ICTS Summer School on Numerical Relativity 2013 – Mathematical formulation

Tutorial # 2

2.1 In class we decomposed the spacetime vector v^a into a spatial part $\gamma^a{}_b v^b$ and a normal part $-n^a n_b v^b$.
(a) Show that the spatial part is indeed spatial by showing that its contraction with the normal vector n^a vanishes.
(b) Show that the normal part is indeed normal by showing that its contraction with the spatial metric γ_{ac} vanishes.

2.2 We can "count" the number of dimensions by taking the trace of the metric. Take the trace of the induced metric γ_{ab} to show that the dimension of the hypersurface Σ is one less than that of the embedding space M.

2.3 Show that the spatial covariant derivative is compatible with the spatial metric γ_{ab}, i.e. show that

$$D_a \gamma_{bc} = 0. \tag{1}$$

2.4 Show that the acceleration of a normal observer, $a_a \equiv n^b \nabla_b n_a$, is related to the lapse α according to

$$a_a = \epsilon D_a \ln \alpha. \tag{2}$$

2.5 In Problem 1.3 you derived the Schwarzschild metric in isotropic form,

$$ds^2 = -\left(\frac{1 - M/(2r)}{1 + M/(2r)}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega^2). \tag{3}$$

Now consider hypersurfaces of constant Schwarzschild time t.
(a) Compute the lapse α and the normal vector n^a.
(b) Compute the spatial metric γ_{ab}.
(c) Show that the extrinsic curvature K_{ab} vanishes.

2.6 Use the example of a mixed-index, rank-2 tensor $T^a{}_b$ to show that we can replace partial derivatives with covariant derivatives in the Lie derivative $\mathcal{L}_{\mathbf{X}} T^a{}_b$.

2.7 Show that

$$D_a D_b V^c = \gamma_a{}^p \gamma_b{}^q \gamma^c{}_r \nabla_p \nabla_q V^r - \epsilon K_{ab} \gamma^c{}_r n^p \nabla_p V^r - \epsilon K_a{}^c K_{bp} V^p. \tag{4}$$