ICTS Summer School on Numerical Relativity 2013 – Mathematical formulation

Tutorial # 4

4.1 (a) Show that

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + 2\,(\delta^i{}_j \partial_k \ln\psi + \delta^i{}_k \partial_j \ln\psi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\partial_l \ln\psi), \tag{1}$$

where the $\bar{\Gamma}^i_{jk}$ are the connection symbols associated with the conformally related metric $\bar{\gamma}_{ij} = \psi^{-4}\gamma_{ij}$.
(b) (Optional) Derive the transformation rules for R_{ij} and R that we quoted in class.

4.2 Consider a traceless tensor T^{ij} and a conformal rescaling

$$T^{ij} = \psi^\alpha \bar{T}^{ij}. \tag{2}$$

Show that the divergence of T^{ij} then transforms according to

$$D_j T^{ij} = \psi^{-10}\bar{D}_j(\psi^{10+\alpha}\bar{T}^{ij}) \tag{3}$$

This motivates our choice of $\alpha = -10$ for the rescaling of the traceless part of the extrinsic curvature, A^{ij}.

4.3 (a) Show that, for conformal flatness, maximal slicing and vacuum, the momentum constraint reduces to

$$\partial^j \partial_j W_i + \frac{1}{3}\partial_i \partial^j W_j = 0 \tag{4}$$

when it is expressed in Cartesian coordinates. Different decompositions of W^i can be used to bring this equation into a set of easier Laplace equations.
(b) Write

$$W_i = \frac{7}{8}V_i - \frac{1}{8}(\partial_i U + x^k \partial_i V_k) \tag{5}$$

where V_i is a new vector and U some scalar. Insert this ansatz into (4), assume that U satisfies

$$\partial^j \partial_j U = 0, \tag{6}$$

and show that V_i must then satisfy

$$\partial^j \partial_j V_i = 0 \tag{7}$$

for the ansatz (5) to satisfy the vector Laplacian (4). That means that we may solve two Laplace equations instead of a vector Laplacian!
(c) Verify that a simple solution to the above equations is given by

$$U = 0, \qquad V_i = -\frac{2P_i}{r} \tag{8}$$

where P_i is a vector with constant coefficients.
(d) Insert this solution into (5) to find W^i. Finally, compute $\bar{A}^{ij}_L$ from W^i. If all went well, you have derived the Bowen-York solution for a boosted black hole with linear momentum P^i.