ICTS Summer School on Numerical Relativity 2013 – Mathematical formulation
Tutorial # 1

1.1 Consider a flat, three-dimensional space in spherical polar coordinates,

$$ds^2 = g_{ab}dx^a dx^b = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1}$$

(a) Show that the only non-vanishing components of the connection are given by

$$\begin{array}{rclcrcl} \Gamma^r_{\theta\theta} & = & -r & \quad & \Gamma^r_{\phi\phi} & = & -r\sin^2\theta \\ \Gamma^\theta_{\phi\phi} & = & -\sin\theta\cos\theta & & \Gamma^\theta_{r\theta} & = & r^{-1} \\ \Gamma^\phi_{r\phi} & = & r^{-1} & & \Gamma^\phi_{\phi\theta} & = & \cot\theta. \end{array} \tag{2}$$

and those that are related to the above by symmetry.
(b) Show that the Laplace operator acting on a scalar function f, defined as

$$\nabla^2 f \equiv \gamma^{ij}\nabla_i\nabla_j f, \tag{3}$$

reduces to the well-known expression

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{4}$$

(c) What is the Riemann tensor for this space?

1.2 Consider a two-dimensional sphere of radius r with metric

$$ds^2 = g_{ab}dx^a dx^b = r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{5}$$

Show that the Riemann tensor can be written as

$$R_{abcd} = \frac{1}{r^2}(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{6}$$

Hint: Confirm that the above expression has the correct symmetries, and think about the number of independent components of the Riemann tensor.

1.3 In class we discussed the Schwarzschild metric

$$ds^2 = -f_0 dt^2 + f_0^{-1} dR^2 + R^2 d\Omega^2, \tag{7}$$

where $f_0 = 1 - 2M/R$ and where R is the *areal* (or *circumferential*) radius. Consider a new radius r that satisfies

$$R = r\left(1 + \frac{M}{2r}\right)^2 \tag{8}$$

(a) Show that under this transformation the Schwarzschild metric (7) takes the form

$$ds^2 = -\left(\frac{1 - M/(2r)}{1 + M/(2r)}\right)^2 dt^2 + \psi^4(dr^2 + r^2 d\Omega^2) \tag{9}$$

and find the "conformal factor" ψ. This form of the metric is called *isotropic*.
(a) Find an expression for r in terms of R. Show that real solutions for r only exist $R \geq 2M$.

1.4 In the previous problem you considered a radial transformation of the Schwarzschild metric. Now consider a temporal transformation to a new time coordinate

$$t' = t + h(R), \tag{10}$$

where the *height function* $h(R)$ describes by how much surfaces of constant new time t' "lift off" surfaces of constant Schwarzschild time t.

(a) Show that in these new coordinates the Schwarzschild metric (7) takes the form

$$ds^2 = -f_0 dt'^2 + 2f_0 h' dt' dR + f_0^{-1}(1 + f_0^2 h'^2) dR^2 + R^2 d\Omega^2, \tag{11}$$

where $h' \equiv dh/dR$.

(b) Find a suitable condition for h so that the Schwarzschild metric takes the *Painlevé-Gullstrand* form

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt'^2 + 2\sqrt{\frac{2M}{R}} dt' dR + dR^2 + R^2 d\Omega^2. \tag{12}$$

Painlevé-Gullstrand coordinates are remarkable in that they leave the spatial part of the metric flat.