Barrier Options

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What are Barrier Options?

- Barrier options are over-the-counter (OTC) contracts typically written on a single underlying risky asset and typically having a single payoff.
- Barrier options have the property that one or more pre-specified flat barriers affect the terms and timing of the final payoff.
- The dependence of the payoff on the barrier(s) is always through the binary random variable which indicates whether or not the barrier has been crossed.
- In general, barrier options have a fixed date at which one starts monitoring the barrier and a later date at which one ends monitoring the barrier. The earlier date at which monitoring starts is often the issuance date and the later date at which monitoring ends is often the expiry date.
- The most liquid barrier options have either a single barrier or else two barriers which straddle the initial spot.

Examples of Single Barrier Options

- One touch payment at expiry
- One touch payment at hit
- Single barrier no touch
- Up/Down and In/Out Call/Put (8 examples). In practice, people bifurcate the 8 examples into those options which knock in or out when the underlying vanilla is in-the-money, versus those which knock in or out when the underlying vanilla is out-of-the-money.

Examples of Double Barrier Options

- Double touch payment at expiry
- Double touch payment at hit
- Double barrier no touch
- Double Knockout Call/Put (2 examples)
- KOKI's - one in-barrier and one-out barrier in either order (4 examples before specifying the form of final payoff)
- While there are other logical possibilities eg. double in, it is fair to say that the above cases cover all of the barrier options that are presently liquid.

Some Variations on Barrier Option Payoffs

- continuous vs discrete monitoring
- barrier windowing (i.e forward start and/or early end)
- knockout rectangles (eg. box options) vs knockout lines.
- sequential barrier options
- wedding cakes
- phoenix options

Digitals and Barrier Options in FX markets

- While barrier options exist in the OTC equity derivatives market, the most liquidity in barrier options is in the OTC foreign exchange (FX) market.
- In FX markets, digitals such as one touch (with payment at expiry) and double no touch are the most liquid form of second generation options.
- Barrier options such as down and outs, up and outs, and double knockouts are also popular.
- All digitals and barriers on currencies involve continuous barrier monitoring.
- Some barrier options are sequential and/or involve windowing, which we don't discuss today.
- Some out barrier options have rebates, which we don't directly discuss today.

Well Known Model-Free Relationships Among Vanillas

- Let's review several well known model-free arbitrage relationships among path-independent options, primarily to establish notation.

$$BP_t(K,T) = \frac{\partial}{\partial K} P_t(K,T).$$

$$BC_t(K,T) = -\frac{\partial}{\partial K} C_t(K,T).$$

$$GC_t(K,T) = C_t(K,T) + K \cdot BC_t(K,T).$$

Model Free Relationships for Single Barriers

- There are also some well-known model-free arbitrage relationships among single barrier options.
- WLOG, we focus on an upper barrier H and set $H > S_0, K$:

$$OTPE_t(T; H) = BCMax_t(H, T).$$

$$SNT_t(T; H) + OTPE_t(T; H) = B_t(T).$$

$$UOBP_t(K, T; H) = BP_t(K, T) - UIBP_t(K, T; H).$$

$$OTPE_t(T; H) = UIBP(H, T; H) + BC(H, T).$$

Model-Free Relationships for Double Barriers

- There are some well known and not so well known model-free arbitrage relationships among double barrier options.

- We set the upper barrier $H > S_0$, $K > L$, where L is the lower barrier:

$$\lim_{L \downarrow -\infty} DNT_t(T; L, H) = SNT_t(T; H).$$

$$\lim_{L \downarrow -\infty} DKOP_t(K, T; L, H) = UOP_t(K, T; H).$$

$$DNT_t(T; L, H) = \lim_{K_c \downarrow L} \lim_{K_p \uparrow H} \frac{DKOC_t(K_c, T; L, H) + DKOP_t(K_p, T; L, H)}{H - L}.$$

Less Well Known Model-Free Relationships

- There are other less well known model-free arbitrage relationships for single barrier options which we introduce:

$$UIP_t(K_p, T; H) = \int_{-\infty}^{K_p} UIBP_t(K, T; H) dK.$$

$$SNT_t(T; H) = \lim_{K \uparrow H} \frac{\partial UOP_t(K, T; H)}{\partial K}.$$

- Assuming a deterministic interest rate $r(t), t \in [0, T]$:

$$OTPH_t(T; H) = OTPE_t(T; H) + \int_t^T r(u) OTPE_t(u; H) du.$$

- The above model-free results imply that there is a static hedge for all (non-windowed single up barrier) digitals and barrier options, once there exists a static hedge for all up-and-in binary puts.

