

Important Results

General Relativity

- Christoffel symbols

$$\Gamma_{bc}^a = \frac{1}{2}g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \quad (1)$$

- Riemann tensor

Definition

$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = R_{cba}^d v_d \quad (2)$$

Compute from

$$R_{bcd}^a = \partial_c \Gamma_{bd}^a - \partial_d \Gamma_{bc}^a + \Gamma_{ec}^a \Gamma_{bd}^e - \Gamma_{ed}^a \Gamma_{bc}^e \quad (3)$$

Symmetries

$$R_{abcd} = -R_{bacd} \quad R_{abcd} = -R_{abdc} \quad R_{abcd} = R_{cdab} \quad (4)$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \quad (5)$$

(also Bianchi identities.) In n dimensions have $n^2(n^2 - 1)/12$ independent components.

- Ricci tensor and Ricci scalar

$$R_{ab} \equiv R_{acb}^c \quad R \equiv g^{ab} R_{ab} \quad (6)$$

Three ways to compute Ricci

$$R_{ab} = \partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{ad}^c \Gamma_{bc}^d \quad (7)$$

$$R_{ab} = \frac{1}{2}g^{cd}(\partial_a \partial_d g_{cb} + \partial_c \partial_b g_{ad} - \partial_a \partial_b g_{cd} - \partial_c \partial_d g_{ab}) + g^{cd}(\Gamma_{ad}^e \Gamma_{ecb} - \Gamma_{ab}^e \Gamma_{ecd}) \quad (8)$$

$$R_{ab} = -\frac{1}{2}g^{cd}\partial_d \partial_c g_{ab} + g_{c(a} \partial_b) \Gamma^c + \Gamma^c \Gamma_{(ab)c} + 2g^{ed} \Gamma_{e(a} \Gamma_{b)cd} + g^{cd} \Gamma_{ad}^e \Gamma_{ecb} \quad (9)$$

where $\Gamma^a \equiv g^{bc} \Gamma_{bc}^a$.

- Einstein tensor

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R \quad (10)$$

- Einstein's equations

$$\boxed{G_{ab} = 8\pi T_{ab}} \quad (11)$$

where T_{ab} is stress-energy tensor, and where we assume $\Lambda = 0$.

The 3 + 1 Decomposition

- normal vector

$$n^a = -\alpha g^{ab} \nabla_b t \quad (12)$$

- induced or spatial metric

$$\gamma_{ab} = g_{ab} + n_a n_b \quad (13)$$

- Extrinsic curvature

$$K_{ab} \equiv -\gamma_a^c \gamma_b^d \nabla_c n_d = -\nabla_a n_b - n_a a_b = -\frac{1}{2} \mathcal{L}_n \gamma_{ab} \quad (14)$$

where $a_a \equiv n^b \nabla_b n_a$ is acceleration of normal observer.

- ADM equations

- Constraint equations

$$R + K^2 + K_{ij}K^{ij} = 16\pi\rho \quad \text{Hamiltonian constraint} \quad (15)$$

$$D_i(K^{ij} - \gamma^{ij}K) = 8\pi S^i \quad \text{momentum constraint} \quad (16)$$

- Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \quad (17)$$

$$\begin{aligned} \partial_t K_{ij} = & \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - D_i D_j \alpha - 8\pi\alpha \left(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho) \right) \\ & + \beta^k \partial_k K_{ij} + K_{ik} \partial_j \beta^k + K_{kj} \partial_i \beta^k \end{aligned} \quad (18)$$

$$(19)$$

Conformal decompositions

- metric

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \quad (20)$$

- extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = \psi^{-2}\bar{A}_{ij} + \frac{1}{3}\psi^4\bar{\gamma}_{ij}K \quad (21)$$

- Hamiltonian constraint

$$\bar{D}^2\psi - \frac{\psi}{8}\bar{R} + \frac{\psi^{-7}}{8}\bar{A}_{ij}\bar{A}^{ij} - \frac{1}{12}\psi^5K^2 = -2\pi\psi^5\rho \quad (22)$$

- momentum constraint

$$\bar{D}_j \bar{A}^{ij} - \frac{2}{3}\psi^6 \bar{\gamma}^{ij} \bar{D}_j K = 8\pi\psi^{10}S^i \quad (23)$$