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1. L^p-Spaces:

Notations:

$\mathbb{R}$: denotes the real number system

$\mathbb{C}$: denotes the complex number system.

$\mathbb{R}^n = \{ x = (x_1, \dots, x_n) : x_i \in \mathbb{R} \}$.

$\mathbb{C}^n = \{ x = (x_1, \dots, x_n) : x_i \in \mathbb{C} \}$.

$x, y \in \mathbb{R}^n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \; ; \quad |z|^2 = \sum_{i=1}^{n} |z_i|^2 .$$

$$|x|_2^2 = |x|^2 = \sum_{i=1}^{n} x_i^2$$

$$|x|_p^p = \sum_{i=1}^{n} |x_i|^p$$

$\mathbb{R}^n_+ = \{ x = (x_1, \dots x_n) \in \mathbb{R}^n : x_n > 0 \}$ - upper half space

$\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} = \{ x = (x_1, \dots x_{n-1}, 0) \in \mathbb{R}^n \}$ - boundary of $\mathbb{R}^n_+$.

$\Omega \subset \mathbb{R}^n$ always denotes a domain, i.e. Connected open set.

$\partial \Omega$ = boundary of Ω.

$dx = dx_1 \dots dx_n$ = Lebesgue measure on $\mathbb{R}^n$.

For a measurable set $A \subset \mathbb{R}^n$ denote

$|A|$ = Lebesgue measure of A.

Regularity of Lebesgue measure. Let A be a measurable set in $\mathbb{R}^n$ with $|A| < \infty$. Given $\epsilon > 0$, $\exists$ Compact set K and an open set V such that $K \subset A \subset V$ and $|V \setminus K| < \epsilon$.

Spaces of Test functions.

Let $0 \le k \le \infty$ be an integer and define

$$C^k(\Omega) = \{ f : \Omega \to \mathbb{R} \ (\mathbb{C}) ; \ f \text{ is } k\text{-times continuously differentiable in } \Omega \}$$

$$C_0^k(\Omega) = \{ f \in C^k(\Omega) : \text{Support of } f \text{ is a Compact subset of } \Omega \}$$

$$\text{Support of } f = \text{Supp } f = \overline{\{x : f(x) \neq 0\}}.$$

For $\alpha = (\alpha_1 \cdots \alpha_n)$, $\alpha_i \ge 0$ α_i are integers. denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

For $f \in C^k(\Omega)$, $0 \le j \le k$ denote

$$|f|_{j,\Omega} = |f|_{j,\infty,\Omega} = \sum_{|\alpha| = j} \sup_{x \in \Omega} |\partial^\alpha f(x)|$$

$$\|f\|_{k,\Omega} = \|f\|_{k,\infty,\Omega} = \sum_{j=0}^{k} |f|_{j,\Omega,0}$$

L^p-spaces: Let $1 \le p \le \infty$ and $f: \mathbb{R}^n \to \mathbb{R}(\mathbb{C})$ be a measurable function. Define

$$\|f\|_{p,\mathbb{R}^n} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p} & \text{if } p < \infty \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| & \text{if } p = \infty. \end{cases}$$

and define

$$L^p(\mathbb{R}^n) = \{ f : \|f\|_{p,\mathbb{R}^n} < \infty \}, \quad \|f\|_p := \|f\|_{p,\mathbb{R}^n}.$$

is a Banach space.

Convolutions: Let $f, g \in L^1(\mathbb{R}^n)$, define

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy.$$

Then we have the

Young's inequality: Let $1 \le p, q \le \infty$ and define r such that

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$$

Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$. Then $f * g \in L^r(\mathbb{R}^n)$ with

$$\|f * g\|_{r,\mathbb{R}^n} \le \|f\|_{p,\mathbb{R}^n} \cdot \|g\|_{q,\mathbb{R}^n}.$$

Proof: Let $h \in L^{r'}$, $\frac{1}{r} + \frac{1}{r'} = 1$. Let $\alpha + \alpha' = 1 = \beta + \beta'$, $0 \le \alpha, \beta \le 1$.

$$I = \left| \int_{\mathbb{R}^n} (f * g)(x) h(x) dx \right| \le \int\int |f(x-y)| |g(y)| |h(x)| dx dy$$

$$= \int\int |f^{\alpha}(x-y) g^{\beta}(y)| \, |f^{\alpha'}(x-y) g^{\beta'}(y) h(x)| dx dy$$

$$\le \int\int

$$\leq \left(\iint |f|^{\alpha r} |x-y|^{\beta r} |g(y)|^{\beta r} \, dx\,dy \right)^{1/r}$$

$$\times \left(\iint |f|^{\alpha' r' s'} |x-y|^{\beta' r' s'} |h(x)|^{r'} \, dx\,dy \right)^{1/r'}$$

$$= \left(\int |f|^{\alpha r} dy \right)^{\frac{1}{r}} \left(\int |y|^{\beta r} \right)^{1/r}$$

$$\times \left(\int h(x)^{r'} dy \left[\int |f|^{\alpha' r' s' (x-y)} g(y)^{\beta' r' s'} dy \right] dx \right)^{1/r'}$$

$$\leq \left(\int |f|^{\alpha r} dx \right)^{\frac{1}{r}} \left(\int |y|^{\beta r} dy \right)^{1/r} \left(\int |h|^{r'} dx \right)^{1/r'}$$

$$\times \left(\int |f|^{\alpha' r' s'} \right)^{\frac{1}{r' s'}} \left(\int |y|^{\beta' r' s'} \right)^{\frac{1}{r' s'}}$$

When $\frac{1}{s} + \frac{1}{s'} = 1$.

Now choose $\alpha, \beta, s,$ such That

$$\alpha r = \alpha' r' s = p. \qquad \alpha + \alpha' = 1$$

$$\beta r = \beta' r' s' = q. \qquad \beta + \beta' = 1$$

Hence

$$\frac{p}{r} + \frac{p}{r' s} = \alpha + \alpha' = 1$$

$$\frac{q}{r} + \frac{q}{r' s'} = \beta + \beta' = 1$$

$$\frac{2}{r} + \frac{1}{r'} \left(\frac{1}{s} + \frac{1}{s'} \right) = \frac{1}{p} + \frac{1}{q}.$$

$$\frac{2}{r} + 1 - \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1.$$

$$I \leq \|f\|_{\frac{p}{r} + \frac{p}{r' s}}^{\frac{q}{r} + \frac{q}{r' s'}} \|g\|_q \|h\|_{r'}$$

$$= \|f\|_p \|g\|_q \|h\|_{r'}.$$

Hence by duality

$$\|f*g\|_r \leq \|f\|_p \|g\|_q.$$

This proves the inequality.

Smoothening operator.

Definition. $L^1_{loc}(\Omega)$ = locally integrable functions defined by

$$L^1_{loc}(\Omega) = \{ f : \Omega \to \mathbb{R}(\mathbb{C}) \text{ s.t. } f \text{ is measurable and for any compact set } K \subset \Omega \int_K |f(x)|\,dx < \infty \}.$$

Mollifiers: Let $0 \leq k \leq \infty$, $\epsilon > 0$, $\rho \in C_0^\infty(\mathbb{R}^n)$ such that

(i) $\rho(x) \geq 0$ for all $x \in \mathbb{R}^n$.

(ii) $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$

Define $\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho(x/\epsilon)$, Then $\{\rho_\epsilon\}$ is called a Mollifiers.

Theorem 1: Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in C_0^k(\mathbb{R}^n)$ where $0 \leq k \leq \infty$. Then

(1) $f * \varphi \in C^k(\mathbb{R}^n)$

(2) If f vanishes outside a closed set A. Then support of $f * \varphi$ $\subset$ A + Supp φ.

(3) If $f \in C^k(\mathbb{R}^n)$, Then $\partial^\alpha (f * \varphi) = \partial^\alpha f * \varphi = f * \partial^\alpha \varphi$ for all $|\alpha| \leq k$

(4) If $f \in C^k(\mathbb{R}^n)$, then for all $|\alpha| \le k$,
$\partial^\alpha (f * \rho_\epsilon) \to \partial^\alpha f$ uniformly on compact subsets as $\epsilon \to 0$.

(5) If $f \in L^p(\mathbb{R}^n)$, $p < \infty$. Then
$\|f * \rho_\epsilon - f\|_p \to 0$ as $\epsilon \to 0$

Proof: Let K = Support of φ. Then

$$(f*\varphi)(x) = \int_{\mathbb{R}^n} f(y)\varphi(x-y)\,dy$$

$$= \int_{x-K} f(y)\varphi(x-y)\,dy$$

Since $f \in L^1_{loc}$ and $x-K$ is compact in $\mathbb{R}^n$ for each fixed x, hence $f*\varphi$ is well defined. From the dominated convergence theorem, it follows that $f*\varphi \in C^0(\mathbb{R}^n)$.

Let A be a closed set and support $f = 0$ on $\mathbb{R}^n \setminus A$. Then

$$(f*\varphi)(x) = \int_{(x-K)\cap K} f(y)\varphi(x-y)\,dy$$

$$= 0 \quad (\text{if } (x-K)\cap A = \emptyset)$$

i.e. $x \notin A+K$. Hence

$$\operatorname{Supp}(f*\varphi) \subset A+K.$$

Let $\varphi \in C_0^k(\mathbb{R}^n)$ with $k \ge 1$. Then for $x \in \mathbb{R}^n \setminus \{0\}$, $h \in \mathbb{R}$, we have

$$\varphi(x+hv)-\varphi(x)=\int_0^1 \frac{d}{dt}\varphi(x+thv)\,dt$$

$$=\int_0^1 h\sum \frac{\partial\varphi}{\partial x_i}(x+thv)v_i\,dt$$

$$=h\int_0^1 \langle\nabla\varphi(x+thv),v\rangle\,dt$$

where $\nabla\varphi(x)=\left(\frac{\partial\varphi}{\partial x_1},\cdots\frac{\partial\varphi}{\partial x_n}\right)$, the gradient of φ.

Hence

$$\frac{(f*\varphi)(x+hv)-(f*\varphi)(x)}{h}=\int_{\mathbb{R}^n} f(y)\left(\frac{\varphi(x+hv-y)-\varphi(x-y)}{h}\right)dx$$

$$=\int_0^1\int_{\mathbb{R}^n} f(y)\langle\nabla\varphi(x+thv-y),v\rangle\,dt\,dx$$

Hence by dominated convergence theorem, we can let $h\to 0$ to obtain

$$\langle\nabla(f*\varphi),v\rangle=\int_{\mathbb{R}^n} f(y)\langle\nabla\varphi(x-y),v\rangle\,dx.$$

and is continuous. Hence $f*\varphi$ is in $C^1(\mathbb{R}^n)$.

and taking $v=e_i=(0,\cdots,1,0,\cdots,0)$, to obtain

$$\frac{\partial}{\partial x_i}(f*\varphi)=f*\frac{\partial\varphi}{\partial x_i}$$

By repeating the same argument, it follows that

$$\partial^\alpha(f*\varphi)=f*\partial^\alpha\varphi$$

for all $|\alpha|\le k$.

Since $f*\varphi(x) = \int_{\mathbb{R}^n} f(x-y)\varphi(y)\,dy$, if $f \in C^k$, Then $\partial^\alpha(f*\varphi) = \partial^\alpha f * \varphi$ $\forall |\alpha| \le k$.

This proves (1), (2) and (3).

Let $A \subset \mathbb{R}^n$ be a compact set. Then $A+K$ is compact

$$f*\rho_\epsilon(x) - f(x) = \int_{\mathbb{R}^n} f(x-y)\rho_\epsilon(y)\,dy - f(x)$$

$$= \int_{\mathbb{R}^n} f(x-\epsilon y)\rho(y)\,dy - f(x)\int_{\mathbb{R}^n}\rho(x)\,dx$$

$$= \int_{\mathbb{R}^n} \big(f(x-\epsilon y) - f(x)\big)\rho(y)\,dy$$

$$= \int_K (f(x-\epsilon y) - f(x))\rho(y)\,dy.$$

Since for $x \in A$, $y \in K$, $x - \epsilon y \in A - \epsilon K$, which is contained in a compact set for $\epsilon \le 1$, hence by uniform continuity,

$$\lim_{\epsilon\to 0} \sup_{\substack{x\in A \\ y\in K}} |f(x-\epsilon y) - f(x)| = 0.$$

Now by dominated convergence theorem,

$$\lim_{\epsilon\to 0} \sup_{x\in A} |f*\rho_\epsilon(x) - f(x)| = 0.$$

Since $\partial^\alpha(f*\rho_\epsilon) = \partial^\alpha f * \rho_\epsilon$, hence if $f \in C^k(\mathbb{R}^n)$, then $\partial^\alpha(f*\rho_\epsilon) = \partial^\alpha f * \rho_\epsilon \to 0$ as $\epsilon \to 0$ uniformly on compact sets. This proves (4)

In order to prove (5), first we have the following

Claim: $C_0^0(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ is dense.

Step 1: Let $A \subset \mathbb{R}^n$ be a measurable set with $|A| < \infty$ and $\epsilon > 0$, Choose $K_1 \subset A \subset U$, K_1 Compact, U open such that: $|U \setminus K_1| < \epsilon$. Let

$$h(x) = \frac{d(x, \mathbb{R}^n \setminus U)}{d(x, \mathbb{R}^n \setminus U) + d(x, K_1)}$$

where $d(x, B)$ is the distance from x to B defined by,

$$d(x, B) = \inf_{y \in B} |x - y|$$

Since d is continuous and hence $h \in C^0(\mathbb{R}^n)$ and $0 \le h \le 1$ with $h(x) = \begin{cases} 0 & \text{if } x \notin U \\ 1 & \text{if } x \in K_1 \end{cases}$

Hence

(i) $|\chi_A(x) - h(x)| \le 2$

(ii) $\chi_A(x) - h(x) = 0$ if $x \notin K \cup (\mathbb{R}^n \setminus U)$.

Hence $\int_{\mathbb{R}^n} |\chi_A(x) - h(x)|^p \le 2^p |U \setminus K_1| < 2^p \epsilon$.

Thus proves that $\|\chi_A - h\|_p \to 0$ as $\epsilon \to 0$ and $h \in C^0(\mathbb{R}^n)$. If A is bounded set, then $h \in C_0^0(\mathbb{R})$.

Step 2: Since simple functions with bounded support are dense in $L^p(\mathbb{R}^n)$. Hence by Step (1), $C_0^0(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ is dense.

