ICTS Summer School on Numerical Relativity 2013 – Mathematical formulation

A brief review of tensor properties

A tensor is a physical object and has an intrinsic meaning independently of coordinates or basis vectors. An example is a rank-1 tensor $\mathbf{A}$ that has a certain length and points in a certain direction independently of the coordinate system in which we express this tensor.

We can expand any tensor in terms of either basis vectors $\mathbf{e}_a$ or basis 1-forms $\tilde{\omega}^a$. Expanding a rank-1 tensor in terms of basis vectors, for example, yields

$$\mathbf{A} = A^a \mathbf{e}_a. \tag{1}$$

This expression deserves several comments. For starters, we have used the Einstein summation rule, meaning that we sum over repeated indices. The "up-stairs" index a on A^a refers to a *contravariant* component of $\mathbf{A}$ – meaning one that is used in an expansion of $\mathbf{A}$ in terms of basis vectors. The index a on the basis vector $\mathbf{e}_a$, on the other hand, does *not* refer to a component of the basis vector – instead it denotes the name of the basis vector (e.g. the basis vector pointing in the x direction). If we wanted to refer to the b-th component of the basis vector $\mathbf{e}_a$, say, we would write $(\mathbf{e}_a)^b$.

An immediate question is whether we always have $(\mathbf{e}_a)^b = \delta_a{}^b$, where $\delta_a{}^b$ is the Kronecker delta. The answer is no – this is true *only* if we are expressing normalized basis vectors in their own coordinate system, for example, if we are expressing Cartesian basis vectors in Cartesian coordinates. In general, however, this is not true. Think, for instance, about Cartesian basis vectors expressed in a spherical polar coordinate system.

We now write the dot product between two vectors as

$$\mathbf{A} \cdot \mathbf{B} = (A^a \mathbf{e}_a) \cdot (B^b \mathbf{e}_b) = A^a B^b \, \mathbf{e}_a \cdot \mathbf{e}_b. \tag{2}$$

Defining the metric as

$$g_{ab} \equiv \mathbf{e}_a \cdot \mathbf{e}_b \tag{3}$$

we obtain

$$\mathbf{A} \cdot \mathbf{B} = A^a B^b g_{ab}. \tag{4}$$

Expanding a rank-1 tensor in terms of 1-forms $\tilde{\omega}^a$ yields

$$\mathbf{B} = B_a \tilde{\omega}^a, \tag{5}$$

where the "down-stairs" index a refers to a *covariant* component. We call the basis 1-forms $\tilde{\omega}^a$ dual to the basis vectors $\mathbf{e}_b$ if

$$\tilde{\omega}^a \cdot \mathbf{e}_b = \delta^a{}_b. \tag{6}$$

This is what we will assume throughout. We can then compute the dot product between $\mathbf{B}$ and $\mathbf{A}$ as

$$\mathbf{A} \cdot \mathbf{B} = (A^a \mathbf{e}_a) \cdot (B_b \tilde{\omega}^b) = A^a B_b \, \mathbf{e}_a \cdot \tilde{\omega}^b = A^a B_b \, \delta_a{}^b = A^a B_a. \tag{7}$$

Since both this expression and (4) have to hold for any tensor $\mathbf{A}$, we can compare the two and identify

$$B_a = g_{ab} B^b. \tag{8}$$

We refer to this operation as "lowering the index of B^a".

We define the inverse metric g^{ab} so that

$$g^{ac} g_{cb} = \delta^a{}_b. \tag{9}$$

We then "raise the index of B_a" using

$$B^a = g^{ab} B_b. \tag{10}$$

We can also show that

$$g^{ab} = \tilde{\omega}^a \cdot \tilde{\omega}^b. \tag{11}$$

Note that we can find the contravariant component of a rank-1 tensor $\mathbf{A}$ by computing the dot product with the corresponding 1-form,

$$A^a = \mathbf{A} \cdot \tilde{\omega}^a. \tag{12}$$

We can verify this expression by inserting the expansion (1) for $\mathbf{A}$, and then using the duality relation (6).

