Introduction to Haken *n*-Manifolds

applied to the Euler characteristic of an aspherical 4-manifold

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Aspherical spaces

Definition

A connected CW complex or manifold is *aspherical* if its universal covering space is contractible. (Or, its higher homotopy groups vanish.)

Standard classical examples

- Circle, tori, flat manifolds
- Surfaces of higher genus, hyperbolic manifolds
- Manifolds of non-positive curvature

Quotes about General Aspherical Manifolds

Wolfgang Lück (Bonn), 2009

"Almost all closed manifolds are aspherical, topologically rigid and asymmetric."

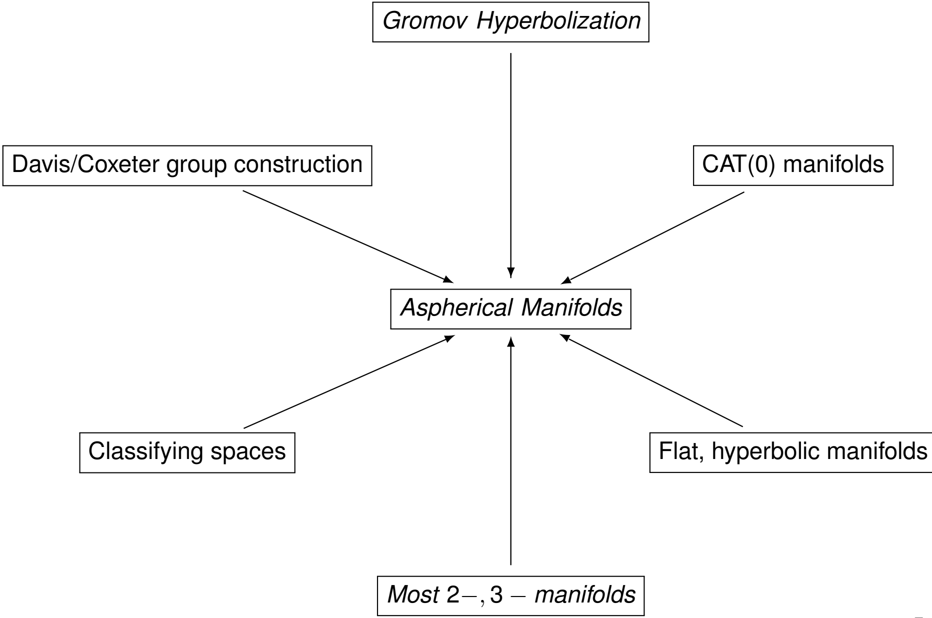
See Lück's survey on Aspherical Manifolds at the Manifold Atlas Project at `www.manifoldatlas.him.uni-bonn.de`

Quotes about General Aspherical Manifolds

Hyam Rubinstein (Melbourne), 2010

“The aspherical world in dimension four is enormous but not much is known about it, . . .

A Bit More Mathematical Context



Two Longstanding Conjectures 1/2

From the 1970s

Euler Characteristic Sign Conjecture (Hopf/Chern/Thurston)



If X^n is a closed aspherical n-manifold, with $n = 2k$, then

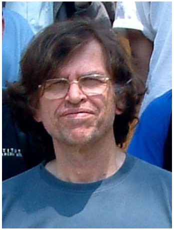
$$(-1)^k \chi(X^n) \geq 0.$$

Model case Product of k aspherical surfaces.

Two Longstanding Conjectures 2/2

From the 1970s

Homology Type Problem (Kan/Thurston)

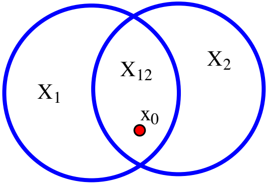
 

Every closed n-manifold (with the obvious exception of S^2 and RP^2) has the same rational homology as an aspherical n-manifold.

Notes True in dimensions ≤ 3. There exist aspherical homology 4-spheres, aspherical homology complex projective planes, and aspherical homology 5-spheres.