Step 3: Let $f \in L^p(\mathbb{R}^n)$. Let $\eta > 0$ such. Choose $h \in C_0^\infty(\mathbb{R}^n)$ s.t. $\|f-h\|_p < \eta/4$

Let ϵ small such that

$$\|h * \rho_\epsilon - h\|_p < \eta/2$$

Then by young's inequality

$$\|f - f*\rho_\epsilon\|_p = \|(f-h)*\rho_\epsilon + h*\rho_\epsilon - h + h - f\|_p$$

$$\leq \|f-h\|_p \|\rho_\epsilon\|_1 + \|h*\rho_\epsilon - h\|_p + \|h-f\|_p$$

$$\leq 2\|f-h\|_p + \|h*\rho_\epsilon - h\|_p$$

$$< \eta$$

Since $\int f_\epsilon(x)\,dx = \frac{1}{\epsilon^n}\int \rho\left(\frac{x}{\epsilon}\right)dx = \int \rho(x)\,dx = 1$

This proves (5). and hence the theorem.

Before going to the next topic, following is the generalization of young's inequality

Theorem 2 (young's inequality): Let $1 \leq p_i \leq \infty$, $i = 1 \cdots L$, and let $1 \leq q \leq \infty$ be such that

$$\sum_{i=1}^{L} \frac{1}{p_i} - \frac{1}{q} = L-1.$$

Let $f_i \in L^{p_i}$, $1 \leq i \leq L$, Then $f_1 * \cdots * f_L \in L^q$.

Proof: By induction. For $L=2$, already proved. Assume that it is true for $L-1$. Then $f_1 * \cdots * f_{L-1} \in L^{q_1}$ where q_1 is given by

$$\sum_{i=1}^{L-1} \frac{1}{p_i} - \frac{1}{q_1} = L-2.$$

Define q by

$$\frac{1}{p_L} + \frac{1}{p_i} - \frac{1}{q} = 1$$

Then $(f_1 * \cdots * f_{L-1}) * f_L \in L^q$. Hence

$$1 - \frac{1}{p_L} + \sum_{i=1}^{L-1} \frac{1}{p_i} - L + 2 - \frac{1}{q}$$

$$\Rightarrow \sum_{i=1}^{L} \frac{1}{p_i} - \frac{1}{q} = L - 1.$$

and

$$\| f_1 * \cdots * f_L \|_q \leq \prod_{i=1}^{L} \| f_i \|_{p_i}.$$

This proves the theorem.

Sobolev Spaces - Weak and Strong derivatives

Weak derivative. Let $f \in L^1_{loc}$. f has weak derivatives of order $k \geq 1$ if there exists $f_\alpha \in L^1_{loc}(\Omega)$ for $|\alpha| \leq k$ such that for all $\varphi \in C_0^\infty(\Omega)$,

$$(-1)^{|\alpha|} \int_\Omega f \, \partial^\alpha \varphi \, dx = \int_\Omega f_\alpha \varphi \, dx.$$

$\{f_\alpha\}$ are called weak derivatives of f & $\partial^\alpha f$ also called Distributional Derivatives.

Properties:

(1) If $f \in C^k(\Omega)$. Then $f_\alpha = \partial^\alpha f$. This follows by integration by parts.

(2) $\{f_\alpha\}$ are unique. Suppose $\exists\ g_\alpha \in L^1_{loc}$ such that

$$\int_\Omega f_\alpha \varphi \, dx = \int_\Omega g_\alpha \varphi \, dx$$

Then

$$\int_\Omega (f_\alpha - g_\alpha) \varphi \, dx = 0$$

Let $\Omega_1 \subset\subset \Omega$ be an open set with $\overline{\Omega_1}$ is compact in Ω. Let $\chi \in C_0^\infty(\Omega)$ such that $\chi \equiv 1$ on Ω_1. Let $h_\alpha = \chi(f_\alpha - g_\alpha)$. Then $h_\alpha \in L^1(\mathbb{R}^n)$ and for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\chi\varphi \in C_0^\infty(\Omega)$ and

$$\int_{\mathbb{R}^n} h_\alpha \varphi \, dx = \int_\Omega (f_\alpha - g_\alpha)\chi\varphi \, dx = 0$$

Since $C_0^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is dense and hence $h_\alpha = 0$ for $|\alpha| \le k$. This proves (2)

Notation :- In view of (1) and (2), we denote

$$f_\alpha = \partial^\alpha f.$$

Example :(1) Let $f(x) = |x|$ and $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$-\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx = -\int_{-\infty}^{0} f(x)\varphi'(x)\,dx - \int_0^\infty f(x)\varphi'(x)\,dx$$

$$= \int_{-\infty}^{0} x\varphi'(x)\,dx - \int_0^\infty x\varphi'(x)\,dx$$

$$= x\varphi(x)\Big]_{-\infty}^{0} - x\varphi(x)\Big]_0^\infty - \int_{-\infty}^{0}\varphi\,dx + \int_0^\infty \varphi\,dx$$

$$= \int_{-\infty}^{\infty} H(x)\varphi(x)\,dx$$

where $H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$. Hence $\frac{df}{dx} = H$ in the Distributional derivative and $H \in L^1_{loc}(\mathbb{R})$.

In general if $f : \mathbb{R} \to \mathbb{R}$ is locally lipschitz continuous function, Then f is absolutely continuous on each bounded interval and by Lebesgue theorem f' exists

Lebesgue's theorem, $f'(x)$ exist a.e. and satisfies

$$-\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx .$$

Lemma 3: Let $k \geq 1$ and $f \in L^1_{loc}(\Omega)$ having weak derivatives upto order k. Let $\chi \in C^k(\Omega)$. Then

(1) For all $\varphi \in C_0^k(\Omega)$, $|\alpha| \leq k$

$$-\int_{\Omega} f\,\partial^\alpha\varphi\,dx = \int_{\Omega} \partial^\alpha f\,\varphi\,dx .$$

$$\frac{\partial}{\partial x_i}(\chi f) = \chi\frac{\partial f}{\partial x_i} + f\frac{\partial \chi}{\partial x_i}$$

(2) Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\chi \in C_0^k(\mathbb{R}^n)$. Then for $|\alpha| \leq k$

$$\partial^\alpha(f\chi) = \partial^\alpha f \chi = f\chi\partial^\alpha\chi .$$

(3) Let $1 \leq p < \infty$ and $\partial^\alpha f \in L^p(\mathbb{R}^n)$ for $|\alpha| \leq k$. Then there exist $g_\epsilon \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{\epsilon \to 0} \|\partial^\alpha f - \partial^\alpha g_\epsilon\|_p = 0$$

Proof: Let $B(0,1) = \{x : |x| < 1\}$ and $\rho \in C_0^\infty(B(1))$ such that

$$\rho \geq 0, \quad \int_{\mathbb{R}^n} \rho(x)\,dx = 1$$

For $\epsilon > 0$, let $\rho_\epsilon(x) = \frac{1}{\epsilon^n}\rho(x/\epsilon)$ be a mollifying sequence. Let $\varphi \in C_0^k(\Omega)$ and $\varphi_\epsilon = \varphi * \rho_\epsilon \in C_0^\infty(\mathbb{R}^n)$. Then

$$\lim_{\epsilon \to 0} \|\partial^\alpha\varphi_\epsilon - \partial^\alpha\varphi\|_\infty = 0 .$$

Since support of $\varphi \subset \Omega$ and $\text{Supp}(\varphi * \rho_\epsilon) \subset$ $\text{Supp}\,\varphi + \text{Supp}(\rho_\epsilon) \subset \text{Supp}\,\varphi + B(\epsilon) \subset \Omega$ if ϵ is

sufficiently small, say $\epsilon < \epsilon_0$ for some fixed ϵ_0 depending on φ. Then

$$(-1)^{|\alpha|}\int_\Omega f\,\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\lim_{\epsilon\to 0}\int_\Omega f\,\partial^\alpha(\varphi_\epsilon)\,dx$$

$$= \lim_{\epsilon\to 0}\int_\Omega \partial^\alpha f\,\varphi_\epsilon\,dx$$

$$= \int_\Omega \partial^\alpha f\,\varphi\,dx.$$

Since $x\varphi \in C_0^k(\Omega)$, hence

$$-\int_\Omega x f\frac{\partial\varphi}{\partial x_i}\,dx = -\int_\Omega f\left[\frac{\partial}{\partial x_i}(x\varphi) - \varphi\frac{\partial x}{\partial x_i}\right]dx$$

$$= \int_\Omega\left[\frac{\partial f}{\partial x_i}x + f\frac{\partial x}{\partial x_i}\right]\varphi\,dx.$$

Hence $$\frac{\partial}{\partial x_i}(xf) = x\frac{\partial f}{\partial x_i} + f\frac{\partial x}{\partial x_i}$$

This proves (1).

Let $\varphi \in C_0^k(\mathbb{R}^n)$, then

$$(-1)^{|\alpha|}\int_{\mathbb{R}^n}(f*\chi)\,\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n}\partial^\alpha\varphi(x)\left(\int_{\mathbb{R}^n} f(y)\chi(x-y)\,dy\right)dx$$

$$= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n}\chi(x-y)\,\partial^\alpha\varphi(x)\,dx\right)dy$$

$$= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(y)\,\partial_y^\alpha\left(\int_{\mathbb{R}^n}\chi(x-y)\,\varphi(x)\,dx\right)dy$$

$$= \int_{\mathbb{R}^n}\partial^\alpha f(y)\left(\int_{\mathbb{R}^n}\chi(x-y)\,\varphi(x)\,dx\right)dy.$$

By induction, for $|\alpha| \le k$, $\partial^\alpha(\chi_R \psi) \to \partial^\alpha \psi$ in $L^p(\mathbb{R}^n)$. Therefore we have

$$\partial^\alpha(\chi_R(f * g_\epsilon)) \to \partial^\alpha f \quad \text{in } L^p(\mathbb{R}^n)$$

as $R \to \infty$, $\epsilon > 0$, and $\chi_R(f * g_\epsilon) \in C_0^\infty(\mathbb{R}^n)$. This proves (3) by taking $R = \frac{1}{\epsilon}$, $g_\epsilon = \chi_{1/\epsilon}(f * g_\epsilon)$. Hence the Lemma.

Sobolev Spaces $W^{m,p}(\Omega)$: Let $m \ge 0$ be an integer and $1 \le p \le \infty$. Define

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega) : \partial^\alpha f \text{ exist for } |\alpha| \le m \text{ and } \partial^\alpha f \in L^p(\Omega) \}.$$

Define the norm

$$\|f\|_{m,p,\Omega} = \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{p,\Omega}$$

Theorem 2: $W^{m,p}(\Omega)$ is a Banach space and for $1 \le p < \infty$, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.

Proof: Let $\{f_k\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then $\{\partial^\alpha f_k\}$ converges in $L^p(\Omega)$ for $|\alpha| \le m$. Let

$$\lim_{k \to \infty} \partial^\alpha f_k = f_\alpha.$$

Claim: $f_\alpha = \partial^\alpha f_0$.

For let $\varphi \in C_0^\infty(\Omega)$, then with

$$(-1)^{|\alpha|} \int_\Omega f_k \partial^\alpha \varphi \, dx = \int_\Omega \partial^\alpha f_k \varphi \, dx,$$

letting $k \to \infty$ to obtain

$$(-1)^{|\alpha|} \int_\Omega f_0 \partial^\alpha \varphi \, dx = \int_\Omega f_\alpha \varphi \, dx.$$

By induction, for $|\alpha| \le k$, $\partial^\alpha(\chi_R \psi) \to \partial^\alpha \psi$ in $L^p(\mathbb{R}^n)$. Therefore we know

$$\partial^\alpha(\chi_R(f * g_\epsilon)) \longrightarrow \partial^\alpha f \quad \text{in } L^p(\mathbb{R}^n)$$

as $R \to \infty$, $\epsilon \to 0$. and $\chi_R(f * g_\epsilon) \in C_0^\infty(\mathbb{R}^n)$. This proves (3), by taking $R = \frac{1}{\epsilon}$, $g_\epsilon = \chi_{1/\epsilon}(f * g_\epsilon)$. Hence the Lemma.

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Theorem 2: $W^{m,p}(\Omega)$ is a Banach space and for $1 \le p < \infty$, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.

Proof: Let $\{f_k\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then $\{\partial^\alpha f_k\}$ converges in $L^p(\Omega)$ for $|\alpha| \le m$. Let

$$\lim_{k \to \infty} \partial^\alpha f_k = f_\alpha.$$

Claim: $f_\alpha = \partial^\alpha f_0$.

For Let $\varphi \in C_0^\infty(\Omega)$, then with

$$(-1)^\alpha \int_\Omega f_k \partial^\alpha \varphi \, dx = \int_\Omega \partial^\alpha f_k \varphi \, dx,$$

letting $k \to \infty$ to obtain

$$(-1)^{|\alpha|} \int_\Omega f_0 \partial^\alpha \varphi \, dx = \int_\Omega f_\alpha \varphi \, dx.$$

Here $f_\alpha = \partial^\alpha f_0$ and $f_k \to f_0$ in $W^{m,p}(\Omega)$.
This proves it is a Banach space.

From (3) of Remark 1, $\exists$ sequence $g_\epsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\|f - g_\epsilon\|_{m,p,\Omega} \to 0$. For arbitrary Ω, see Adams book on Sobolev Spaces. This proves the theorem.

As a consequence of this, we have the alternate definition as follows:

Strong derivatives in $L^p(\Omega)$:- Let $m \geq 0$ be an integer and $f \in L^p(\Omega)$. f is said to have strong L^p-derivatives of order m in Ω if there exist a sequence $g_k \in C^\infty(\Omega) \cap L^p(\Omega)$ such that $\partial^\alpha g_k \in L^p(\Omega)$ and

(i) $g_k \to f$ in $L^p(\Omega)$

(ii) $\{\partial^\alpha g_k\}$ is cauchy sequence in $L^p(\Omega)$.

Remark: From theorem 1, weak and strong derivatives are same in $W^{m,p}(\Omega)$ and

$W^{m,p}(\Omega) =$ closure of $C^\infty(\Omega) \cap W^{m,p}(\Omega)$.

Definition: $W_0^{m,p}(\Omega)$:-

$W_0^{m,p}(\Omega) =$ closure of $C_0^\infty(\Omega) \cap W^{m,p}(\Omega)$.

Lemma 3: Let $u \in W^{1,p}(\Omega)$. Then $|u| \in W^{1,p}(\Omega)$ and

$$\frac{\partial}{\partial x_i}|u| = \operatorname{sign}(u)\frac{\partial u}{\partial x_i} \quad \text{---(*)}$$

In order to prove (5), first we have the following

Claim: $C_0^0(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ is dense.