Static Hedging of Up-and-In Binary Puts

- Suppose that the underlying is a forward FX rate F, so that it has zero risk-neutral drift under the forward measure.
- Further suppose that the process is skip-free, i.e. no jumps over the barrier.
- Let τ denote the first passage time to the higher barrier H ($\tau = \infty$ if never hit).
- Finally assume that if $\tau < T$, then at time τ, the conditional risk-neutral density governing F_T is symmetric about $F_\tau = H$. Hence:

$$BP_\tau(K, T) = BC_\tau(2H - K, T).$$

- As a result, for $t \in [0, T \wedge \tau]$, no arbitrage implies:

$$UIBP_t(K, T; H) = BC_t(2H - K, T).$$

Implications for Up-and-In Put

- Recall that when the price is a skip-free symmetric martingale:

$$UIBP_t(K, T; H) = BC_t(2H - K, T), \qquad t \in [0, T \wedge \tau].$$

- Also recall that:

$$UIP_t(K_p, T; H) = \int_{-\infty}^{K_p} UIBP_t(K, T; H) dK.$$

- Hence, integrating the top equation in K from K_p down implies:

$$UIP_t(K_p, T; H) = \int_{-\infty}^{K_p} BC_t\left(2H - K, T\right) dK.$$

- But recall that:

$$BC_t(K, T) = -\frac{\partial}{\partial K} C_t(K, T).$$

- So by the fundamental theorem of calculus:

$$UIP_t(K_p, T; H) = C_t\left(2H - K_p, T\right).$$

Implications for a One Touch

- Again recall that when the price is a skip-free symmetric martingale:

$$UIBP_t(K,T;H) = BC_t(2H-K,T), \qquad t \in [0, T \wedge \tau].$$

- Also recall that:

$$OTPE_t(T;H) = UIBP(H,T;H) + BC_t(H,T).$$

- Hence setting $K = H$ in the top equation implies:

$$OTPE_t(T;H) = 2BC_t(H,T), \qquad t \in [0, T \wedge \tau].$$

- In words, the risk-neutral probability of hitting a barrier is twice the probability of finishing beyond it.

Geometric Brownian Martingale

- An (old) argument against the price symmetry assumption is that positive probability of $F_T > 2H$ when $F_\tau = H$ implies positive probability that $F_T < 0$.
- To allow unlimited upside while imposing zero probability of nonpositive F_T, suppose that we instead assume that the forward price follows geometric Brownian motion (Black model).
- Assuming no arbitrage, there is a probability measure (called dollar measure) $\mathbb{D}$ equivalent to the physical probablity measure under which F is a martingale.

SDE's for Forward price and its Reciprocal

- Suppose that under $\mathbb{D}$, the forward price F solves the SDE:

$$dF_t = \sigma F_t dW_t, \qquad t > 0.$$

- The solution to this SDE is well known to be:

$$F_t = F_0 e^{\sigma W_t - \sigma^2 t/2}, \qquad t > 0.$$

- Consider the reciprocal of F:

$$R_t \equiv 1/F_t = R_0 e^{-\sigma W_t + \sigma^2 t/2}, \qquad t > 0,$$

where $R_0 \equiv \frac{1}{F_0}$.

- Under $\mathbb{D}$, the reciprocal of the forward price R solves the SDE:

$$dR_t = \sigma^2 R_t dt - \sigma R_t dW_t, \qquad t > 0.$$

- The upward drift is due to the convexity of the function $y(x) = 1/x$.

SDE's Under Measure Change

- Recall that under $\mathbb{D}$, the reciprocal R of the forward price F solves the SDE:

$$dR_t = \sigma^2 R_t dt - \sigma R_t dW_t, \qquad t > 0.$$

- Letting $M_t \equiv \frac{F_t}{F_0} = e^{\sigma W_t - \sigma^2 t/2}$, we have: $\frac{dM_t}{M_t} = \sigma dW_t, \qquad t > 0,$ so M is a $\mathbb{D}$ martingale. As F_t has mean F_0 under $\mathbb{D}$, M_t has mean 1 under $\mathbb{D}$.

- Suppose that we use M_T to define a new probability measure $\mathbb{P}$ called pound measure by:

$$\frac{d\mathbb{P}}{d\mathbb{D}} = M_T = \frac{F_T}{F_0} = e^{\sigma W_T - \sigma^2 T/2}.$$

- Then by Girsanov's theorem, there exists a $\mathbb{P}$ standard Brownian motion B such that under $\mathbb{P}$, R solves the SDE:

$$\begin{aligned} dR_t &= \sigma^2 R_t dt + \frac{1}{M_t} d\langle M, R\rangle_t - \sigma R_t dB_t, \\ &= -\sigma R_t dB_t, \qquad t > 0. \end{aligned}$$

A Pair of SDE's

- Recall that under the \$ measure $\mathbb{D}$, the forward price F solves the SDE:

$$\frac{dF_t}{F_t} = \sigma dW_t, \qquad t > 0,$$

  where W is a standard Brownian motion under $\mathbb{D}$.

- Also recall that under the pound measure $\mathbb{P}$, the reciprocal R of the forward price solves the SDE:

$$\frac{dR_t}{R_t} = -\sigma dB_t, \qquad t > 0,$$

  where B is a standard Brownian motion under $\mathbb{P}$.