Still, the two conjectures can't both be true!

Whitehead's Theorem



J.H.C. Whitehead,1939

When the union of aspherical subspaces is aspherical. Suppose a cell complex X can be expressed as the union of two aspherical subcomplexes X_1 and X_2 such that each component of the intersection X_{12} is aspherical and the inclusion induced homomorphisms $\pi_1(X_{12}, x_0) \to \pi_1(X, x_0)$ and $\pi_1(X_{12}, x_0) \to \pi_1(X, x_0)$ are injective for all choices of base point $x_0 \in X_{12}$. Then X is aspherical.

Idea The universal covering of X can be constructed from contractible pieces with pairwise intersections contractible. Then the homology groups, hence higher homotopy groups are trivial.

Recollections about Haken 3-Manifolds

(By Halmos) 1964

2005

Haken, early 1960s

a classical Haken manifold is a 3-manifold that can be cut open successively along "incompressible" (in particular π_1-injective) surfaces, until one obtains a finite collection of 3-cells. Originally called *sufficiently large 3-manifolds*. Aspherical 3-manifolds.

Waldhausen, 1968

Accessible details and a rigidity theorem for Haken 3-manifolds.

The Idea of a Haken *n*-Manifold

in higher dimensions

Developed by B. Foozwell and H. Rubinstein (Melbourne)

Roughly speaking, a Haken *n*-manifold is an *n*-manifold that can be cut open successively along "essential" codimension-one submanifolds, until one obtains a finite collection of *n*-cells.

A few more details of the rather complicated precise definition will come later.

The Idea of a Haken n-Manifold

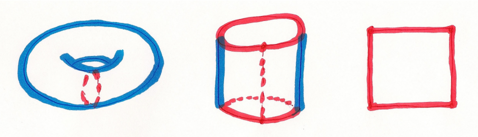
To make this idea work a fair amount of technical preparation is required, building on ideas of K. Johannson from the 1970s, using "boundary patterns".

Intuitively

one can say that Haken n-manifolds are designed to facilitate proofs by induction, both on the dimension of the manifold and on the length of a hierarchy (= number of steps required to reduce to n-cells).

Picture to have in mind

Torus $\to$ Annulus $\to$ 4-sided disk.



Examples of Haken n-Manifolds

Surfaces

A connected topological surface is a Haken 2-manifold if and only if it is not a 2-sphere or projective plane.

3-Manifolds

Any 3-manifold that admits an incompressible surface is Haken. Now it is known that any hyperbolic 3-manifold is “virtually Haken”.

Products and Fiber Bundles

Products of Haken manifolds admit Haken structures.

Basic Properties of Haken n-Manifolds

Asphericity

A Haken n-manifold is aspherical. It's universal covering is contractible. Foozwell

Universal covering

The (interior of the) universal covering of a Haken n-manifold is homeomorphic to euclidean space. Foozwell

Note The exotic Davis n-manifolds, $n \geq 4$, are aspherical but not Haken or even virtually Haken.

Word problem

The fundamental group of a Haken n-manifold has a solvable word problem. Foozwell

Some technicalities (with details suppressed)

Boundary Patterns based on ideas of Klaus Johannson

A boundary pattern on a manifold M^n is a collection of $(n-1)$-submanifolds with boundary of ∂M^n (the "facets") such that the intersection of any k of them is an $(n-k)$-submanifold of ∂M^n.

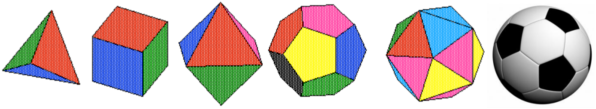
Note If two $(n-1)$-faces intersect at all, then they intersect in a common $(n-2)$-face.

More technicalities

Haken n-cells

- Haken 0-cell = point,
- Haken 1-cell = interval.
- Haken 2-cell= polygon with at least 4 sides. No triangles
- A Haken n-cell is a topological cell with a complete boundary pattern in which each facet is a Haken $(n-1)$-cell.