Step 1: Let $A \subset \mathbb{R}^n$ be a measurable set with $|A| < \infty$ and $\epsilon > 0$, Choose $K_1 \subset A \subset U$, K_1 Compact, U open such that: $|U \setminus K_1| < \epsilon$. Let

$$h(x) = \frac{d(x, \mathbb{R}^n \setminus U)}{d(x, \mathbb{R}^n \setminus U) + d(x, K_1)}$$

where $d(x, B)$ is the distance from x to B defined by,

$$d(x, B) = \inf_{y \in B} |x - y|$$

Since d is continuous and hence $h \in C^0(\mathbb{R}^n)$ and $0 \le h \le 1$ with $h(x) = \begin{cases} 0 & \text{if } x \notin U \\ 1 & \text{if } x \in K_1 \end{cases}$

Hence

(i) $|\chi_A(x) - h(x)| \le 2$

(ii) $\chi_A(x) - h(x) = 0$ if $x \notin K \cup (\mathbb{R}^n \setminus U)$.

Hence $\int_{\mathbb{R}^n} |\chi_A(x) - h(x)|^p \le 2^p |U \setminus K_1| < 2^p \epsilon$.

Thus proves that $\|\chi_A - h\|_p \to 0$ as $\epsilon \to 0$ and $h \in C^0(\mathbb{R}^n)$. If A is bounded set, then $h \in C_0^0(\mathbb{R})$.

Step 2: Since simple functions with bounded support are dense in $L^p(\mathbb{R}^n)$. Hence by Step (1), $C_0^0(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ is dense.

Step 3: Let $f \in L^p(\mathbb{R}^n)$. Let $\eta > 0$ such. Choose $h \in C_0^\infty(\mathbb{R}^n)$ s.t. $\|f-h\|_p < \eta/4$

Let ϵ small such that

$$\|h * \rho_\epsilon - h\|_p < \eta/2$$

Then by young's inequality

$$\|f - f*\rho_\epsilon\|_p = \|(f-h)*\rho_\epsilon + h*\rho_\epsilon - h + h - f\|_p$$

$$\leq \|f-h\|_p \|\rho_\epsilon\|_1 + \|h*\rho_\epsilon - h\|_p + \|h-f\|_p$$

$$\leq 2\|f-h\|_p + \|h*\rho_\epsilon - h\|_p$$

$$< \eta$$

Since $\int f_\epsilon(x)\,dx = \frac{1}{\epsilon^n}\int \rho\left(\frac{x}{\epsilon}\right)dx = \int \rho(x)\,dx = 1$

This proves (5). and hence the theorem.

Before going to the next topic, following is the generalization of young's inequality

Theorem 2 (young's inequality): Let $1 \leq p_i \leq \infty$, $i = 1 \cdots L$, and let $1 \leq q \leq \infty$ be such that

$$\sum_{i=1}^{L} \frac{1}{p_i} - \frac{1}{q} = L-1.$$

Let $f_i \in L^{p_i}$, $1 \leq i \leq L$, Then $f_1 * \cdots * f_L \in L^q$.

Proof: By induction. For $L=2$, already proved. Assume that it is true for $L-1$. Then $f_1 * \cdots * f_{L-1} \in L^{q_1}$ where q_1 is given by

$$\sum_{i=1}^{L-1} \frac{1}{p_i} - \frac{1}{q_1} = L-2.$$

Define q by

$$\frac{1}{p_L} + \frac{1}{p_i} - \frac{1}{q} = 1$$

Then $(f_1 * \cdots * f_{L-1}) * f_L \in L^q$. Hence

$$1 - \frac{1}{p_L} + \sum_{i=1}^{L-1} \frac{1}{p_i} - L + 2 - \frac{1}{q}$$

$$\Rightarrow \sum_{i=1}^{L} \frac{1}{p_i} - \frac{1}{q} = L - 1.$$

and

$$\| f_1 * \cdots * f_L \|_q \leq \prod_{i=1}^{L} \| f_i \|_{p_i}.$$

This proves the theorem.

Sobolev Spaces - Weak and Strong derivatives

Weak derivative. Let $f \in L^1_{loc}$. f has weak derivatives of order $k \geq 1$ if there exists $f_\alpha \in L^1_{loc}(\Omega)$ for $|\alpha| \leq k$ such that for all $\varphi \in C_0^\infty(\Omega)$,

$$(-1)^{|\alpha|} \int_\Omega f \, \partial^\alpha \varphi \, dx = \int_\Omega f_\alpha \varphi \, dx.$$

$\{f_\alpha\}$ are called weak derivatives of f & $\partial^\alpha f$ also called Distributional Derivatives.

Properties:

(1) If $f \in C^k(\Omega)$. Then $f_\alpha = \partial^\alpha f$. This follows by integration by parts.

(2) $\{f_\alpha\}$ are unique. Suppose $\exists\ g_\alpha \in L^1_{loc}$ such that

$$\int_\Omega f_\alpha \varphi \, dx = \int_\Omega g_\alpha \varphi \, dx$$

Then

$$\int_\Omega (f_\alpha - g_\alpha) \varphi \, dx = 0$$

Let $\Omega_1 \subset\subset \Omega$ be an open set with $\overline{\Omega_1}$ is compact in Ω. Let $\chi \in C_0^\infty(\Omega)$ such that $\chi \equiv 1$ on Ω_1. Let $h_\alpha = \chi(f_\alpha - g_\alpha)$. Then $h_\alpha \in L^1(\mathbb{R}^n)$ and for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\chi\varphi \in C_0^\infty(\Omega)$ and

$$\int_{\mathbb{R}^n} h_\alpha \varphi \, dx = \int_\Omega (f_\alpha - g_\alpha)\chi\varphi \, dx = 0$$

Since $C_0^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is dense and hence $h_\alpha = 0$ for $|\alpha| \le k$. This proves (2)

Notation :- In view of (1) and (2), we denote

$$f_\alpha = \partial^\alpha f.$$

Example :(1) Let $f(x) = |x|$ and $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$-\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx = -\int_{-\infty}^{0} f(x)\varphi'(x)\,dx - \int_0^\infty f(x)\varphi'(x)\,dx$$

$$= \int_{-\infty}^{0} x\varphi'(x)\,dx - \int_0^\infty x\varphi'(x)\,dx$$

$$= x\varphi(x)\Big]_{-\infty}^{0} - x\varphi(x)\Big]_0^\infty - \int_{-\infty}^{0}\varphi\,dx + \int_0^\infty \varphi\,dx$$

$$= \int_{-\infty}^{\infty} H(x)\varphi(x)\,dx$$

where $H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$. Hence $\frac{df}{dx} = H$ in the Distributional derivative and $H \in L^1_{loc}(\mathbb{R})$.

In general if $f : \mathbb{R} \to \mathbb{R}$ is locally lipschitz continuous function, Then f is absolutely continuous on each bounded interval and by Lebesgue theorem f' exists

Lebesgue's theorem, $f'(x)$ exist a.e. and satisfies

$$-\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx .$$

Lemma 3: Let $k \geq 1$ and $f \in L^1_{loc}(\Omega)$ having weak derivatives upto order k. Let $\chi \in C^k(\Omega)$. Then

(1) For all $\varphi \in C_0^k(\Omega)$, $|\alpha| \leq k$

$$-\int_{\Omega} f\,\partial^\alpha\varphi\,dx = \int_{\Omega} \partial^\alpha f\,\varphi\,dx .$$

$$\frac{\partial}{\partial x_i}(\chi f) = \chi\frac{\partial f}{\partial x_i} + f\frac{\partial \chi}{\partial x_i}$$

(2) Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\chi \in C_0^k(\mathbb{R}^n)$. Then for $|\alpha| \leq k$

$$\partial^\alpha(f\chi) = \partial^\alpha f \chi = f\chi\partial^\alpha\chi .$$

(3) Let $1 \leq p < \infty$ and $\partial^\alpha f \in L^p(\mathbb{R}^n)$ for $|\alpha| \leq k$. Then there exist $g_\epsilon \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{\epsilon \to 0} \|\partial^\alpha f - \partial^\alpha g_\epsilon\|_p = 0$$

Proof: Let $B(0,1) = \{x : |x| < 1\}$ and $\rho \in C_0^\infty(B(1))$ such that

$$\rho \geq 0, \quad \int_{\mathbb{R}^n} \rho(x)\,dx = 1$$

For $\epsilon > 0$, let $\rho_\epsilon(x) = \frac{1}{\epsilon^n}\rho(x/\epsilon)$ be a mollifying sequence. Let $\varphi \in C_0^k(\Omega)$ and $\varphi_\epsilon = \varphi * \rho_\epsilon \in C_0^\infty(\mathbb{R}^n)$. Then

$$\lim_{\epsilon \to 0} \|\partial^\alpha\varphi_\epsilon - \partial^\alpha\varphi\|_\infty = 0 .$$

Since support of $\varphi \subset \Omega$ and $\text{Supp}(\varphi * \rho_\epsilon) \subset$ $\text{Supp}\,\varphi + \text{Supp}(\rho_\epsilon) \subset \text{Supp}\,\varphi + B(\epsilon) \subset \Omega$ if ϵ is

sufficiently small, say $\epsilon < \epsilon_0$ for some fixed ϵ_0 depending on φ. Then

$$(-1)^{|\alpha|}\int_\Omega f\,\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\lim_{\epsilon\to 0}\int_\Omega f\,\partial^\alpha(\varphi_\epsilon)\,dx$$

$$= \lim_{\epsilon\to 0}\int_\Omega \partial^\alpha f\,\varphi_\epsilon\,dx$$

$$= \int_\Omega \partial^\alpha f\,\varphi\,dx.$$

Since $x\varphi \in C_0^k(\Omega)$, hence

$$-\int_\Omega x f\frac{\partial\varphi}{\partial x_i}\,dx = -\int_\Omega f\left[\frac{\partial}{\partial x_i}(x\varphi) - \varphi\frac{\partial x}{\partial x_i}\right]dx$$

$$= \int_\Omega\left[\frac{\partial f}{\partial x_i}x + f\frac{\partial x}{\partial x_i}\right]\varphi\,dx.$$

Hence $$\frac{\partial}{\partial x_i}(xf) = x\frac{\partial f}{\partial x_i} + f\frac{\partial x}{\partial x_i}$$

This proves (1).

Let $\varphi \in C_0^k(\mathbb{R}^n)$, then

$$(-1)^{|\alpha|}\int_{\mathbb{R}^n}(f*\chi)\,\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n}\partial^\alpha\varphi(x)\left(\int_{\mathbb{R}^n} f(y)\chi(x-y)\,dy\right)dx$$

$$= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n}\chi(x-y)\,\partial^\alpha\varphi(x)\,dx\right)dy$$

$$= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(y)\,\partial_y^\alpha\left(\int_{\mathbb{R}^n}\chi(x-y)\,\varphi(x)\,dx\right)dy$$

$$= \int_{\mathbb{R}^n}\partial^\alpha f(y)\left(\int_{\mathbb{R}^n}\chi(x-y)\,\varphi(x)\,dx\right)dy.$$

By induction, for $|\alpha| \le k$, $\partial^\alpha(\chi_R \psi) \to \partial^\alpha \psi$ in $L^p(\mathbb{R}^n)$. Therefore we have

$$\partial^\alpha(\chi_R(f * g_\epsilon)) \to \partial^\alpha f \quad \text{in } L^p(\mathbb{R}^n)$$

as $R \to \infty$, $\epsilon > 0$, and $\chi_R(f * g_\epsilon) \in C_0^\infty(\mathbb{R}^n)$. This proves (3) by taking $R = \frac{1}{\epsilon}$, $g_\epsilon = \chi_{1/\epsilon}(f * g_\epsilon)$. Hence the Lemma.

Sobolev Spaces $W^{m,p}(\Omega)$: Let $m \ge 0$ be an integer and $1 \le p \le \infty$. Define

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega) : \partial^\alpha f \text{ exist for } |\alpha| \le m \text{ and } \partial^\alpha f \in L^p(\Omega) \}.$$

Define the norm

$$\|f\|_{m,p,\Omega} = \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{p,\Omega}$$

Theorem 2: $W^{m,p}(\Omega)$ is a Banach space and for $1 \le p < \infty$, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.

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From (3) of Remark 1, $\exists$ sequence $g_\epsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\|f - g_\epsilon\|_{m,p,\Omega} \to 0$. For arbitrary Ω, see Adams book on Sobolev Spaces. This proves the theorem.

As a consequence of this, we have the alternate definition as follows:

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Remark: From theorem 1, weak and strong derivatives are same in $W^{m,p}(\Omega)$ and

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Definition: $W_0^{m,p}(\Omega)$:-

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Lemma 3: Let $u \in W^{1,p}(\Omega)$. Then $|u| \in W^{1,p}(\Omega)$ and

$$\frac{\partial}{\partial x_i}|u| = \operatorname{sign}(u)\frac{\partial u}{\partial x_i} \quad \text{---(*)}$$

Proof. Let $\epsilon > 0$ and $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. Define

$$u_\epsilon = (u^2+\epsilon^2)^{1/2} \in C^\infty(\Omega).$$

Then $$\frac{\partial u_\epsilon}{\partial x_i} = \frac{u}{(u^2+\epsilon^2)^{1/2}} \frac{\partial u}{\partial x_i}$$

Let $\varphi \in C_0^\infty(\Omega)$, Then

$$-\int_\Omega u_\epsilon \frac{\partial \varphi}{\partial x_i} dx = \int_\Omega \frac{u}{(u^2+\epsilon^2)^{1/2}} \frac{\partial u}{\partial x_i} \varphi \, dx$$

Letting $\epsilon \to 0$ we obtain

$$-\int_\Omega |u| \frac{\partial \varphi}{\partial x_i} dx = \int_\Omega (\operatorname{Sign} u) \frac{\partial u}{\partial x_i} \varphi \, dx \qquad (*)$$

hence $(*)$ holds if $u \in C^1(\Omega)$. Let $u \in W^{1,p}(\Omega)$ and $u_k \to u$ in $W^{1,p}(\Omega)$ with $u_k \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. Then

$$-\int_\Omega (u_k^2+\epsilon^2)^{1/2} \frac{\partial \varphi}{\partial x_i} dx = \int \frac{u_k}{(u_k^2+\epsilon^2)^{1/2}} \frac{\partial u_k}{\partial x_i} \varphi \, dx.$$

Since $$\left| \frac{u_k}{(u_k^2+\epsilon^2)^{1/2}} \frac{\partial u_k}{\partial x_i} \right| \le \left| \frac{\partial u_k}{\partial x_i} \right|$$

$$\frac{u_k}{(u_k^2+\epsilon^2)^{1/2}} \frac{\partial u_k}{\partial x_i} \to \frac{u}{(u^2+\epsilon^2)^{1/2}} \frac{\partial u}{\partial x_i} \quad \text{a.e.}$$

hence by dominated Convergence theorem, we have letting $k \to \infty$ to obtain

$$-\int_\Omega (u^2+\epsilon^2)^{1/2} \frac{\partial \varphi}{\partial x_i} dx = \int_\Omega \frac{u}{(u^2+\epsilon^2)^{1/2}} \frac{\partial u}{\partial x_i} \varphi \, dx.$$

Now letting $\epsilon \to 0$ to obtain the Lemma.

