G}_K \equiv m^2(K)T\mathcal{D}_{KK} + (K - F_0)\mathcal{D}_K - \mathcal{I}.$$

- The last slide showed that this operator annihilates put prices:

$$\mathcal{G}_K P(K) = 0, \qquad K \in \mathbb{R}.$$

- For $K \neq F_0$, you can check that $\mathcal{G}_K$ factorizes as:

$$\mathcal{G}_K = \left[\frac{m^2(K)T}{F_0 - K}\mathcal{D}_K - \mathcal{I}\right]\left[\mathcal{I} - (K - F_0)\mathcal{D}_K\right].$$

- When the inner operator acts on $P(K)$, it produces Optionality $\mathcal{O}(K)$. It follows that $\mathcal{O}(K)$ solves a simple 1st order linear ODE:

$$\mathcal{O}'(K) + \frac{K - F_0}{m^2(K)T}\mathcal{O}(K) = 0, \qquad K \in \mathbb{R}.$$

# Formula for Optionality and PDF

- The solution to the first order ODE for Optionality is:

$$\mathcal{O}(K) = p \times e^{-\int_{-\infty}^{K} \frac{I-F_0}{m^2(I)T} dI},$$

where $p$ is a positive constant.

- The money vol definition implies that the PDF $q(K)$ is related to the Optionality curve via:

$$q(K) = \frac{\mathcal{O}(K)}{m^2(K)T} = \frac{p \times e^{-\int_{-\infty}^{K} \frac{I-F_0}{m^2(I)T} dI}}{m^2(K)T}.$$

- The positive constant is determined by requiring that the PDF integrates to one:

$$q(K) = \frac{e^{-\int_{-\infty}^{K} \frac{I-F_0}{m^2(I)T} dI} / (m^2(K)T)}{\int_{-\infty}^{\infty} e^{-\int_{-\infty}^{J} \frac{I-F_0}{m^2(I)T} dI} / (m^2(J)T) dJ}.$$

# Put Option Pricing Formula

- The last slide shows that the PDF is given in closed form by:

$$q(K) = \frac{e^{-\int_{-\infty}^{K} \frac{I-F_0}{m^2(I)T} dI} / (m^2(K)T)}{\int_{-\infty}^{\infty} e^{-\int_{-\infty}^{J} \frac{I-F_0}{m^2(I)T} dI} / (m^2(J)T) dJ}.$$

- Recall that put prices solve a linear 2nd order ODE:

$$m^2(K)TP''(K) + (K - F_0)P'(K) - P(K) = 0, \qquad K \in \mathbb{R},$$

- Since $P''(K) = q(K)$, put prices are given in closed form by:

$$P(K) = (K - F_0) \int_{-\infty}^{K} q(J)dJ + m^2(K)Tq(K), \qquad K \in \mathbb{R}.$$

# Boundary Conditions for Call and Put

- One can check that the put pricing formula on the last slide satisfies the boundary conditions $\lim_{K \downarrow -\infty} P(K) = 0$ and $\lim_{K \uparrow \infty} P(K) \sim K - F_0$.
- Call prices solve the same ODE, but have complementary boundary conditions $\lim_{K \downarrow -\infty} C(K) \sim F_0 - K$ and $\lim_{K \uparrow \infty} C(K) = 0$. The solution is determined by put call parity and the put formula on the last slide.
- We conclude that put and call prices can each be analytically expressed in terms of a given money vol curve.
- However, the fastest way to numerically determine an option price curve is to approximate the ODE using finite differences and solve an initial value problem. The closed form formula can be used to generate the initial value and slope. This finite difference scheme will actually be faster than converting implied volatilities to option prices.

# Arbitrage-free Option Prices

- Recall that put and call prices each solve a linear 2nd order ODE:

$$Tm^2(K)V''(K) + (K - F_0)V'(K) - V(K) = 0, \qquad K \in \mathbb{R},$$

- More generally, for $T > 0$, linear 2nd order ODE's of the form:

$$Tm^2(K)V''(K) + c(K)V'(K) - V(K) = 0, \qquad K \in \mathbb{R},$$

  are said to be "of positive type" by Birkhoff and Kotin (1967).
- These ODE's have a positive solution. Moreover, so long as a certain integral diverges, they also have one increasing and one decreasing solution.
- In our special case, the put is an increasing solution and the call is a decreasing solution. Our specialization also has the property that the 2nd derivative of both solutions has the same sign as the Wronskian (or Optionality) of these two linearly independent solutions.
- Since Optionality is non-negative, option prices are arbitrage-free

# Generalized Bachelier Formula for Option Prices

- If the money vol curve $m(K)$ is constant at $a$, then the formula for $q(K)$ reduces to a normal PDF with mean $F_0$ and variance $a^2T$.
- Analogously, the put option pricing formula:

$$P(K) = (K - F_0)\int_{-\infty}^{K} q(J)dJ + m^2(K)Tq(K), \qquad K \in \mathbb{R},$$

  then reduces to the Bachelier put formula.
- As a result, we refer to the option pricing formulae as generalized Bachelier formulae, when the money vol curve $\{m(K), K \in \mathbb{R}\}$ is not constant.

# Consistent Dynamics for the Underlying Forward Price

- Having converted the money vol curve into an arbitrage-free option price curve at one maturity, one naturally wonders whether one can construct continuous time dynamics for the underlying forward price which are consistent with these curves.
- It is a numerically difficult problem to find a time homogeneous local vol function which meets a given arbitrage-free set of options prices across strikes.
- However, the problem of finding consistent dynamics is just the Skorohod stopping problem. As the underlying is a martingale, we just need to find a stopping time for SBM such that option prices can be represented as a solution to a 2nd order ODE.

# Madan Yor Brownian Scaling Solution

- Fortunately, Madan and Yor (2002) have shown that a consistent SDE for the underlying forward price is:

$$dF_t = b\left(\frac{F_t - F_0}{\sqrt{t}}\right) dW_t, \qquad t \in [0, T],$$

  where $b(z) \equiv m(K), z \equiv \frac{K-F_0}{\sqrt{T}}$ is the money vol considered as a function of the moneyness measure $z$.
- These dynamics can be used to generate option prices at other maturities, to generate greeks, and to value exotics.
- An interesting open problem is whether consistent jump dynamics can also be found (LVG won't work).

# Optionality and Volatility

- In a setting where option prices across strikes are given at a single maturity, we developed a natural concept called Optionality and gave it several interpretations.
- When optionality is normalized by the PDF and time to maturity, we obtained a second new concept called the “money variance rate”. We showed that the money variance rate is non-negative if and only if option prices at one maturity are arbitrage-free.
- The approach is much simpler than one recently proposed in Schweizer and Wissel (2008).
- Future research can explore whether this approach can be extended when arbitrage-free option prices are given at two or more maturities.
- Alternatively, one can explore imposing dynamics on the money vol curve, as is commonly done for local and implied volatilities.

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