Examples



- An n-cube is Haken,
- But a simplex is not Haken
- A dodecahedron is Haken,
- But an icosahedron does not even have a valid boundary pattern

Cutting Open

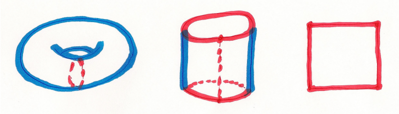
Induced boundary pattern

Let M^n have a boundary pattern and let $F^{n-1} \subset M^n$ be a codimension-one proper submanifold intersecting the facets and their faces transversely.

$$\text{Set } N^n := M^n | F^{n-1} = M^n \text{ cut open along } F^{n-1}.$$

Then N^n inherits a boundary pattern with facets given as the components of $G^{n-1}|\partial F^{n-1}$, where G^{n-1} varies over the facets of M^n, together with two copies of F^{n-1}.

Picture



Hierarchies

Hierarchy

For an n-manifold M^n (with “complete” and “useful” boundary pattern) is a sequence

$$(M_0, F_0), (M_1, F_1), \ldots, (M_k, F_k)$$

of “good pairs”, such that

- $M_0 = M^n$
- $M_{i+1} = M_i | F_i$, and
- $M_{k+1} := M_k | F_k$ is a disjoint union of Haken n-cells.

Haken n-manifold

A compact n-manifold with complete useful boundary pattern and a corresponding hierarchy.

Aside on Haken manifolds with boundary

Best case

Aspherical boundary that is π_1-injective-e.g. nontrivial knot exterior

May have

- non-aspherical boundary–e.g. n-ball, $n \geq 3$
- non π_1-injective boundary-e.g. 2-disk, solid handlebody

In general

However, the facets must be both aspherical and π_1-injective.

Current Directions for Haken n-Manifolds

Recent Theorem (Foozwell-Rubinstein, 2012)

If X^3 is a closed Haken 3-manifold, then there is a compact Haken 4-manifold Y^4 such that $\partial W^4 = X^3$ with $\pi_1 X^3 \to \pi_1 W^4$ injective.

Characteristic submanifolds?

Is there anything like the JSJ decomposition of prime 3-manifolds for higher-dimensional Haken n-manifolds?

Rigidity Conjecture for Haken n-manifolds?

If X^n is a compact Haken n-manifold without boundary, Y^n is a closed n-manifold, and $f : Y^n \to X^n$ is a homotopy equivalence, then f is homotopic to a homeomorphism.

Change Gears

Study the Euler Characteristic Problem for Haken 4-Manifolds.

Euler Characteristic of a Haken 4-Manifold

Part of joint project with Steve Klee.

Theorem (“90%”)

If X^4 is a closed Haken 4-manifold, then $\chi(X^4) \geq 0$.

Keep in mind

If X^4 is a compact Haken 4-manifold with boundary, then $\chi(X^4)$ may be negative. E.g., the n-fold boundary connected sum

$$S^1 \times I^3 \natural S^1 \times I^3 \natural \ldots \natural S^1 \times I^3$$

has $\chi = 1 - n$.

Euler Characteristic of a Haken 4-Manifold

Manifolds with boundary

Need a quantity $\varphi(\partial X^{4})$ depending on the boundary and the boundary pattern such that

$$\chi(X^4) \geq \varphi(\partial X^4)$$

Desired Properties/Plan of attack

- If X^4 is a Haken 4-cell, then $\varphi(\partial X^4) \leq 1$. (Beginning of induction.)
- If $Y^4 = X^4 | G^3$, then

$$\begin{aligned} \chi(X^4) &= \chi(Y^4) - \chi(G^3) \quad \text{(Sum Formula)} \\ &\geq \varphi(\partial Y^4) - \chi(G^3) \quad \text{(Inductive Hypothesis)} \\ &\geq \varphi(\partial X^4) \quad \text{(Transformation Law)} \end{aligned}$$