Corollary 4: Let $u \in W^{1,p}(\Omega)$ and $u = u^+ - u^-$. Then $u^\pm \in W^{1,p}(\Omega)$ and

$$\frac{\partial u^+}{\partial x_i} \frac{\partial u^-}{\partial x_i} = 0 \quad \text{a.e.}$$

Proof: $u^+ = \frac{|u|+u}{2}$, $u^- = \frac{|u|-u}{2} \in W^{1,p}(\Omega)$.

$$\frac{\partial u^\pm}{\partial x_i} = \frac{1}{2}(\operatorname{sign} u \pm 1)\frac{\partial u}{\partial x_i}$$

$$\frac{\partial u^+}{\partial x_i}\frac{\partial u^-}{\partial x_i} = \frac{1}{4}((\operatorname{sign} u)^2 - 1)\left(\frac{\partial u}{\partial x_i}\right)^2 = 0.$$

This proves the Corollary.

Sobolev Embedding Theorems:

Start with $n=1$. Let $u \in W^{1,p}(\mathbb{R}) \cap C^1(\mathbb{R})$. Then for $x, y \in \mathbb{R}$, we have

$$u(x) - u(y) = \int_x^y \frac{du}{dt}\,dt$$

$$|u(x) - u(y)| \le \int_x^y \left|\frac{du}{dt}\right| dt$$

$$\le \left(\int_x^y \left|\frac{du}{dt}\right|^p dt\right)^{1/p} |x-y|^{1/p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$

Hence

$$|u(x) - u(y)| \le \|u\|_{1,p,\mathbb{R}} |x-y|^{1/p'}$$

This implies u is uniformly continuous. In fact

$$\sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{1/p'}} \le \|u\|_{1,p,\mathbb{R}}$$

and hence u belongs to Hölder space of exponent $1/p'$.

where Hölder spaces are defined as follows:

Definition $C^{0,\alpha}(\Omega)$: Let $0 \le \alpha \le 1$ and $u \in C^0(\Omega)$. u is said to be in $C^{0,\alpha}(\Omega)$ if

$$\|u\|_{\infty} = \sup_{x \in \Omega} |u(x)| < \infty$$

$$\sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} < \infty.$$

and the norm in $C^{0,\alpha}(\Omega)$ is defined by

$$\|u\|_{0,\alpha,\Omega} = \|u\|_{\infty} + \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}$$

Now the question is, for $n > 1$, does $W^{1,p}(\Omega)$ are continuous?

Example: Let $n=2$, $\Omega = B(1) \subset \mathbb{R}^2$ Let

$$u(x) = \log\left(\log \frac{2}{|x|}\right) \quad \text{for } x \in B(1).$$

Clearly $u(0)=\infty$. Let $\varphi = \varphi(|x|) \in C_0^{\infty}(B(1))$. Since u is radial, it is enough to look at radial derivation of u. Then

$$-\int_{B(1)} u \frac{\partial \varphi}{\partial x} dx = -2\pi \int_0^1 u \frac{d\varphi}{dr} r\, dr$$

$$= -2\pi \int_0^1 \log\left(\log \frac{2}{r}\right) \frac{d\varphi}{dr} r\, dr$$

$$= -2\pi \left[\varphi r \log \log \frac{2}{r}\right]_0^1$$

$$+ 2\pi \int_0^1 \frac{1}{\left(\log \frac{2}{r}\right) r} \varphi\, r\, dr$$

Hence $\frac{\partial u}{\partial x} = -\frac{1}{r \log 2/r}$.

$$\int_0^1 \left|\frac{\partial u}{\partial x}\right| x\, dx = \int_0^1 \frac{dx}{\left(\log \frac{2}{x}\right)} < \infty.$$

Hence $u \in W^{1,1}(B(1))$ and $u(0) = \infty$. Hence u need not be bounded.

Theorem: Let $p < n$ and $p^* = \frac{np}{n-p}$. Then

$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ and p^* is optimal.

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \le C_N \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{1/p}.$$

Proof: First we prove this for $p = 1$ and $u \in C_0^\infty(\mathbb{R}^n)$

$$u(x_1 \cdots x_n) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1 \cdots x_{i-1}, y_i, x_{i+1} \cdots x_n)\, dy_i$$

$$|u(x_1 \cdots x_n)| \le \int_{-\infty}^{\infty} |Du(x_1 \cdots x_{i-1}, y_i, x_{i+1} \cdots x_n)|\, dy_i,$$

with $|Du| = \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|$

$$|u|^{n/n-1} \le \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du(x_1 \cdots x_{i-1}, y_i, x_{i+1} \cdots x_n)|\, dy_i\right)^{1/n-1}$$

$$\int_{-\infty}^{\infty} |u(x_1 \cdots x_n)|^{n/n-1} dx_1 \le \left(\int_{-\infty}^{\infty} |Du(y_1, x_2 \cdots x_n)|\, dy_1\right)^{1/n-1}$$

$$\times \int_{-\infty}^{\infty} \prod_{i=2}^{n-1} \left(\int_{-\infty}^{\infty} |Du(x_1, \cdots y_i, \cdots x_n)|\, dy_i\right)^{1/n-1} dx_1.$$

From Hölder's inequality we have

$$\int_{-\infty}^{\infty} |u(x_1,x_{n'})|^{n/n-1} dx_1 \le \left(\int_{-\infty}^{\infty} |Du(y_1,x_2-x_{n'})|\,dy_1\right)^{1/n-1}$$

$$\times \prod_{i=2}^{n-1} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2,y_i,x_{i+1},\dots x_{n'})|\,dy_i\,dx_1\right)^{1/n-1}$$

$$= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,y_2)|\,dy_2\,dx_1\right)^{1/n-1}$$

$$\times \left(\int_{-\infty}^{\infty} |Du(y_1,x_2-x_{n'})|\,dy_1\right)^{1/n-1}$$

$$\times \prod_{i\neq 2,1} \left(\int_{-\infty}^{\infty} |Du(x_1,x_2,y_i,x_{i+1},\dots x_{n'})|\,dy_i\,dx_1\right)^{1/n-1}$$

Again integrating w.r.t. x_2 and applying Hölder to obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x_1,x_{n'})|^{n/n-1} dx_1\,dx_2 \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2,x_3-x_{n'})|\,dx_1\,dx_2\right)^{1/n-1}$$

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2,\dots x_{n'})|\,dx_1\,dx_2\right)^{1/n-1}$$

$$\prod_{i\neq 1,2} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(x_1,x_2,\dots y_i,\dots x_{n'})|\,dz\,dx_1\,dy_i\right)^{1/n-1}$$

and again integrating w.r.t. $x_3,\dots x_n$ and applying Hölder to obtain

$$\int_{\mathbb{R}^n} |u(x)|^{n/n-1} dx \le \left(\int_{\mathbb{R}^n} |Du|\,dx\right)^{n/n-1}$$

Let $u = |v|^\gamma$, $Du = \gamma |v|^{\gamma-2} v \, Dv$, $\gamma > 1$,

Then

$$\left(\int_{\mathbb{R}^n} |v|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} \gamma |v|^{\gamma-1} |Dv| \, dx$$

$$\le \gamma \left(\int_{\mathbb{R}^n} |v|^{p'(\gamma-1)} dx \right)^{1/p'} \left(\int_{\mathbb{R}^n} |Dv|^p dx \right)^{1/p}$$

Now choose γ such that $\frac{1}{p} + \frac{1}{p'} = 1$

$$\frac{\gamma n}{n-1} = p'(\gamma - 1)$$

$$\gamma \left(\frac{n}{n-1} - p' \right) = p', \quad \gamma = \frac{p'}{\left(\frac{n}{n-1} - p' \right)} = \frac{p(n-1)}{n-p} > 1$$

Hence $\frac{\gamma n}{n-1} = \frac{p' n}{n(1-p') + p} = \frac{n}{n\left(\frac{1}{p'} - 1\right) + 1}$

$$= \frac{n}{-\frac{n}{p} + 1} = \frac{np}{n-p} = p^*$$

$$\left(\int_{\mathbb{R}^n} |v|^{p^*} \right)^{1/p^*} \le C \left(\int_{\mathbb{R}^n} |Dv|^p \right)^{1/p}$$

To show that p^* is optimal, suppose $\exists \, q > 1$ s.t.

$$\left(\int_{\mathbb{R}^n} |v|^q dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |Dv|^p dx \right)^{1/p}.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$. Let $v = \varphi(\lambda x)$. Then

$$Dv = \lambda \, D\varphi(\lambda x)$$

$$\left(\int_{\mathbb{R}^n} |v|^q dx \right) = \int_{\mathbb{R}^n} |\varphi(\lambda x)|^q dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |\varphi|^q dx.$$

$$\int_{\mathbb{R}^n} |Dv|^p dx = \frac{\lambda^p}{\lambda^n} \int |D\varphi|^p dx.$$

Hence $\forall \lambda > 0$, we have

$$\frac{1}{\lambda^{n/q}} \|\varphi\|_q \le C \frac{\lambda}{\lambda^{n/p}} \|D\varphi\|_p$$

$$\Longrightarrow \frac{n}{q} = \frac{n}{p} - 1 \quad \text{or} \quad q = \frac{np}{n-p} = p^*.$$

This proves the theorem.

Theorem 6. Now assume that $p > n$. Then there exist $C > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$

$$\|u\|_{0,\gamma} \le C \|u\|_{1,p,\mathbb{R}^n}$$

where $\gamma = 1 - \frac{n}{p}$.

Proof. By density, we can assume that $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$. Denote $B(x,r) = \{y : |x-y| < r\}$ but $|w| = 1$. Then for $0 \le s < r$, $x + sw \in B(x,r)$ and by mean value theorem we have

$$u(x+sw) - u(x) = \int_0^s \frac{d}{dt} u(x+tw)\, dt$$

$$= \int_0^s \langle \nabla u(x+tw), w \rangle\, dt$$

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)|\, d\sigma(w) \le \int_{\partial B(0,1)} \int_0^s |\nabla u(x+tw)|\, dt\, d\sigma(w)$$

$$= \int_0^s \left(\int_{\partial B(0,1)} |\nabla u(x+tw)|\, d\sigma(w) \right) dt$$

$$= \int_0^s \int_{\partial B(0,1)} |\nabla u(x+tw)| \cdot \frac{t^{n-1}\, d\sigma(w)\, dt}{t^{n-1}}$$

$$= \int_0^r \int_{\partial B(0,1)} |\nabla u(y)| \frac{t^{n-1}\, dt\, d\sigma(w)}{|x-y|^{n-1}}$$

When $y = x + tw$, $t = |x-y|$.

Multiplying this by $s^{n-1} ds$ and integrate it obtain

$$\int_{B(x,r)} |u(y)-u(x)|\, dy = \int_0^r \int_{\partial B(0,1)} |u(x+sw)-u(x)|\, s^{n-1}\, ds\, d\sigma(w)$$

$$\leq \int_0^r s^{n-1} \left(\int_0^r \int_{\partial B(0,1)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy \right) ds$$

$$= \frac{r^n}{n} \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy$$

Hence

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)-u(x)|\, dy \leq C_n \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy$$

i.e

$$\fint_{B(x,r)} |u(y)-u(x)|\, dy \leq C_n \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy. \quad -(*)$$

From This we deduce The Theorem. Let $x \in \Omega^{th}$, Then

$|u(x)| = |u(x)-u(y)+u(y)| \leq |u(x)-u(y)| + |u(y)|$

Integrating this over $B(x,r)$ and using $(*)$ to obtain

$$|u(x)| \le \fint_{B(x,r)} |u(y)-u(x)|\,dy + \fint_{B(x,r)} |u(y)|\,dy$$

$$\le C_n \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy + \fint_{B(x,r)} |u(y)|\,dy$$

$$\le C_n \left(\int_{B(x,r)} |\nabla u(y)|^p dy\right)^{1/p} \left(\int_{B(x,r)} \frac{dy}{|x-y|^{(n-1)p'}}\right)^{1/p'} + \left(\fint_{B(x,r)} |u(y)|^p dy\right)^{1/p}.$$

Let $C_1 = |\partial B(0,1)|$. Then

$$\int_{B(x,r)} \frac{dy}{|x-y|^{(n-1)p'}} = \int_{\partial B(0,1)} \left(\int_0^r \frac{t^{n-1}\,dt}{t^{(n-1)p'}}\right) d\sigma(w)$$

$$= C_1 \int_0^r \frac{dt}{t^{(n-1)\left(\frac{p}{p-1}-1\right)}}$$

$(n-1)\left(\frac{p}{p-1}-1\right) = \frac{(n-1)}{p-1} < 1$ iff $n-1 < p-1$ iff $p > n$. Hence

$$\left(\int_{B(x,r)} \frac{dy}{|x-y|^{(n-1)p'}}\right)^{1/p'} = C_2 \left(r^{1-\frac{n-1}{p-1}}\right)^{1/p'} = C_2\, r^{\frac{p-n}{p-1}\left(\frac{p-1}{p}\right)} \quad (*)$$

$$= C_2\, r^{\left(1-\frac{n}{p}\right)}.$$

Hence

$$|u(x)| \le C_n C_2\, r^{1-\frac{n}{p}} \|u\|_{1,p,\mathbb{R}^n} + \frac{C_3}{r^{n/p}} \|u\|_{p,B(x,r)}$$

Hence taking sup, and taking sup on x to obtain

$$\|u\|_{\infty} \le c \|u\|_{1,p,\mathbb{R}^n}$$

for some $c > 0$.

Let $x, y \in \mathbb{R}^n$ and $r = |x-y|$. Then for $z \in B(x,r) \cap B(y,r) = \Omega$, we have

$$|u(x) - u(y)| \le |u(x) - u(z)| + |u(z) - u(y)|$$

integrating on Ω to obtain

$$|u(x) - u(y)| \le \fint_{\Omega} |u(x) - u(z)|\,dz + \fint_{\Omega} |u(z) - u(y)|\,dz$$

Since $|\Omega| \ge c\,|B(x,r)|$, $|\Omega| \ge c\,|B(y,r)|$, and hence from (*) and (**) we have

$$|u(x) - u(y)| \le \frac{|B(x,r)|}{|\Omega|} \fint_{B(x,r)} |u(x) - u(z)|\,dz + \frac{|B(y,r)|}{|\Omega|} \fint_{\Omega} |u(y) - u(z)|$$

$$\le C \left\{ \fint_{B(x,r)} |u(x) - u(z)|\,dz + \fint_{B(y,r)} |u(y) - u(z)|\,dz \right\}$$

$$\le C_4\, r^{1 - \frac{n}{p}} \left\{ \left(\int_{B(x,r)} |\nabla u|^p\,dz \right)^{1/p} + \left(\int_{B(y,r)} |\nabla u|^p\,dz \right)^{1/p} \right\}$$

$$\le 2 C_4\, r^{1 - \frac{n}{p}} \|u\|_{1,p,\mathbb{R}^n} \qquad (***)$$

Hence

$$\frac{|u(x) - u(y)|}{|x-y|^{1 - \frac{n}{p}}} \le c \|u\|_{1,p,\mathbb{R}^n}$$

Since $C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$, the theorem follows.