- Hence for any t, $\frac{F_t}{F_0}$ has the same law under $\mathbb{D}$ as $\frac{R_t}{R_0}$ has under $\mathbb{P}$.

Implications for European Claims

- Let's assume zero interest rates in the two countries.
- Since $\frac{F_T}{F_0}$ has the same law under $\mathbb{D}$ as $\frac{R_T}{R_0}$ has under $\mathbb{P}$, it takes just as many dollars to create the payoff $f\left(\frac{F_T}{F_0}\right)$ as it takes pounds to create the payoff $f\left(\frac{R_T}{R_0}\right)$:

$$E^{\mathbb{D}} f\left(\frac{F_T}{F_0}\right) = E^{\mathbb{P}} f\left(\frac{R_T}{R_0}\right).$$

Example: Vanilla Calls

- Recall that it takes just as many dollars to create the payoff $f\left(\frac{F_T}{F_0}\right)$ as it takes pounds to create the payoff $f\left(\frac{R_T}{R_0}\right)$:

$$E^{\mathbb{D}} f\left(\frac{F_T}{F_0}\right) = E^{\mathbb{P}} f\left(\frac{R_T}{R_0}\right).$$

- For example, if $f(x) = (F_0 x - K_c)^+$, then:

$$E^{\mathbb{D}} (F_T - K)^+ = E^{\mathbb{P}} \left(F_0 \frac{R_T}{R_0} - K_c\right)^+.$$

- What happens if we express the quantity on the right in dollars rather than pounds?

Call Put Symmetry

- Recall our example of a vanilla call:

$$E^{\mathbb{D}}(F_T - K_c)^+ = E^{\mathbb{P}}\left(F_0\frac{R_T}{R_0} - K_c\right)^+.$$

- Since $\frac{d\mathbb{P}}{d\mathbb{D}} = \frac{F_T}{F_0}$, it follows that $\frac{d\mathbb{D}}{d\mathbb{P}} = \frac{R_T}{R_0}$, and hence we have:

$$\begin{aligned}
E^{\mathbb{D}}(F_T - K_c)^+ &= E^{\mathbb{P}}\frac{R_T}{R_0}1\left(F_0 - \frac{K_c R_0}{R_T}\right)^+ \\
&= K_c R_0 E^{\mathbb{D}}1\left(\frac{F_0}{K_c R_0} - \frac{1}{R_T}\right)^+ \\
&= \frac{K_c}{F_0}E^{\mathbb{D}}\left(\frac{F_0^2}{K_c} - F_T\right)^+.
\end{aligned}$$

- This is call put symmetry. So long as it holds at the first passage time to a barrier, it can be used to find a static hedge for a down-and-in call.

Static Hedge of a Down-and-In Call

- Consider a down-and-in call (DIC) written on the forward FX rate F.
- Letting K denote the strike and L denote the lower barrier $L < F_0$, the payoff at the maturity date T is:

$$DIC_T(K, L, T) = 1(\tau < T)(F_T - K)^+,$$

where τ is the first passage time of F to L ($\tau = \infty$ if L is never hit).

- Suppose we sell a DIC at $t = 0$ and buy $\frac{K}{L}$ vanilla puts struck at $\frac{L^2}{K}$.
- If F stays above L over $[0, T]$, then the puts expire worthless as does the DIC.
- If F hits L before T, then at $\tau \in [0, T]$, call put symmetry implies that the \$ received from selling the $\frac{K}{L}$ puts is exactly the \$ needed to buy 1 vanilla call struck at K_c:

$$E_\tau^{\mathbb{D}}(F_T - K_c)^+ = \frac{K_c}{L} E_\tau^{\mathbb{D}} \left(\frac{L^2}{K_c} - F_T \right)^+.$$

Implications for Pricing of a Down-and-In Call

- Recall that $\frac{K}{L}$ vanilla puts struck at $\frac{L^2}{K}$ has the same payoff at $\tau \wedge T$ as a down-and-in call with strike K and barrier $L < F_0$.

- It follows from no arbitrage that:

$$DIC_0(K, L, T) = \frac{K}{L} P_0 \left(\frac{L^2}{K}, T \right).$$

- So long as L is skipfree and call put symmetry holds at τ, the hedge succeeds even if F does not follow geometric Brownian motion.

- The static hedging results can be extended to constant rates, a drifting underlying, asymmetric dynamics, and other single and double barrier options (see my website for details).

- In particular, if we independently randomize the instantaneous volatility when the underlying has zero drift, then call put symmetry still holds and hence so do the above results.

Summary of Barrier Options

- We first developed many model-free results that relate payoffs and values of different barrier options to each other.

- We then showed how to (semi-)statically hedge the payoffs of some single barrier options with vanilla options when barriers are skip-free and when the risk-neutral distribution of the terminal forward FX rate is symmetric at the first passage time(s).

- As the conditions leading to exact replication never hold in practice for either classical dynamic replication or for semi-static option replication, it is worth noting that nothing prevents doing both.