Under a change of basis, i.e. when we transform from one coordinate system x^a to another, say $x^{b'}$, basis vectors and basis 1-forms transform according to

$$\mathbf{e}_{b'} = M^a_{\ b'}\mathbf{e}_a \tag{13}$$

$$\tilde{\omega}^{b'} = M^{b'}_{\ a}\tilde{\omega}^a \tag{14}$$

where $M^{b'}_{\ a}$ is the transformation matrix and $M^a_{\ b'}$ its inverse, so that

$$M^{a'}_{\ c}M^c_{\ b'} = \delta^{a'}_{\ b'}. \tag{15}$$

Note that vectors and 1-forms transform in "inverse ways". This guarantees that the duality relation (6) also holds in the new coordinate system,

$$\tilde{\omega}^{a'} \cdot \mathbf{e}_{b'} = (M^{a'}_{\ c}\tilde{\omega}^c) \cdot (M^d_{\ b'}\mathbf{e}_d) = M^{a'}_{\ c}M^d_{\ b'}(\tilde{\omega}^c \cdot \mathbf{e}_d) = M^{a'}_{\ c}M^d_{\ b'}\delta^c_{\ d} = M^{a'}_{\ c}M^c_{\ b'} = \delta^{a'}_{\ b'}. \tag{16}$$

The components of a vector then transform according to

$$A^{b'} = \mathbf{A} \cdot \tilde{\omega}^{b'} = \mathbf{A} \cdot (M^{b'}_{\ a}\tilde{\omega}^a) = M^{b'}_{\ a}A^a \tag{17}$$

and similarly

$$B_{b'} = M^a_{\ b'}B_a. \tag{18}$$

The fact that contravariant and covariant components transform in "inverse" ways guarantees that the dot product (7) is invariant under coordinate transformations,

$$A^{b'}B_{b'} = M^{b'}_{\ a}A^a M^c_{\ b'}B_c = \delta^c_{\ a}A^a B_c = A^a B_a, \tag{19}$$

as it is supposed to be.

We can generalize all the above concepts to higher-rank tensors. A rank-n tensor can be expanded into n basis vectors or 1-forms, and we transform the components of a rank-n tensor with n copies of the transformation matrix or its inverse.

For transformations between *coordinate bases*, for which the basis vectors are tangent to coordinate lines, we have

$$M^{b'}_{\ a} \equiv \frac{\partial x^{b'}}{\partial x^a} = \partial_a x^{b'}. \tag{20}$$

As an illustration of the above concepts, consider the components of a displacement vector dx^a, which measures the displacement between two points expressed in a coordinate system x^a. To compute the components of this vector in a different coordinate system, say a primed coordinate system $x^{b'}$, we use the chain rule to obtain

$$dx^{b'} = \frac{\partial x^{b'}}{\partial x^a}dx^a = M^{b'}_{\ a}dx^a \tag{21}$$

where we have used (20) in the last step. As expected, the components of dx^a transform like the vector components in (17).

As an example of a 1-form, consider the components of the gradient $\partial f/\partial x^a$ of a function f, again expressed in some coordinate system x^a. To transform to a new coordinate system $x^{b'}$ we again use the chain rule, but this time we obtain

$$\frac{\partial f}{\partial x^{b'}} = \frac{\partial x^a}{\partial x^{b'}}\frac{\partial f}{\partial x^a} = M^a_{\ b'}\frac{\partial f}{\partial x^a} \tag{22}$$

as in (18). We see that the "inverse" transformation of the components of a gradient are a result of the chain rule.

Finally, consider the difference df in the function values f at two (close) points. Clearly, this difference is an invariant, i.e. independent of coordinate choice. We can express this difference as the dot product between the vector displacement vector dx^a between the two points and the 1-form $\partial f/\partial x^a$,

$$df = \frac{\partial f}{\partial x^a}dx^a. \tag{23}$$

As an exercise, apply the above transformation rules for the components of vectors and 1-form to show that df is indeed invariant under a coordinate transformation.