Theorem 7. Let $p > n$. Then u is a.e. differentiable.

Proof: Let E = Lebesgue points of ∇u. Then $|\mathbb{R}^n \setminus E| = 0$ and for all $x \in E$

$$\lim_{\delta x \to 0} \int_{B(x,\delta x)} |\nabla u(x) - \nabla u(y)|\, dy = 0.$$

Since $p > n$, $u \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma = 1 - \frac{n}{p}$. Let $x \in E$ and

$$v(y) = u(y) - u(x) - \nabla u(x)(y - x)$$

Then

$$\nabla_y v = \nabla u(y) - \nabla u(x), \quad v(x) = 0$$

Hence from (* **) of Theorem 6, we have $|x - y| \le 2\delta x$

$$|u(y) - u(x) - \nabla u(x)(y-x)| = |v(y) - v(x)|$$

$$\le c\,|x-y|^{1-\frac{n}{p}} \left\{ \left(\int_{B(x,\delta x)} |\nabla v(z)|^p \right)^{1/p} + \left(\int_{B(y,\delta x)} |\nabla v(z)|^p \right)^{1/p} \right\}$$

$$\frac{|u(y) - u(x) - \nabla u(x)(y-x)|}{|x-y|} \le c_1 \left\{ \left(\int_{B(x,\delta x)} |\nabla u(x) - \nabla u(z)|^p \right)^{1/p} + \left(\int_{B(x,2\delta x)} |\nabla u(x) - \nabla u(z)|^p \right)^{1/p} \right\}$$

$$= o(1) \quad \text{as } \delta x \to 0$$

Since $x \in E$. Hence u is diff at x. This proves the Theorem.

Density results.

Let $\Omega \subset \mathbb{R}^n$ be a domain and $d(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$ given by

$$d(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y|.$$

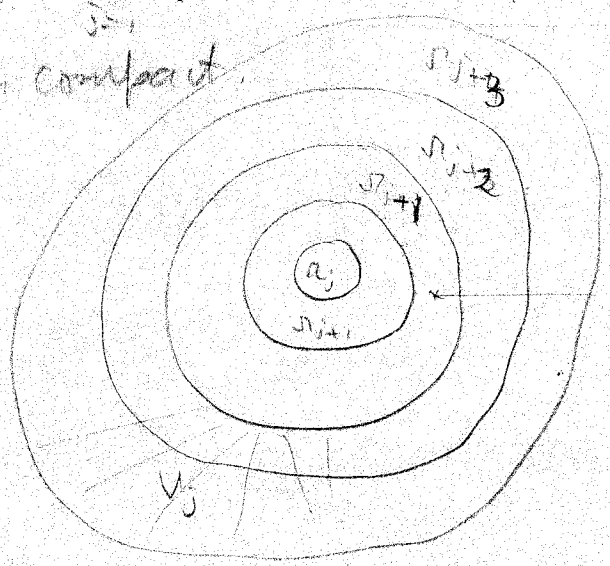
Put $j \in \mathbb{N}$ and define

$$\Omega_j = \left\{ x \in \Omega : |x| < j,\ d(x, \partial\Omega) > \frac{1}{j} \right\}$$

Then we have

(i) $\Omega_j \subset \Omega_{j+1} \subset \cdots$, $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$

(ii) $\overline{\Omega_j} \subset \Omega_{j+1}$, $\overline{\Omega_j}$ is compact.



For $x \in \Omega_{j+1}$, $y \in \partial\Omega_{j+3}$, $z \in \partial\Omega$.

Then

$$|x - z| \le |x - y| + |y - z|$$

$$\frac{1}{j+1} \le d(x, \partial\Omega) \le d(x, \partial\Omega_{j+3}) + d(y, \partial\Omega).$$

$$= d(x, \partial\Omega_{j+3}) + \frac{1}{j+3}$$

Hence $d(x, \partial\Omega_{j+3}) \ge \frac{1}{j+1} - \frac{1}{j+3} = \frac{2}{(j+1)(j+3)}$.

Put $V_j = \Omega_{j+3} \setminus \overline{\Omega}_{j+1}$, for $j \ge 3$. Then

(i) $\overline{V_j} \subset \overline{\Omega}_{j+3}$ is compact

(ii) $V_j \cap V_k = \phi$ if $k \ge j+3$ or $k \le j-3$.

(iii) $\Omega = \bigcup_{j=1}^{\infty} V_j$ is a locally finite covering.

i.e. $\#\{ j : V_j \cap V_k \neq \phi \} \le 6$.

Let $\{\chi_j\}$ be a C_0^∞ partition of unity subordinate to $\{V_j\}$. i.e. $\sum \chi_j(x) = 1 \quad \forall x \in \Omega$.

Let $u \in W^{m,p}(\Omega)$ and $u_j = \chi_j u$. Then

$$u = \sum_{j=1}^{\infty} u_j$$

Let $\{\rho_\epsilon\}$ be a mollifying sequence with $\operatorname{supp} \rho \subset B(0,1)$. Then $u_j \in W^{1,p}(\mathbb{R}^n)$ since $\operatorname{supp} u_j \subset V_j$ and choose ϵ_j such that for given $\delta > 0$

$$\begin{cases} \|\rho_{\epsilon_j} * u_j - u_j\|_{m,p,\mathbb{R}^n} \leq \delta/2^{j+1} \\ \operatorname{supp} \rho_{\epsilon_j} * u_j \subset B(0,\epsilon_j) + V_j \subset \Omega_{j+4} \setminus \overline{\Omega}_j \end{cases}$$

Let

$$v_\delta = \sum (\rho_{\epsilon_j} * u_j) \in C^\infty(\Omega)$$

Then $v_\delta \in W^{m,p}(\Omega)$

$$\|v_\delta - u\|_{m,p,\Omega} \leq \sum \|\rho_{\epsilon_j} * u_j - u_j\|_{m,p,\Omega} \leq \delta \sum \frac{1}{2^{j+1}} = \delta$$

Therefore we have the following

Lemma 8: $C^\infty(\Omega) \cap W^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega)$ is dense.

Under suitable conditions on Ω we will show that $C_0^\infty(\overline{\Omega}) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$. This we will prove in several steps.

E is called an extension operator. $\exists\ C>0$ such that for each $|\alpha| \le m$, $x_n < 0$, we have

$$|D^\alpha Eu(x)| < C \sum_{j=1}^{m+1} |(D^\alpha u)(x', -jx_n)|$$

$$\|D^\alpha Eu(x)\|_{p,\mathbb{R}^n_-} \le C \sum_{j=1}^{m+1} \|D^\alpha u(x', -jx_n)\|_{p,\mathbb{R}^n_-}$$

and

$$\int_{\mathbb{R}^n_-} |D^\alpha u(x', -jx_n)|^p dx' dx_n = \frac{1}{j} \int_{\mathbb{R}^n_+} |D^\alpha u(x)|^p dx$$

Hence

$$\|D^\alpha Eu(x)\|_{p,\mathbb{R}^n_-} \le C(m+1) \|D^\alpha u\|_{p,\mathbb{R}^n_+}$$

Hence

$$\boxed{\|Eu\|_{m,p,\mathbb{R}^n} \le C \|u\|_{m,p,\mathbb{R}^n_+}}$$

Hence we have the following

Lemma 9. (Extension operator): There exist a linear operator $E: C^m(\mathbb{R}^n_+) \to C^m(\mathbb{R}^n)$ and a constant $C>0$ such that

$$\begin{cases} Eu(x) = u(x) \text{ if } x \in \mathbb{R}^n_+ \\ \|Eu\|_{m,p,\mathbb{R}^n} \le C \|u\|_{m,p,\mathbb{R}^n_+}. \end{cases}$$

Furthermore if $u \in C_0^m(\overline{\mathbb{R}^n_+})$ (i.e. u is having compact support in $\overline{\mathbb{R}^n_+}$), then $Eu \in C_0^m(\mathbb{R}^n)$.

Extension operators

Let $u \in C^m(\overline{\mathbb{R}^n_+})$. $\alpha_j \in \mathbb{R}$ and define Eu on $\mathbb{R}^n$ as follows. For $x \in \mathbb{R}^n$, denote $x = (x', x_n)$.

$$Eu(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n > 0 \\ \sum_{j=1}^{m+1} \alpha_j u(x', -jx_n) & \text{if } x_n < 0 \end{cases}$$

where α_j's are chosen such that

$$\sum_{j=1}^{m+1} (-j)^k \alpha_j = 1 \quad \forall \ 0 \le k \le m. \quad (*)$$

Let $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & (-2) & \cdots & -(m+1) \\ (-1)^2 & (-2)^2 & \cdots & (-(m+1))^2 \\ \vdots & \vdots & & \vdots \\ (-1)^m & (-2)^m & \cdots & (-(m+1))^m \end{pmatrix}$.

Since $\det(A) \neq 0$, Therefore $\exists$ unique (α_j) such that $(*)$ is satisfied.

In view of $(*)$ Eu satisfies $\forall\ x' \in \mathbb{R}^{n-1}$,

$$\left(\frac{\partial}{\partial x_n}\right)^j Eu(x', 0+) = \left(\frac{\partial}{\partial x_n}\right)^j Eu(x', 0-)$$

For all $0 \le j \le m$.

Furthermore for $|\alpha| \le m$, Eu satisfies

$$D^\alpha Eu(x', 0+) = D^\alpha Eu(x', 0-)$$

Hence $Eu \in C^m(\mathbb{R}^n)$ with $Eu|_{\overline{\mathbb{R}^n_+}} = u$.

Corollary 10 Let $g: \mathbb{R}^{n-1} \to \mathbb{R}$ be a C^m-function satisfying $\|D^\alpha g\|_\infty \le C \ \forall\ |\alpha| \le m$. Let

$$\Omega_+ = \{x \in \mathbb{R}^n : x_n > g(x_1)\},\ \Omega_- = \{x \in \mathbb{R}^n : x_n < g(x_1)\},$$

$$\partial\Omega = \{x \in \mathbb{R}^n : x_n = g(x_1)\}.$$

Then there exist an extension operator $\tilde{E}: C^\infty(\Omega_+) \to C^\infty(\mathbb{R}^n)$ and a $C_1 > 0$ such that for $u \in C^\infty(\bar{\Omega}_+)$,

$$\tilde{E}u(x) = u(x) \quad \text{for } x \in \Omega_+ \qquad \text{---(1)}$$

$$\|\tilde{E}u\|_{m,p,\mathbb{R}^n} \le C \|u\|_{m,p,\Omega_+} \qquad \text{---(2)}$$

Furthermore if $u \in C_0^m(\bar{\Omega}_+)$, Then $Eu \in C_0^m(\Omega_-)$.

Proof: Define $\phi: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi(x) = (x_1, x_n - g(x_1))$$

$$\phi^{-1}(y) = (y_1, y_n + g(x_1))$$

are C^m-maps and $|D^\alpha \phi(x)| + |D^\alpha \phi^{-1}(y)| \le C$ $\forall\ x, y$ and $\alpha \ne 0$. Furthermore

$$\phi(\Omega_+) = \mathbb{R}^n_+$$

$$\phi(\partial\Omega_+) = \partial\mathbb{R}^n_+.$$

Hence $\phi^{-1}: C^\infty(\bar{\Omega}_+) \to C^\infty(\bar{\mathbb{R}}^n_+)$ is a bijective map given by

$$\phi^{-1}(v) = v \circ \phi^{-1}$$

Define $\tilde{E}: C^\infty(\bar{\Omega}_+) \to C^\infty(\mathbb{R}^n)$ by

$$\tilde{E}(u) = (\phi \circ E \circ \phi^{-1})(u)$$

Since $\|D^\alpha g\|_\infty \le C$ $\forall\ |\alpha| \le m$ and hence $\tilde{E}$ satisfies (1) and (2) and hence the Corollary.

Definition: $\Omega \subset \mathbb{R}^n$ is said to have C^m boundary (or smooth boundary of order m) if

$\forall\, y \in \partial\Omega$, $\exists\ a, r > 0$ and a coordinate system $(z_1, \dots, z_n)$ and $g : \mathbb{R}^{n-1} \to \mathbb{R}$ a C^m function such that

(i) $\|g\|_{m,\infty} \le a$

(ii) $\Omega \cap B(y, r) = \{ z \in B(y, r) : z_n > g(z') \}$ and $(y', g(y')) = y$.

(iii) $\partial\Omega \cap B(y, r) = \{ z \in B(y, r) : z_n = g(z') \}$.

Then we have the following

Theorem 11: Let Ω be a bounded domain with smooth boundary of order m. Then $\exists$ a linear map $E_\Omega : C^m(\bar{\Omega}) \to C_c^m(\mathbb{R}^n)$ such that

$$(\tilde{E}_\Omega u)(x) = u(x) \text{ if } x \in \Omega \quad \text{--- (1)}$$

$$\|\tilde{E}_\Omega u\|_{m,p,\mathbb{R}^n} \le C \|u\|_{m,p,\Omega} \quad \text{--- (2)}$$

Proof: For each $y \in \partial\Omega$, let $r(y) > 0$, $g_y : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfy (i), (ii) and (iii) of the above definition. Then $\{ B(y, r(y)) \}_{y \in \partial\Omega}$ is a cover and hence it has a finite subcover $V_i = B(y_i, r(y_i))$. Now choose $V_0 \subset\subset \Omega$ such that $\{V_0, V_i\}_{i \ge 1}$ is a cover for Ω. Let $\{\chi_i\}$ be a partition of unity

subordinate to this cover. From the previous Corrollary, there exist extension operators

$E_i : C_0^m(\Omega \cap B(y_i, \frac{r(y_i)}{2})) \to C_0^m(\mathbb{R}^n)$ satisfying (1) and (2). Denote $E_0 : C_0^\infty(V_0) \to C_0^\infty(\mathbb{R}^n)$ the inclusion mapping. Then define E_Ω by

$$(E_\Omega u)(x) = \sum_{i=0}^{N} E_i(\chi_i u)(x), \quad \text{with } \#I = N+1.$$

Then E_Ω satisfies (1) and (2) and this proves the theorem.

Theorem 12: $W^{m,k}(\mathbb{R}^n_+) \cap C_0^\infty(\overline{\mathbb{R}^n_+}) \subseteq W^{m,k}(\mathbb{R}^n_+)$ is dense for $1 \le p < \infty$.

