ICTS Summer School on Numerical Relativity 2013 – Mathematical formulation

Tutorial # 3

3.1 Consider a two-dimensional sphere embedded in a flat, three-dimensional, as in the example we discussed in class. Show that the geometries of the two spaces satisfy Ricci's equation.

3.2 Show that the Lie derivative of the projection operator $\gamma^a_{\ b}$ along αn^a vanishes,

$$\mathcal{L}_{\alpha\mathbf{n}}\gamma^a_{\ b} = 0, \tag{1}$$

and that this is not true in general for the Lie derivative along n^a alone. This result implies that spatial tensors that are Lie dragged along αn^a remain spatial.

3.3 Show that the spatial Christoffel symbols

$$\Gamma^a_{bc} = \frac{1}{2}\gamma^{ad}(\partial_b\gamma_{dc} + \partial_c\gamma_{db} - \partial_d\gamma_{bc}) \tag{2}$$

are spatial projections of the spacetime Christoffel symbols,

$$\Gamma^a_{bc} = \gamma^a_{\ d}\,\gamma^e_{\ b}\,\gamma^f_{\ c}\,{}^{(4)}\Gamma^d_{ef}. \tag{3}$$

3.4 Show that the determinant $g = \det(g_{ab})$ of the spacetime metric g_{ab} can be written as

$$\sqrt{-g} = \alpha\sqrt{\gamma}, \tag{4}$$

where $\gamma = \det(\gamma_{ij})$ is the determinant of the spatial metric γ_{ij}.
Hint: Recall that for any square matrix A_{ij} the inverse can be computed from $(A^{-1})_{ij}$ = cofactor of $A_{jk}/\det(A_{ij})$.

3.5 Show that the determinant of the spatial metric γ satisfies

$$\partial_t \ln\gamma^{1/2} = -\alpha K + D_i\beta^i, \tag{5}$$

while the mean curvature $K \equiv \gamma^{ij}K_{ij}$ satisfies

$$\partial_t K = -D^2\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_i K. \tag{6}$$

3.6 (a) Revisit the Painlevé-Gullstrand metric of Problem 1.4 and identify the lapse α, the shift β^i and the spatial metric γ_{ij}.
(b) Compute the extrinsic curvature K_{ij} and the Ricci tensor R_{ij}.
(c) (Optional) Verify that this solution satisfies the constraint and evolution equations with $\partial_t K_{ij} = 0$.