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Euler Characteristic of a Haken 4-Manifold

Form of a possible φ-function

$$\varphi(\partial X^4) = \sum_{k=0}^{3} r_k f_k(X^4) + \sum_{k=0}^{3} s_k \sum_{F^k < X^4} \chi(F^k) + \sum_{k=0}^{3} t_k b_k(\partial X^4)$$

Theorem

If there is such a function, then it can be expressed in the form

$$\varphi(\partial X^4) = a f_0(X^4) + b f_1(X^4) + \frac{1}{4} \sum_{F^3 < X^4} \chi(F^3)$$

where $a + 2b = -1/16$.

Proof Evaluate on "known" Haken 4-manifolds.

Euler Characteristic of a Haken 4-Manifold

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Euler Characteristic of a Haken 4-Manifold

Theorem (not all details written down yet)

If φ has the form

$$\varphi(\partial X^4) = af_0(X^4) + bf_1(X^4) + \frac{1}{4} \sum_{F^3 < X^4} \chi(F^3)$$

where $a + 2b = -1/16$, then for any compact Haken 4-manifold

$$\chi(X^4) \geq \varphi(\partial X^4).$$

Remark

For Haken 4-cells this inequality is dual in a certain sense to the Charney-Davis inequality for flag triangulations of the 3-sphere.

Euler Characteristic of a Haken 4-Manifold

To prove

$$\chi(X^4) \geq \varphi(\partial X^4) = af_0(X^4) + bf_1(X^4) + \frac{1}{4}\sum_{F^3 < X^4} \chi(F^3)$$

Proof by induction on length of hierarchy, outline only

- Beginning of induction. ("Generic" case done; details remain to be worked through.) Reduces to

  If X^4 is a Haken 4-cell, then $f_0(X^4) \geq 4f_3(X^4) - 16$.

- Inductive step. (Complete.) If $Y^4 = X^4 | G^3$ is the next step in the hierarchy, then

$$\begin{aligned}
\chi(X^4) &= \chi(Y^4) - \chi(G^3) \quad \text{(Sum Formula for } \chi\text{)}\checkmark \\
&\geq \varphi(\partial Y^4) - \chi(G^3) \quad \text{(Inductive Hypothesis)}\checkmark \\
&\geq \varphi(\partial X^4) \quad \text{(Estimate how } \varphi \text{ can change)}\checkmark
\end{aligned}$$

Further Directions

Higher dimensional versions

Find a function φ such that for any X^6, Haken 6-manifold with boundary pattern one has $-\chi(X^6) \geq \varphi(\partial X^6)$.

Known 2-D version is a fun exercise

$$-\chi(X^2) \geq 2f_0(X^2) - 3f_1(X^2) + 3b_0(\partial X^2)$$

Alternatively

Find a Haken 6-manifold without boundary such that $\chi(X^6) > 0$.
Find an aspherical homology 6-sphere!

Somehow there is more room for things to go “wrong” in dimensions > 4.

References for Haken *n*-manifolds

Just search for Foozwell

- Bell Foozwell, Haken *n*-manifolds, Ph.D. thesis, Melbourne University, 2007. Available at `sites.google.com/site/bellfoozwell/Home/research`
- Bell Foozwell and Hyam Rubinstein, Introduction to the theory of Haken *n*-manifolds, Proceedings of the Conference on Topology and Geometry in Dimension Three: Triangulations, Invariants, and Geometric Structures. Contemporary Mathematics, 560, AMS, 2011. Search for Foozwell at `books.google.com`.
- Bell Foozwell, The Universal Covering Space of a Haken *n*-Manifold, preprint, 2011. `arXiv:1108.0474v3`.

Happy Birthday, Ravi!

Portrait of the Mathematical Artist as a Young Man

Ravi Kulkarni, late 1970s at Indiana University