Proof:

Step 1: Let $f \in L^p(\mathbb{R}^n)$, then

$$\lim_{|h| \to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx = 0.$$

To prove this, let $\epsilon > 0$ and choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\|\varphi - f\|_p < \epsilon/2$. Then

$$|f(x+h) - f(x)| \le |f(x+h) - \varphi(x+h)| + |\varphi(x+h) - \varphi(x)| + |\varphi(x) - f(x)|$$

Denote $\tau_h f(x) = f(x+h)$, then

$$\|\tau_h f - f\|_p \le \|\tau_h(f - \varphi)\|_p + \|\tau_h \varphi - \varphi\|_p + \|\varphi - f\|$$
$$\le 2\|f - \varphi\|_p + \|\tau_h \varphi - \varphi\|$$

$$\leq \epsilon + \|\tau_h\varphi - \varphi\|_p.$$

$$\therefore \overline{\lim_{|h|\to 0}} \|\tau_h f - f\|_p \leq \epsilon + \lim_{|h|\to 0} \|\tau_h\varphi - \varphi\|_p$$

Since $\varphi \in C_0^0(\mathbb{R}^n)$, hence by dominated convergence theorem, $\|\tau_h\varphi - \varphi\|_p \to 0$ as $h \to 0$. Hence

$$\overline{\lim_{|h|\to 0}} \|\tau_h f - f\|_p \leq \epsilon$$

Since ϵ is arbitrary, this proves Step 2.

Step (2): Let $u \in C^m(\mathbb{R}^n_+) \cap W^{m,p}(\mathbb{R}^n_+)$. Let $\epsilon > 0$. Let $e_n = (0, \cdots, 0, 1)$ and $\tau_\epsilon u(x) = u(x + \epsilon e_n)$. Then $\tau_\epsilon u \in C^m(\overline{\mathbb{R}^n_+})$ and $D^\alpha(\tau_\epsilon u) = \tau_\epsilon(D^\alpha u)$. Hence

$$\|D^\alpha(\tau_\epsilon u) - D^\alpha u\|_{p,\mathbb{R}^n_+} = \int_{\mathbb{R}^n_+} |D^\alpha u(x + \epsilon e_n) - D^\alpha u(x)|^p dx$$

Let $v_\alpha(x) = \begin{cases} D^\alpha u(x) & \text{if } x_n > 0 \\ 0 & \text{if } x_n < 0, \end{cases}$

Then $v_\alpha \in L^p(\mathbb{R}^n)$ and by Step (1)

$$\int_{\mathbb{R}^n} |v_\alpha(x + \epsilon e_n) - v_\alpha(x)|^p dx \longrightarrow 0 \text{ as } \epsilon \to 0.$$

On the other hand, we have

$$\int_{\mathbb{R}^n_+} |D^\alpha v(x + \epsilon e_n) - D^\alpha v(x)|^p dx = \int_{\mathbb{R}^n} |v_\alpha(x + \epsilon e_n) - v_\alpha(x)|^p dx - \int_{-\epsilon < x_n < 0} |D^\alpha v(x + \epsilon e_n)|^p dx.$$

$$\to 0 \text{ as } \epsilon \to 0$$

Hence $\eta_\varepsilon u \to u$ in $W^{m,p}(\mathbb{R}^n_+)$ and $\eta_\varepsilon u \in C^m(\overline{\mathbb{R}^n_+})$ as $\varepsilon \to 0$. Since $C^m(\mathbb{R}^n_+) \cap W^{m,p}(\mathbb{R}^n_+)$ is dense in $W^{m,p}(\mathbb{R}^n_+)$, we therefore have $C^m(\overline{\mathbb{R}^n_+}) \cap W^{m,p}(\mathbb{R}^n_+)$ is dense in $W^{m,p}(\mathbb{R}^n_+)$. This proves the theorem.

Following the same argument as in theorem 11, it follows that

Theorem 13: Let Ω be a bounded open set with smooth boundary of order m. Then $C^m(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ for all $1 \le p < \infty$.

Rellich compactness result: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary of order m. Let $E_\Omega : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ be an extension operator. Let $\overline{\Omega} \subset \Omega_1$, and $\overline{\Omega_1}$ is compact. Let $\chi \in C_0^\infty(\Omega_1)$ such that $\chi = 1$ on $\overline{\Omega}$. Let $E = \chi E_\Omega$. Then E is an extension operator satisfying

(1) $u \in C^m(\overline{\Omega}) \Rightarrow \operatorname{Supp} Eu \subset\subset \operatorname{Supp}\chi = K \subset\subset \Omega_1$.

(2) $\|Eu\|_{m,p,\mathbb{R}^n} \le C \|u\|_{m,p,\Omega}$

(3) $Eu = u$ on Ω.

Definition: Denote $W_0^{m,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with $W^{m,p}(\Omega)$ norm.

Theorem 14 Let $1 \le p < \infty$ and $m \ge 0$ be an integer. Then

(1) Let $mp < n$ and $p^* = \frac{np}{n-mp}$. Then

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$$

(2) Let Ω be a bounded domain with smooth boundary of order m. Let $mp \le n$. Then for each $1 \le q \le \frac{np}{n-mp}$, $u \in W^{m,p}(\Omega)$

(i) $\|u\|_{q,\Omega} \le c_q \|u\|_{m,p,\Omega}$.

(ii) If $mp = n$, then $\|D^k u\|_{q,\Omega} \le c \|u\|_{m,p,\Omega}$ $\forall\, q < \infty$ $\forall\, k \le m-1$

Proof. Let $mp < n$ and denote

$$D^m u = (D^\alpha u)_{|\alpha|=m}$$

$$\|D^m u\|_{p,\mathbb{R}^n} = \sum_{|\alpha|=m} \|D^\alpha u\|_{p,\mathbb{R}^n}$$

By Sobolev imbedding (1) is true for $m=1$. Assume it is true for $m-1$. Let $u \in W^{m,p} \Rightarrow Du \in W^{m-1,p}$ and by induction $Du \in L^q$ where $q = \frac{np}{n-(m-1)p}$ and

$$\|Du\|_{q,\mathbb{R}^n} \le c \|Du\|_{m-1,p,\mathbb{R}^n} \le c \|u\|_{m,p,\mathbb{R}^n}.$$

but $p^* = \frac{nq}{n-q} = \frac{n \frac{np}{n-(m-1)p}}{n - \frac{np}{n-(m-1)p}} = \frac{np}{n-mp}$

and by Sobolev embedding

$$\|u\|_{p^*,\mathbb{R}^n} \le c\,\|Du\|_{q_1,\mathbb{R}^n} \le c\,\|u\|_{m,p,\mathbb{R}^n}.$$

This proves (1).

(2) First we recall Sobolev imbedding. For $u \in C_0^1(\mathbb{R}^n)$, $r \ge 0$ we have

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{rn}{n-1}}\right)^{\frac{n-1}{n}} \le r\left(\int_{\mathbb{R}^n} |u|^{(r-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\right)^{1/p}.$$

First $n = p$, then

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{rn}{n-1}}\right)^{\frac{n-1}{n}} \le r\left(\int_{\mathbb{R}^n} |u|^{(r-1)\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \left(\int_{\mathbb{R}^n} |Du|^p\right)^{1/p}.$$

First $\operatorname{Supp} u \subset K \subset\subset \mathbb{R}^n$ a compact set. Then for $l \ge 0$, $r - l \ge 0$ we have

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{rn}{n-1}}\right)^{\frac{n-1}{n}} \le r(r-1)\cdots(r-l)\left(\int_{\mathbb{R}^n} |u|^{(r-l)\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \left(\int_{\mathbb{R}^n} |Du|^p\right)^{1/p}. \quad -(*)$$

Let $\Omega \subset \Omega_1$, and $E: W^{m,p}(\Omega) \to W_0^{m,p}(\Omega_1)$ be the extension operator as constructed earlier.

Corollary 1. Let $mp \le n$. $u \in W^{m,p}(\Omega)$ and $v = Eu \in W_0^{m,p}(\Omega_1) \subset W^{m,p}(\mathbb{R}^n)$. Hence $Eu \in L^{p^*}(\mathbb{R}^n)$. Since $\operatorname{Supp} Eu \subset \Omega_1$, and hence for $1 \le q \le p^*$, let $q_1 = p^*/q$. Then

$$\int_\Omega |u|^q\,dx \le \int_{\Omega_1} |Eu|^q\,dx \le \left(\int |Eu|^{p^*}\right)^{q/p^*} |\Omega_1|^{1/q_1}$$

$$\leq c_1 |\Omega_1|^{1/q'} \|Eu\|_{m,p}$$

$$\leq c_2^q \|u\|^q_{m,p,\Omega}$$

where $c_2 = c(p, \Omega_1, n) > 0$.

i.e. $$\|u\|_{q,\Omega} \leq c_2 \|u\|_{m,p,\Omega}.$$

Case (ii) $mp = n$.

First consider the case when $m = 1$.

Let $u \in W^{1,n}(\Omega)$ then $Eu \in W_0^{1,n}(\Omega_1) \subset W^{1,n}(\mathbb{R}^n)$ and $u \in L^n(\Omega)$ $Eu \in L^n(\mathbb{R}^n)$ then $u \in L^q(\Omega)$ for all $1 \leq q \leq n$. Let $\alpha \geq 1$ and define γ and ℓ by

$$\gamma = \alpha(n-1), \quad \ell = (\alpha-1)(n-1)$$

Then by (xx)

$$\left(\int_\Omega |u|^{\alpha n}\right)^{\frac{n-1}{n}} \leq \left(\int_{\mathbb{R}^n} |Eu|^{\alpha n}\right)^{\frac{n-1}{n}}$$

$$\leq c \left(\int_{\mathbb{R}^n} |Eu|^n\right)^{\frac{n-1}{n}} \left(\int_{\mathbb{R}^n} |DEu|^n\right)^{(\alpha-1)\left(\frac{n-1}{n}\right)}$$

i.e.

$$\left(\int_\Omega |u|^{\alpha n}\right)^{\frac{1}{\alpha n}} \leq c \left(\int_{\mathbb{R}^n} |Eu|^n\right)^{\frac{1}{\alpha n}} \left(\int_{\mathbb{R}^n} |DEu|^n\right)^{\frac{\alpha-1}{\alpha n}}$$

$$\leq c \left(\int |DEu|^n\right)^{\frac{1}{\alpha n}} \left(\int |DEu|^n\right)^{\frac{\alpha-1}{\alpha n}}$$

$$= c \left(\int_{\mathbb{R}^n} |DEu|^n\right)^{1/n}$$

$$\leq c \|Eu\|_{1,n,\mathbb{R}^n} \leq c \|u\|_{1,n,\Omega}$$

where $C = C(k,n,\Omega) > 0$. This is true $\forall\, \alpha \in \mathbb{N}$ and since $|\Omega| < \infty$ and hence $u \in L^q(\Omega)$ $\forall\, 1 \le q < \infty$.

(iii) Let $mp = n$ and $m > 1$. Let $u \in W^{m,p}(\Omega)$. Then for $0 \le k < m$, $D^{k-1}u \in L^p$, $D(D^k u) \in L^p$ and hence $D^{k-1}u \in W^{1,p}(\Omega) \Rightarrow D^{k-1}u \in L^q(\Omega)$ $\forall\, q \ge 1$. This proves the theorem.

Interpolation inequalities:

Lemma 1: Let $1 \le p < \infty$. Then $\exists\, C > 0$ such that for all $u \in C^2[0,1]$, we have for $I = [0,1]$

$$|u'(0)|^p \le C \left\{ \|u\|^p_{0,p,I} + \|u''\|^p_{0,p,I} \right\}$$

Proof: Let $0 \le x \le \frac{1}{3}$, $\frac{2}{3} \le y \le 1$. Then by mean value theorem, there exist $\xi \in [x,y]$ such that

$$|u'(\xi)| = \left| \frac{u(x) - u(y)}{x - y} \right| \le 3\left(|u(x)| + |u(y)|\right).$$

$$u'(0) = u'(\xi) + \int_0^{\xi} u''(t)\,dt$$

$$|u'(0)| \le |u'(\xi)| + \int_0^1 |u''(t)|\,dt$$

$$\le 2^{\frac{p-1}{p}} \left(|u(\xi)|^p + \left(\int_0^1 |u''(t)|\,dt \right)^p \right)^{1/p}$$

$$\le 2^{\frac{p-1}{p}} \left(3^p \left(|u(x)|^p + |u(y)|^p + \left(\int_0^1 (u'(t))^p\,dt \right) \right) \right)^{1/p}.$$

$$|u(0)|^p \le 2^{p-1} \left(3^p \left(|u(x)|^p + |u(y)|^p \right) + \int_0^1 |u''(t)|^p\,dt \right)$$

$$\frac{1}{3^2}|u'(0)|^p = \int_0^{1/3}\int_{2/3}^{1}|u'(0)|^p\,dxdy \le 2^{p-1}\left(3^p\int_0^{1/3}\int_{2/3}^{1}|u(x)|^p dxdy\right.$$

$$+\int_0^{1/3}\int_{2/3}^{1}|u(y)|^p dxdy$$

$$\left.+\int_0^{1/3}\int_{2/3}^{1}\int_0^1|u''(t)|^p\,dxdydt\right)$$

$$\le 2^{p-1}3^{p-1}\left(\|u\|^p_{0,p,I}+\|u''\|^p_{0,p,I}\right)$$

$$|u'(0)|^p \le C\left(\|u\|^p_{0,p,I}+\|u''\|^p_{0,p,I}\right).$$

This proves the lemma.

Lemma 2. Let $\delta>0$, $I=[0,\delta]$. Then $\exists\, C>0$ s.t. for all $u\in C^2(I)$, we have

$$|u'(0)|^p \le \frac{C}{\delta}\left(\delta^{-p}\|u\|^p_{0,p,I}+\delta^p\|u''\|^p_{0,p,I}\right).$$

Proof. Let $g(t)=u(\delta t)$ for $t\in[0,1]$, then

$$g'(t)=\delta u'(\delta t),\quad g''(t)=\delta^2 u''(\delta t),$$

$$\int_0^1|g(t)|^p dt=\frac{1}{\delta}\int_0^\delta|u(\theta)|^p d\theta$$

$$\int_0^1|g''(t)|^p dt=\frac{\delta^{2p}}{\delta}\int_0^\delta|u''(\theta)|^p d\theta.$$

Hence by Lemma 1, we have

$$|u'(0)|^p \le \frac{C}{\delta}\left\{\delta^{-p}\|u\|^p_{0,p,I}+\delta^p\|u''\|^p_{0,p,I}\right\}.$$

This proves the Lemma.

Definition. For $u \in C^m(\Omega)$ and $0 \le k \le m$, denote

$$|u|_{k,p,\Omega} = \begin{cases} \sum_{|\alpha|=k} |D^\alpha u|_{0,p,\Omega} & \text{if } p < \infty \\ \operatorname{ess\,sup}_{x \in \Omega} \sum_{|\alpha|=k} |D^\alpha u(x)| & \text{if } p = \infty \end{cases}$$

Lemma 3. $\exists\, C > 0$ such that for all $u \in C_0^2(\mathbb{R}^n)$ for $0 < \rho$, we have

$$|u|_{1,p,\mathbb{R}^n} \le C \left\{ \rho\, |u|_{0,p,\mathbb{R}^n} + \rho^{-1} |u|_{2,p,\mathbb{R}^n} \right\}$$

Proof:- Let $S^{n-1} = \{ w \in \mathbb{R}^n ;\ |w| = 1 \}$ the unit sphere. Let $x \in \mathbb{R}^n$, $w \in S^{n-1}$, Consider

$$v(t) = u(x + tw) \qquad t \in [0, \rho]$$

Then

$$v'(t) = \langle \nabla u(x+tw), w \rangle = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x+tw) w_i$$

$$v''(t) = \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(x+tw)\, w_i w_j$$

Hence by Lemma 2, we have

$$|\langle \nabla u(x), w \rangle|^p \le \frac{C}{\rho} \left\{ \rho^{-p} \int_0^{\rho} |u(x+tw)|^p dt + \rho^p \int_0^{\rho} |D^2 u(x+tw)|^p dt \right\}$$

Observe that $\exists\, C > 0$ such that $\forall\, v \in \mathbb{R}^n$

$$\int_{S^{n-1}} |\langle v, w \rangle|^p dw \ge C |v|^p.$$

Hence

$$|\nabla u(n)|^p \le C \int_{S^{n-1}} |\langle \nabla u(n), \omega \rangle|^p d\omega$$

$$\le \frac{C}{\delta} \Big\{ \delta^{-p} \int_{S^{n-1}} \int_0^{\delta} |u(n+t\omega)|^p dt\, d\omega + \delta^p \int_{S^{n-1}} \int_0^{\delta} |D^2 u(n+t\omega)|^p dt\, d\omega \Big\}$$

Hence by change of variables to obtain

$$\int_{\mathbb{R}^n} |\nabla u(n)|^p dn \le \frac{C}{\delta} \Big\{ \delta^{-p} \int_{S^{n-1}} \int_0^{\delta} \Big(\int_{\mathbb{R}^n} |u(n+t\omega)|^p dn \Big) dt\, d\rho + \delta^p \int_{S^{n-1}} \int_0^{\delta} \Big(\int_{\mathbb{R}^n} |D^2 u(n+t\omega)| dn \Big) dt\, d\rho \Big\}$$

$$\Rightarrow \boxed{\|u\|^p_{1,p,\mathbb{R}^n} \le C \Big\{ \delta^{-p} \|u\|^p_{0,p,\mathbb{R}^n} + \delta^p |u|^p_{2,p,\mathbb{R}^n} \Big\}}$$

While $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a \ge 0, b \ge 0$, $0 \le \alpha \le 1$, hence

$$|u|_{1,p,\mathbb{R}^n} \le C \Big\{ \delta^{-1} \|u\|_{0,p,\mathbb{R}^n} + \delta |u|_{2,p,\mathbb{R}^n} \Big\}.$$

This proves the Lemma.

Corollary 4: Let $1 \le k \le m-1$, and $0 < \delta$. Then $\exists\, C > 0$ s.t. $\forall\, u \in W^{m,p}(\mathbb{R}^n)$

$$|u|_{k,p,\mathbb{R}^n} \le C \Big\{ \delta^k |u|_{0,p,\mathbb{R}^n} + \delta^{-(m-k)} |u|_{m,p,\mathbb{R}^n} \Big\} \quad (*)$$

Proof: Proof is by induction: Denote $|u|_{k,p} = |u|_{k,p,\mathbb{R}^n}$,

Then from Lemma 3, we have.

$$|u|_{1,p} \le C\{\rho|u|_{0,p} + \rho^{-1}|u|_{1,p}\}.$$

Hence

$$|u|_{2,p} = |Du|_{1,p} \le C\{\rho|Du|_{0,p} + \rho^{-1}|Du|_{2,p}\}$$
$$\le C\{\rho|u|_{1,p} + \rho^{-1}|u|_{3,p}\}$$
$$\le C\{\rho\{\epsilon|u|_{0,p} + \epsilon^{-1}|u|_{2,p}) + \rho^{-1}|u|_{3,p}\}$$
$$\le C\{\rho C|u|_{0,\epsilon} + \rho\epsilon^{-1}|u|_{2,p} + \rho^{-1}|u|_{3,p}\}.$$

Now choose ϵ s.t. $C\rho\epsilon^{-1} = \frac{1}{2}$. Then $\epsilon = C\rho$ and

$$(1 - C\rho\epsilon^{-1})|u|_{2,p} \le C\{\rho^2|u|_{0,p} + \rho^{-1}|u|_{3,p}\}$$

i.e. $$|u|_{2,p} \le 2C\{\rho^2|u|_{0,p} + \rho^{-1}|u|_{3,p}\}.$$

$$|u|_{1,p} \le C\{\rho|u|_{0,p} + \rho^{-1}|u|_{2,p}\}$$
$$\le C\{\rho|u|_{0,p} + \rho^{-1}(\epsilon^2|u|_{0,p} + \epsilon^{-1}|u|_{3,p})\}$$
$$\le C\{(\rho + \rho^{-1}\epsilon^2)|u|_{0,p} + (\rho\epsilon)^{-1}|u|_{3,p}\}.$$

Let $\epsilon = \rho$, Then

$$|u|_{1,p} \le C\{\rho|u|_{0,p} + \rho^{-2}|u|_{3,p}\}.$$

Hence assume that (*) holds for $1 \le k \le m-1$, that $u \in C^{m+1}$, and $1 \le k \le m$. Then for $k \le m$,

$$|u|_{k,p} = |Du|_{k-1,p} \le C\{\rho^{k-1}|Du|_{0,p} + \rho^{-(m-k+1)}|Du|_{m,p}\}$$
$$\le C\{\rho^{k-1}|u|_{1,p} + \rho^{-(m-k+1)}|u|_{m+1,p}\}$$
$$\le C\{\rho^{k-1}\{\epsilon|u|_{0,p} + \epsilon^{-(k-1)}|u|_{k,p}\} + \epsilon^{-(m-k+1)}|u|_{m+1,p}\}$$

Choose $C\epsilon^{-(k-1)}\rho^{(k-1)} = \frac{1}{2}$, Then

$$\left(1-\frac{1}{2}\right)|u|_{k,p} \le c\left\{\varepsilon^{k}|u|_{0,p}+\varepsilon^{-(m+1-k)}|u|_{m,p}\right\}$$

This proves the corrollary.

Corrollary 5: $\forall\ 0\le k\le m$

$$|u|_{k,p,\mathbb{R}^n} \le c\,|u|_{0,p,\mathbb{R}^n}^{1-k/m}\,|u|_{m,p,\mathbb{R}^n}^{k/m} \le c\,|u|_{0,p,\mathbb{R}^n}^{1-k/m}\,\|u\|_{m,p,}^{k/m}$$

Proof:

$$|u|_{k,p,\mathbb{R}^n} \le c\left\{\varepsilon^{k}|u|_{0,p,\mathbb{R}^n}+\varepsilon^{-(m-k)}|u|_{m,p,\mathbb{R}^n}\right\}$$

Choose ε such that

$$\varepsilon^{k}|u|_{0,p,\mathbb{R}^n} = \varepsilon^{-(m-k)}|u|_{m,p,\mathbb{R}^n}$$

i.e $$\varepsilon = \left(\frac{|u|_{m,p,\mathbb{R}^n}}{|u|_{0,p,\mathbb{R}^n}}\right)^{1/m}$$

Hence $$\varepsilon^{k}|u|_{0,p,\mathbb{R}^n} = |u|_{0,p,\mathbb{R}^n}^{1-k/m}\,|u|_{m,p,\mathbb{R}^n}^{k/m}$$

Therefore

$$|u|_{k,p,\mathbb{R}^n} \le c\,|u|_{0,p,\mathbb{R}^n}^{1-k/m}\,\|u\|_{m,p,\mathbb{R}^n}^{k/m}$$

holds for $1\le k\le m-1$. For $k=0$ or $k=m$, the inequality holds trivially by taking $C=1$. This proves the corrollary.

Interpolation theorem 6: Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary of order k. Let $0\le k\le m$. Then $\exists\ c>0$ s.t. $\forall\ u\in W^{m,p}(\Omega)$,

$$|u|_{k,p,\Omega} \le c\,|u|_{0,p,\Omega}^{1-k/m}\,\|u\|_{m,p,\Omega}^{k/m}.$$

Proof. Let $E: W^{m,p}(\Omega)\to W^{m,p}(\mathbb{R}^n)$ be an extension operator. Then from corrollary 5, we have

$$|u|_{k,p,\Omega} \le |Eu|_{k,p,\mathbb{R}^n}$$
$$\le C\,|Eu|_{0,p,\mathbb{R}^n}^{1-k/m}\,|Eu|_{m,p,\mathbb{R}^n}^{k/m}$$
$$\le C\,|u|_{0,p,\Omega}^{1-k/m}\,\|u\|_{m,p,\Omega}^{k/m}$$

This proves the theorem.

Compactness in L^p spaces.

Theorem 7. Let $\{f_k\} \subset L^p(\mathbb{R}^n)$ be a sequence such that

(i) $\|f_k\|_p \le M$ for some $M > 0$, $\forall k \ge 1$

(ii) Tight: Given $\eta > 0$, $\exists\ R(\eta) > 0$ such that for all $k \ge 1$

$$\int_{|x| > R(\eta)} |f_k|^p \, dx < \eta.$$

(iii) $\forall\ \eta > 0$, $\exists\ \delta > 0$ such that for all $k \ge 1$

$$\int_{\mathbb{R}^n} |f_k(x) - f_k(x+h)|^p < \eta \quad \forall\ |h| < \delta.$$

Then there exist a convergent subsequence of $\{f_k\}$.

Proof: Let $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(x/\epsilon)$ be mollifiers, i.e. $\phi_1(x) \ge 0$, $\operatorname{supp}\phi_1 \subset B(0,1)$ and $\|\phi_1\|_1 = 1$. Let

$$f_{k,\epsilon} = f_k * \phi_\epsilon.$$

Then for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$f_{k,\epsilon}(x) - f_k(x) = \int_{\mathbb{R}^n} (f_k(x - \epsilon y) - f(x)) g(y)\, dy$$

$$|f_{k,\epsilon}(x) - f_k(x)| \leq \int |f_k(x - \epsilon y) - f(x)|\, g(y)^{1/p} g(y)^{1/p'} dy$$

$$\leq \left(\int |f_k(x - \epsilon y) - f(x)|^p g(y)\, dy \right)^{1/p} \left(\int g(y)\, dy \right)^{1/p'}$$

Hence

$$|f_{k,\epsilon}(x) - f_k(x)|^p \leq \int |f_k(x - \epsilon y) - f(x)|^p g(y)\, dy$$

$$\int |f_{k,\epsilon}(x) - f_k(x)|^p \leq \int \left(\int_{B(0,1)} |f_k(x - \epsilon y) - f(x)|^p dx \right) g(y)\, dy$$

$|f(x - \epsilon y) - x| = |\epsilon y| \leq \epsilon$ for $|y| \leq 1$. Hence from the hypothesis (iii) we have given $\eta > 0$, $\exists\ \delta > 0$ such that for all $0 < \epsilon < \delta$, $\forall\ k \geq 1$

$$\int |f_{k,\epsilon}(x) - f_k(x)|^p dx \leq \eta \qquad \text{---(1)}$$

Claim: $\forall\ k > 0$, $\{f_{k,\epsilon}\}$ is uniformly bounded and uniformly continuous family of functions.

$$|f_{k,\epsilon}(x) - f_{k,\epsilon}(z)| = \left| \int (f_k(x - \epsilon y) - f_k(z - \epsilon y)) g(y)\, dy \right|$$

$$\leq \left(\int |f_n(x-\epsilon y) - f_k(z-\epsilon y)|^p \rho(y)\, dy \right)^{1/p}$$

$$\leq \left(\int_{B(0,1)} |f_k(x-\epsilon y) - f_k(z-\epsilon y)|^p \, dy \right)^{1/p}$$

$$\leq \frac{1}{\epsilon^{n/p}} \left(\int_{B(0,\epsilon)} |f_k(x-y) - f_k(z-y)|^p \, dy \right)^{1/p}$$

Hence from (iii), $\forall\ \eta > 0$, $\exists\ \delta > 0$ such that whenever $|(x-y)-(z-y)| = |x-z| < \delta$, and for $\forall\ k \geq 1$

$$|f_{k,\epsilon}(x) - f_{k,\epsilon}(z)|^p \leq \frac{\eta}{\epsilon^n} \quad -(2)$$

From Hypothesis (ii), we have for $\eta > 0$, choose $R(\eta) > 0$ such that $\forall\ k$

$$\int_{|y| > R(\eta) - \epsilon} |f_k(y)|^p \, dy < \eta.$$

Then for $0 < \epsilon \leq 1$, $|x| > R(\eta)$, we have

$$|f_{k,\epsilon}(x)|^p \leq \int |f_k(y)|^p \rho_\epsilon(x-y)\, dy$$

$$= \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} |f_k(y)|^p \, dy$$

$$\leq \frac{1}{\epsilon^n} \int_{|y| > R(\eta) - \epsilon} |f_k(y)|^p \, dy$$

$$< \eta/\epsilon^n.$$

Hence $f_{k,\epsilon}(n) \to 0$ as $|n| \to \infty$ uniformly in k. and

$$\|f_{k,\epsilon}\|_\infty^p \le \int_{\mathbb{R}^n} |f_k|(n-\epsilon y)|^p dy \le \frac{M}{\epsilon^n}$$

Hence $\{f_{k,\epsilon}\}$ is a uniformly bounded equicontinuous family for each fixed $\epsilon > 0$. Skip from the claim.

Furthermore we have for all $k \ge 1$

$$\int_{|n|>R(\eta)} |f_{k,\epsilon}(n)|^p dn \le \frac{1}{\epsilon^n} \int_{B(0,\epsilon)} \left(\int_{|n|>R(\eta)} |f(n-\epsilon y)|^p dn \right) dy$$

$$\le \frac{C}{\epsilon^n} \int_{|n|>R(\eta)-\epsilon} |f(z)|^p dz$$

$$\le \frac{C}{\epsilon^n} \eta \qquad -(3)$$

Put $\epsilon = \frac{1}{m} = \epsilon_m$ and $f_{k,m} = f_{k,\epsilon_m}$. Then by Arzela-Ascoli and Cantor's diagonalization process, we can extract a subsequence still denote by $\{f_{k,m}\}$ such that for each fixed m, $\{f_{k,m}\}$ is cauchy sequence in $C^0(\mathbb{R}^n)$.

From (3), $\forall \eta > 0$, and $m \ge 1$, choose $R_0 = R_0(m,3)$ and $R(m,\eta) > 0$ such that for all $l, k \ge R_0$,

$$\left(\int_{|x|>R(m,\eta)} |f_{k,m}(x)|^p dx\right)^{1/p} < \eta^{1/p} \qquad (4)$$

$$\left(\int_{|x|\le R(m,\eta)} |f_{k,m}(x) - f_{l,m}(x)|^p dx\right)^{1/p} \le \eta^{1/p} \qquad (5)$$

Hence we know for $k, l \ge k_0(m,\eta)$

$$\|f_{k,m} - f_{l,m}\|_p^p \le \int_{|x|\le R(m,\eta)} |f_{km} - f_{l,m}|^p dx$$

$$\le C\left(\int_{|x|\ge R(m,\eta)} |f_{k,m}|^p dx \int_{|x|\ge R(m,\eta)} |f_{l,m}|^p dx\right)$$

$$\le (C_1\eta)^p \qquad -(6)$$

Let $\eta > 0$. Then from (1) choose $m \ge 1$ such that $\forall\ k \ge 1$

$$\|f_{k,m} - f_k\|_p < \eta$$

Let $k_0 = k_0(m,\eta)$ be as defined above. Then for $k, l \ge k_0$ we have

$$\|f_k - f_l\|_p \le \|f_k - f_{km}\|_p + \|f_{km} - f_{l,m}\|_p + \|f_{l,m} - f_l\|_p$$

$$< \eta + C_1\eta + \eta = (2 + C_1)\eta.$$

Hence $\{f_k\}$ is cauchy in L^p and this proves the theorem.

As an immediate consequence of this, we have the following

Theorem 8 (Rellich-Kondrachov): Let Ω be a bounded domain with smooth boundary of order m. Let $\{u_k\} \subset W^{1,p}(\Omega)$ be a bounded sequence. Then for $1 \le q < p^* \equiv \frac{np}{n-p}$, $\{u_k\}$ has a convergent subsequence in $L^q(\Omega)$.

Proof: Let $\overline{\Omega} \subset \Omega_1$ and $\overline{\Omega_1}$ is compact in $\mathbb{R}^n$. Let $E: W^{1,p}(\Omega) \to W_0^{1,p}(\Omega_1)$ be an extension operator. It is enough to show that $\{Eu_k\}$ has convergent subsequence in $L^q(\Omega_1)$. Hence without loss of generality we can assume that

$$\begin{cases} \{u_k\} \subset \overset{\circ}{C}{}^1_0(\mathbb{R}^n), \\ \operatorname{Supp} u_k \subset \Omega_1 \\ \|u_k\|_{1,p,\mathbb{R}^n} \le M \quad \forall k. \end{cases}$$

From the mean value theorem, we have

$$u_k(x+h) - u_k(x) = \int_0^1 \frac{d}{dt} u_k(x+th)\, dt$$

$$= \int_0^1 \langle \nabla u_k(x+th), h \rangle dt.$$

$$|u_k(x+h) - u_k(x)|^p \le |h|^p \int_0^1 |\nabla u_k(x+th)|^p dt$$

$$\int_{\mathbb{R}^n} |u_k(x+h) - u_k(x)|^p \le h \int_0^1 \int_{\mathbb{R}^n} |\nabla u_k(x+th)|^p \, dx \, dt$$

$$\le h \int_{\mathbb{R}^n} |\nabla u_k(x)|^p \, dx$$

$$= h \, \|u_k\|_{1,p}^p$$

$$\le h^p M^p \quad -(1)$$

Since $\operatorname{supp} u_k \subset \Omega_1$, hence choose $R_0 > 0$ such $\Omega_1 \subset B(0, R_1)$. Then $\forall\, k$

$$\int_{|x| > R_1} |u_k(x)|^p \, dx = 0.$$

Hence $\{u_k\} \subset L^p(\mathbb{R}^n)$ satisfies the hypothesis of the Theorem 7. Hence $\{u_k\}$ has a convergent subsequence in $L^p(\mathbb{R}^n)$ and hence in $L^p(\Omega_1)$.

In order to complete the proof of the theorem, we need the following

Interpolation inequality: Let $(X, \mathcal{A}, \mu)$ be a measure space and $L^p(X, \mu)$ denote the L^p-Space. Let $p_1 < q < p_2$, and $\frac{1}{q} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Then for $u \in L^{p_1} \cap L^{p_2}$, we have

$$\|u\|_{q,X} \le \|u\|_{p_1,X}^{\theta} \, \|u\|_{p_2,X}^{1-\theta}.$$

Assuming this, we complete the proof of the theorem.

By Sobolev imbedding, we have $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and hence $u_k \in L^p \cap L^{p^*}$. Let $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$. Then

$$\|u_k - u_\ell\|_q \le \|u_k - u_\ell\|_p^{\theta} \, \|u_k - u_\ell\|_{p^*}^{1-\theta}$$

$$\le c \, \|u_k - u_\ell\|_p^{\theta} \, \|u_k - u_\ell\|_{1,p}^{1-\theta}$$

$$\le c M^{1-\theta} \, \|u_k - u_\ell\|_p^{\theta}.$$

Hence $\{u_k\}$ is cauchy in L^q. Now let $1 \le q < p$. Since Ω_1 is bounded, we have

$$\|u_k - u_\ell\|_q^q = \int_{\Omega_1} |u_k - u_\ell|^q \, dx$$

$$\le \left(\int_{\Omega_1} |u_k - u_\ell|^p \right)^{q/p} |\Omega_1|^{\frac{p}{p-q}}$$

$$\to 0 \quad \text{as } k, \ell \to \infty.$$

Hence $\{u_k\}$ is cauchy in L^q for all $1 \le q < p^*$. This proves the theorem.

Proof of Interpolation inequality: Let $\lambda > 0$. Then

$$\int_\Omega |u|^q \, dx = \int_{|u| \le \lambda} |u|^q \, dx + \int_{|u| \ge \lambda} |u|^q \, dx.$$

$$\leq \lambda^{q-p_1} \int_{|u|<\lambda} |u|^{p_1} dx + \frac{1}{\lambda^{p_2-q}} \int_{|u|\geq\lambda} |u|^{p_2}$$

$$\leq \lambda^{q-p_1} \|u\|_{p_1,x}^{p_1} + \frac{1}{\lambda^{p_2-q}} \|u\|_{p_2,x}^{p_2}$$

Now choose λ such that

$$\lambda^{q-p_1} \|u\|_{p_1,x}^{p_1} = \frac{1}{\lambda^{p_2-q}} \|u\|_{p_2,x}^{p_2}$$

or

$$\lambda^{p_2-p_1} = \frac{\|u\|_{p_2,x}^{p_2}}{\|u\|_{p_1,x}^{p_1}}$$

Then $\lambda^{q-p_1} \|u\|_{p_1,x}^{p_1} = \|u\|_{p_1,x}^{p_1} \left(\frac{\|u\|_{p_2,x}^{p_2}}{\|u\|_{p_1,x}^{p_1}} \right)^{\frac{q-p_1}{p_2-p_1}}$

$$\|u\|_{q,x}^{q} \leq \|u\|_{p_1,x}^{p_1\left(1-\frac{q-p_1}{p_2-p_1}\right)} \|u\|_{p_2,x}^{p_2\left(\frac{q-p_1}{p_2-p_1}\right)}$$

$$\|u\|_{q,x} \leq \|u\|_{p_1,x}^{\frac{p_1}{q}\left(1-\frac{q-p_1}{p_2-p_1}\right)} \|u\|_{p_2,x}^{\frac{p_2}{q}\left(\frac{q-p_1}{p_2-p_1}\right)}$$

$$= \|u\|_{p,x}^{\theta} \|u\|_{p_2,x}^{1-\theta}$$

Thus proves the inequality.

Corollary 9. Let $n \geq 1$, and $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary of order m. Then let $p^* = \frac{np}{n-mp}$ and $q \geq 1$, $0 \leq k \leq m-1$. Then

(1) If $q < p^* = \frac{np}{n-mp}$, then

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact.}$$

(2) $$W^{m,p}(\Omega) \hookrightarrow W^{k,p}(\Omega) \text{ is compact.}$$

(3) If $p > n$, Then

$$W^{1,p}(\Omega) \hookrightarrow C^0(\Omega) \text{ is compact.}$$

Proof. Let $\{u_k\}$ be a bounded sequence in $W^{m,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$. Hence $\{u_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Hence $\{u_k\}$ has a convergent sequence in $L^p(\Omega)$, still denoted by u_k.

(1) If $p \leq q < p^*$, Then by interpolation inequality,

$$\|u_k - u_\ell\|_{q,\Omega} \leq C \|u_k - u_\ell\|_{p,\Omega}^{1-\theta} \|u_k - u_\ell\|_{m,p,\Omega}^{\theta}$$

$$\leq C \|u_k - u_\ell\|_{p,\Omega}^{1-\theta} \to 0 \text{ as } k, \ell \to \infty.$$

Hence $\{u_k\}$ converges in $L^q(\Omega)$

Since Ω is bounded, then for $1 \le q \le p$, we have

$$\int_\Omega |u_k - u_l|^q \, dx \le \left(\int_\Omega |u_k - u_l|^p \, dx \right)^{q/p} |\Omega|^{\frac{p-q}{p}}$$

$$\to 0 \quad \text{as } k, l \to \infty.$$

(2) Let $0 \le k \le m-1$. Then

$$|u_r - u_s|_{k,p,\Omega} \le c \, |u_r - u_s|_{p,\Omega}^{1-k/m} \, \|u_r - u_s\|_{m,p,\Omega}^{k/m}$$

$$\le c \, |u_r - u_s|_{p,\Omega}^{1-k/m}$$

$$\to 0 \quad \text{as } r, s \to \infty.$$

(3) $p > n \Rightarrow W^{1,p}(\Omega) \hookrightarrow C^{0, 1-n/p}(\Omega) \hookrightarrow C^0(\Omega)$

is compact by Arzela-Ascoli Theorem.

Skip from the Corollary.

Poincaré inequality.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary of order m. Let

$W_0^{m,p}(\Omega) =$ closure of $C_0^\infty(\Omega)$ w.r.t. m,p norm.

in a closed subset of $W^{m,p}(\Omega)$.

Theorem (Poincaré's inequality): $\exists\ C = C(n) > 0$ such that for all $u \in W_0^{1,p}(\Omega)$ $(1 \le p < \infty)$

(i) $$\left(\int_\Omega |u|^p dx\right) \le C(\Omega) \int_\Omega |\nabla u|^p dx.$$

(ii) $\forall\ u \in W^{1,p}(\Omega)$,

$$\int_\Omega |u - u_\Omega|^p dx \le C(n) \int_\Omega |\nabla u|^p dx$$

where $u_\Omega = \fint_\Omega u(x)\,dx$, the mean value of u on Ω.

Proof: Since Ω is bounded, there exist $M > 0$ such that

$$\Omega \subset \{x : -M < x_n < M\}$$

Then

$$u(x', x_n) = \int_{-M}^{x_n} \frac{\partial u}{\partial t}(x', t)\,dt$$

$$|u(x', x_n)|^p \le \left(\int_{-M}^{M} \left|\frac{\partial u}{\partial t}(x', t)\right|^p dt\right)(2M)^{1/p'}$$

$$\int_\Omega |u|^p dx = \int_{\mathbb{R}^n} |u(x', x_n)|^p dx' dx_n = \int_{\mathbb{R}^{n-1}} \int_{-M}^{M} |u(x', x_n)|^p dx_n dx'$$

$$\le (2M)^{1/p} (2M) \int_{\mathbb{R}^{n-1}} \int_{-M}^{M} \left|\frac{\partial u}{\partial t}(x', x_n)\right|^p dx dx_n$$

$$\le (2M)^{1+1/p} \int_\Omega |Du|^p dx.$$

Thm proven (i)

(ii) Suppose (ii) is false. Then $\exists$ sequence $u_k \in W^{1,p}(\Omega)$ such that as $k \to \infty$

$$\int_\Omega |Du_k|^p \, dx \to 0$$

$$\int_\Omega |u_k - u_{k,0}|^p \, dx = 1$$

but $v_k = u_k - u_{k,0}$, then $|v_k|_{1,p,\Omega} \to 0$, $|v_k|_{p,\Omega} = 1$

Hence $\|v_k\|_{1,p,\Omega} = |v_k|_{p,\Omega} + |v_k|_{1,p,\Omega} \leq 2 \quad \forall k$

$\therefore k \geq k_0$. Hence by Rellich's theorem, $\exists$ a convergent subsequence still denoted by v_k converges to v in $L^p(\Omega)$

Hence $|v|_{p,\Omega} = \lim_{k \to \infty} |v_k|_{p,\Omega} = 1$

$W^{1,p}(\Omega)$ is reflexive Banach space, hence $v_k \to v$ in $W^{1,p}(\Omega)$ but $\ell : W^{1,p}(\Omega) \to \mathbb{R}$ be defined by

$$\ell(w) = \int |\nabla v|^{p-2} \langle \nabla w \cdot \nabla v \rangle$$

$$|\ell(w)| \leq \int |\nabla v|^{p-1} |\nabla w| \leq \left(\int |\nabla v|^p\right)^{\frac{p-1}{p}} \left(\int |\nabla w|^p\right)^{1/p}$$

Hence $\ell \in (W^{1,p}(\Omega))^*$. Also

$$|\ell(v_k)| \leq |v|_{1,p,\Omega}^{p-1} |v_k|_{1,p,\Omega} \to 0 \text{ as } k \to \infty.$$

Hence

$$0 = \lim_{k\to\infty} I(u_k) = I(u) = \int_\Omega |\nabla u|^p \, dx$$

$$\Rightarrow \nabla u = 0$$

$$\Rightarrow u \text{ is constant}, \quad u = \alpha.$$

Since $\int_\Omega u_k = 0 \Rightarrow \alpha = \int u = 0 \Rightarrow \underline{u \equiv 0}$.

But $\int |u|^p \, dx = 1$ which is a contradiction. This proves the theorem.

Trace theorems

Fractional Sobolev Spaces. Let $0 < s < 1$, Then define $W^{s,p}(\Omega)$ as follows:

Let $u : \Omega \to \mathbb{R}$ be a measurable function and define

$$\|u\|^p_{s,p,\Omega} = \int_\Omega |u|^p \, d\sigma + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n-1+sp}} \, d\sigma(x) \, d\sigma(y)$$

$$W^{s,p}(\partial\Omega) = \{u : \|u\|_{s,p,\Omega} < \infty\}.$$

Theorem (Trace theorem).

Let $\gamma_0 : C^0(\bar{\Omega}) \to C^0(\partial\Omega)$ defined by

$$\gamma_0(u) = u|_{\partial\Omega}$$

Called the Trace map.